TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 182, August 1973

ON THE GREEN'S FUNCTION FOR THE BIHARMONIC EQUATION IN AN INFINITE WEDGE

BY

JOSEPH B. SEIF

ABSTRACT. The Green's function for the biharmonic equation in an infinite angular wedge is considered. The main result is that if the angle $\underline{\alpha}$ is less than $\alpha_1 \cong 0.812\pi$, then the Green's function does not remain positive; in fact it oscillates an infinite number of times near zero and near ∞. The method uses a number of transformations of the problem including the Fourier transform. The inversion of the Fourier transform is accomplished by means of the calculus of residues and depends on the zeros of a certain transcendental function. The distribution of these zeros in the complex plane gives rise to the determination of the angle α_1. A general expression for the asymptotic behavior of the solution near zero and near infinity is obtained. This result has the physical interpretation that if a thin elastic plate is deflected downward at a point, the resulting shape taken by the plate will have ripples which protrude above the initial plane of the plate.

1. **Introduction.** In this paper we consider the Green's function for the problem

$$\Delta\Delta u = f \quad \text{in } D, \qquad u = \partial u/\partial n = 0 \quad \text{on } \partial D, \tag{1.1}$$

for a certain two dimensional region D, where Δ denotes the 2-dimensional Laplacian, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The Green's function $G(x, x_0)$ is the solution of (1.1) where $f = \delta(x - x_0)$. The Green's function solves problem (1.1) in the sense that for any f the solution u is given by the integral operator

$$u(x) = \int_D G(x, x_0) f(x_0)\, dx_0.$$

It has the physical interpretation that if a thin elastic plate, clamped at ∂D, is given a unit load at the point x_0, then $G(x, x_0)$ is the resulting deflection at the point x from the original plane of the plate.

This problem is related to several classical questions in the calculus of variations. Szegö [8] showed for the homogeneous problem (1.1) with $f = 0$, that for all D of given area such that the first eigenfunction has no nodal lines, the

Received by the editors October 17, 1972.
AMS (MOS) subject classifications (1970). Primary 31B30; Secondary 35J40, 35B99.
Key words and phrases. Biharmonic equation, Green's function, asymptotic expansion.

circle has the lowest eigenvalue. A nodal line is a curve through the interior of D on which $u = 0$.

Numerical work by Bauer and Reiss [1] indicates that, for a square, nodal lines appear near the corner. Duffin and Shaffer [3] announced that for an annulus with inner radius r_0 and outer radius 1, the principal eigenfunction has a diametral nodal line if r_0 is small enough.

Hadamard [5] has conjectured that the Green's function for problem (1.1) is positive if D is bounded and convex. The physical interpretation is that a clamped plate will be deflected downward under the influence of a downward point load. Duffin [2] showed that G becomes negative if D is an infinite strip. Loewner and Szegö (unpublished) have exhibited bounded regions for which the conjecture is false, but these regions are not convex. The statement is true for a circle. Garabedian [4] has shown that for an ellipse whose major axis is not even twice as long as the minor axis, the Green's function takes negative values. This gives a conclusive counterexample to Hadamard's conjecture.

Osher [7] has shown that if D is the quarter plane, the sign of the Green's function oscillates infinitely often as $r \longrightarrow 0$ and $r \longrightarrow \infty$.

In this paper we generalize Osher's result to the case where D is an infinite wedge with angle α, $0 < \alpha < \pi$. We find an angle α_1 at which the behavior seems to change i.e. for $\alpha < \alpha_1$ the behavior near 0 and ∞ is the same as in Osher's case, but for $\alpha > \alpha_1$ the Green's function does not oscillate as we approach 0 or ∞ along radial lines ($\alpha \cong 0.812\pi$). However, we have not yet been able to show that the Green's function is actually positive for $\alpha > \alpha_1$ nor have we determined the behavior of G for $\alpha = \alpha_1$.

The idea of this paper is similar to Osher [7]. In §2 the problem is stated, D is mapped into an infinite strip and it turns out that the coefficients of the differential equation remain constant. Thus we apply a Fourier transform to one of the variables to obtain an ordinary differential equation with constant coefficients. To recapture the solution of the original equation we want to deform the path of integration of the inverse Fourier transform into the complex plane and so, in §3, we locate the poles with respect to the Fourier variable of the solution of this ordinary differential equation. In §4 we compute the residues at these poles. This is the step that we have not yet succeeded in doing for $\alpha = \alpha_1$. Finally, we state and prove our main theorems in §5.

It would be very interesting to find a direct physical intepretation of the angle α_1.

2. **Calculation of the solution.** The problem is now to solve

$$\begin{aligned} &\Delta\Delta u = f(r, \theta), \qquad 0 < r < \infty,\ 0 < \theta < \alpha, \\ &u(r, 0) = u(r, -\alpha) = u'(r, 0) = u'(r, -\alpha) = 0. \end{aligned} \tag{2.1}$$

We are here referring to the X, Y-plane in rectangular coordinates or the r, θ-plane in polar coordinates. We make the complex change of variables for $Z = X + iY$: $-\ln Z = \eta + i\zeta$ or

$$Z = e^{-(\eta + i\zeta)}, \qquad X = e^{-\eta}\cos\zeta, \qquad Y = e^{-\eta}\sin\zeta. \tag{2.2}$$

The equation (2.1) then becomes

$$(\partial^2/\partial\eta^2 + \partial^2/\partial\zeta^2)((\partial/\partial\eta + 2)^2 + \partial^2/\partial\zeta^2)\hat{u} = e^{-4\eta}g(e^{-\eta}\cos\zeta, -e^{-\eta}\sin\zeta) \tag{2.3}$$

with $f(r, \theta)$ in the region $-\infty < \eta < +\infty$, $-\alpha < \zeta < 0$ and the boundary conditions $\hat{u}(\eta, 0) = \hat{u}(\eta, -\alpha) = \hat{u}_\zeta(\eta, 0) = \hat{u}_\zeta(\eta, -\alpha) = 0$. We now apply a Fourier transform with respect to η, and call the new variable k. The equation now becomes

$$(\partial^2/\partial\zeta^2 - k^2)(\partial^2/\partial\zeta^2 + (ik + 2)^2)v(k, \zeta) = h(k, \zeta) \tag{2.4}$$

where $v(k, \zeta) = \mathfrak{F}_\eta u(\eta, \zeta)$ is the Fourier transform of u, where

$$h(k, \zeta) = \mathfrak{F}e^{-4\eta}g(e^{-\eta}\cos\zeta, -e^{-\eta}\sin\zeta)$$

with the boundary conditions $v(k, 0) = v(k, -\alpha) = v'(k, 0) = v'(k, -\alpha) = 0$ (here "$'$" denotes differentiation with respect to ζ). This approach was used by Kondrat'ev [6] in his work on more general elliptic equations in conical regions, and by Osher in [7].

So we have now transformed our equation to an ordinary differential equation with constant coefficients. The roots of its characteristic equation are $\pm k$ and $\pm(k - 2i)$. The solution is given by

$$v(k, \zeta) = v_0(k, \zeta) + v_0(k, -\alpha)(\sinh k\zeta/\sinh k\alpha) + CH(k, \zeta) + DH(k, \zeta + \alpha) \tag{2.5}$$

where

$$H(k, \zeta) \equiv -\frac{\sinh(k - 2i)\alpha}{\sinh k\alpha}\sinh k\zeta + \sinh(k - 2i)\zeta, \tag{2.6}$$

and

$$v_0(k, \zeta) = \frac{i}{4}\int_0^\zeta h(t)\left[\frac{\sinh k(t - \zeta)}{k(k - i)} - \frac{\sinh(k - 2i)(t - \zeta)}{(k - 2i)(k - i)}\right]dt \tag{2.7}$$

is some solution of the inhomogeneous equation. (2.5) also automatically satisfies the boundary conditions $v(k, 0) = v(k, -\alpha) = 0$ and we must choose C and D so that the remaining boundary conditions are satisfied. We solve a system of two linear equations

$$H'(0)C + H'(\alpha)D = -\frac{k}{\sinh k\alpha}v_0(-\alpha),$$
$$H'(-\alpha)C + H'(0)D = -v_0'(-\alpha) - k\coth k\alpha v_0(-\alpha)$$

(here and elsewhere we suppress the k dependence of C, D, H, and v_0). We denote by $E(k)$ the determinant of this system i.e. (using the fact that $H'(-\alpha) = H'(\alpha)$)

$$E(k) = [H'(0)]^2 - [H'(-\alpha)]^2$$

(2.8)

$$= -4\,\frac{\sinh (k-2i)\alpha}{\sinh k\alpha}\,[(k-i)^2 \sin^2 \alpha - \sinh^2 (k-i)\alpha]$$

(see the Appendix for this computation). We now get

$$C = \frac{1}{E(k)}\left[(-H'(0) + H'(\alpha)\cosh k\alpha)\,\frac{kv_0(-\alpha)}{\sinh k\alpha} + H'(\alpha)v_0'(-\alpha)\right],$$

(2.9)

$$D = \frac{1}{E(k)}\left[(H'(\alpha) - H'(0)\cosh k\alpha)\,\frac{kv_0(-\alpha)}{\sinh k\alpha} - H'(0)v_0'(-\alpha)\right].$$

We summarize these results in

Lemma 2.1. *The unique solution of problem* (2.4) *is given by equations* (2.5)–(2.9).

Our object is now to examine the solution of the original problem (2.1) by applying the inverse Fourier to the solution of (2.4). We shall see below that the asymptotic behavior of this solution depends on the poles of $v(k, \zeta)$ as an analytic function of k, in particular on the zeros of

$$p(k) = (k-i)^2 \sin^2\alpha - \sinh^2(k-i)\alpha. \tag{2.10}$$

We shall find these zeros in the next section.

3. The zeros of $p(k)$. We shall locate the zeros of $p(k)$ in the complex plane or, equivalently and more simply, we shall locate the zeros of

$$q(z) = z^2 \sin^2 \alpha - \sinh^2 \alpha z \tag{3.1}$$

where α is fixed, $0 < \alpha < \pi$.

Lemma 3.1. *The zeros of $q(z)$ are symmetric about the origin and the real axis (and hence also the imaginary axis) in the complex z-plane, i.e.*,

$$q(z) = 0 \tag{3.2}$$

if and only if (a) $q(-z) = 0$ *and* (b) $q(\bar{z}) = 0$.

The proof is immediate from the defining equation (3.1).

Because of Lemma 3.1 we need only look for the zeros of $q(z)$ in the first quadrant, so if we set $z = s + it$ we may assume that $s \geq 0$ and $t \geq 0$. Also it is obvious from equation (3.1) that $q(z)$ has purely imaginary zeros at 0, $\pm i$. We shall show that for $\alpha < \alpha_1$ (defined below) these are the only imaginary roots, while for $\alpha \geq \alpha_1$ there are other imaginary roots. The asymptotic behavior of $u(r, \theta)$ as $r \to 0$ and as $r \to \infty$ depends very heavily on which of these cases occur. At this point it will be useful to familiarize ourselves with the behavior of the function $|(\sin u)/u|$.

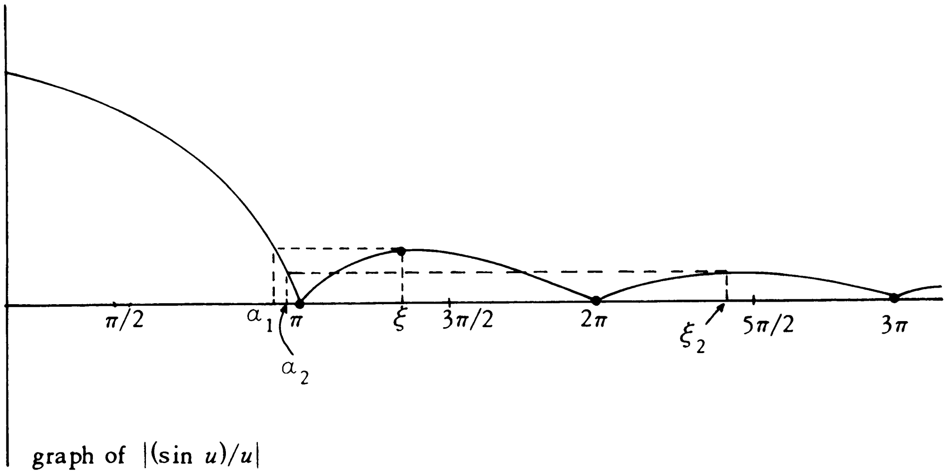

Figure 1

We note that this function is positive in each interval of the real line $(n\pi, (n+1)\pi)$. It is zero at the endpoints of these intervals. Within each such interval the function attains its maximum at a single point ξ_n, and this is the only critical point in the interval. Each point ξ_n satisfies the transcendental equation

$$\xi_n = \tan \xi_n. \tag{3.3}$$

The maximum of $|(\sin u)/u|$ in each of these intervals decreases with increasing n and has the value $\cos \xi_n$. The values ξ_n approach the midpoints of the intervals with increasing n. The first of these points, i.e. the solution of (3.3) lying in the interval $(\pi, 2\pi)$, we call ξ or $\xi = \xi_1$.

Definition 3.2. α_1 is the unique solution in the interval $(0, \pi)$ of

$$\alpha_1^{-1} \sin \alpha_1 = -\xi^{-1} \sin \xi \tag{3.4}$$

where $\xi = \xi_1$ is the unique root of (3.4) in the interval $(\pi, 2\pi)$ $(\alpha_1 \cong 0.812\pi)$. More generally, we define α_n to be the unique solution in the interval $(0, \pi)$ of

$$\alpha_n^{-1} \sin \alpha_n = (-1)^n \xi_n^{-1} \sin \xi_n \tag{3.5}$$

where ξ_n is the unique root of (3.3) in the interval $(n\pi, (n+1)\pi)$. (*Note.* $\alpha_1 < \alpha_2 < \cdots$ and $\alpha_n \to \pi$ as $n \to \infty$.)

Location of zeros of $p(k)$ in the complex plane

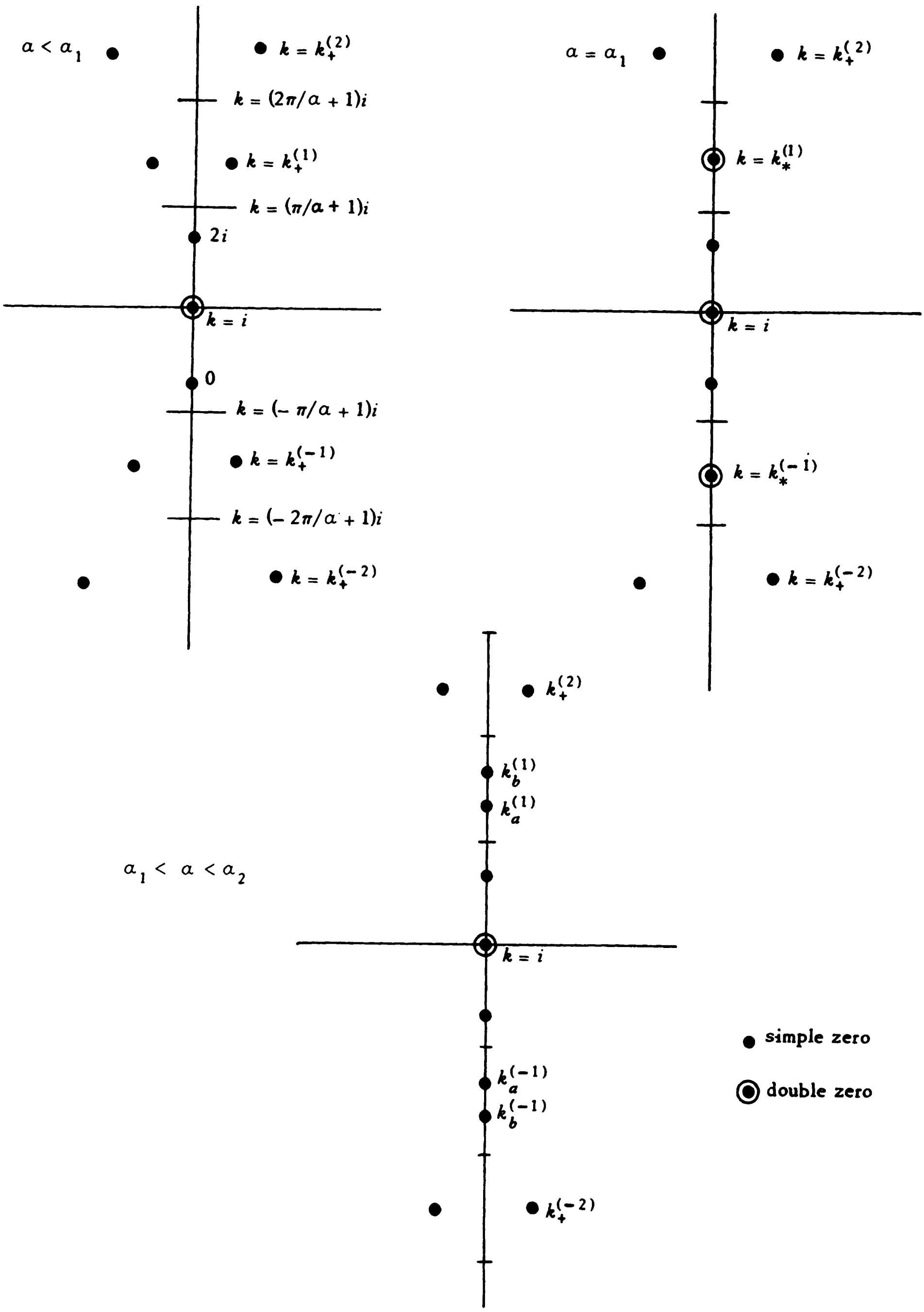

Figure 2

Lemma 3.2. *$q(z)$ has the following zeros and no others: For all α, $z = 0$ is a double root and $z = \pm i$ is a simple root. For the other roots (now considered only for $s \geq 0$, $t \geq 0$) we have the following cases:*

Case I. *If $\alpha < \alpha_1$, there is one simple root $z = s + it$ in each strip $n\pi/\alpha < t < (n + 1/2)\pi/\alpha$, $n = 1, 2, 3, \cdots$, with $s = \operatorname{Re} z > 0$. (Of course there are also the three symmetric roots given by Lemma 3.1.)*

Case II. *If $\alpha_k < \alpha < \alpha_{k+1}$, there are two simple purely imaginary roots in each interval $n\pi/\alpha < t < (n + 1)\pi/\alpha$, $n = 1, 2, 3, \cdots, k$, and (as in Case I) a simple complex root with $\operatorname{Re} z > 0$ in each strip $n\pi/\alpha < t < (n + 1/2)\pi/\alpha$, $n = k + 1, k + 2, \cdots$.*

Case III. *If $\alpha = \alpha_k$ then $q(z)$ has a double root at $z = \xi_k i/\alpha_k$, two simple purely imaginary roots in each interval $n\pi/\alpha < t < (n + 1)\pi/\alpha$, $n = 1, \cdots, k - 1$, and a simple root with $\operatorname{Re} z > 0$ in each strip $n\pi/\alpha < t < (n + 1/2)\pi/\alpha$, $n = k + 1, k + 2, \cdots$.*

Proof. 1. It is immediately obvious that $z = 0$ is a double root, and $z = \pm i$ is a root for any α.

2. To find the complex roots we obtain from (3.2)

$$z \sin \alpha = \pm \sinh \alpha z. \tag{3.6}$$

We write the real and imaginary parts of (3.6) to get

$$s \sin \alpha = \pm \sinh \alpha s \cos \alpha t, \tag{3.7}$$

$$t \sin \alpha = \pm \cosh \alpha s \sin \alpha t. \tag{3.8}$$

Now (3.8) gives

$$\cosh \alpha s = \pm (t \sin \alpha)/\sin \alpha t \tag{3.9}$$

which tells us immediately that

$$\pm(t \sin \alpha)/\sin \alpha t \geq 1. \tag{3.10}$$

In particular (3.10) implies that $\pm\sin \alpha t > 0$, so that if we take the + sign in (3.6) we have $2n\pi < \alpha t < (2n + 1)\pi$ and if we take the minus sign $(2n - 1)\pi < \alpha t < 2n\pi$ for some n. Now (3.9) yields

$$s = \alpha^{-1} \cosh^{-1}(\pm(t \sin \alpha)/\sin \alpha t) \tag{3.11}$$

where we take the positive branch of $\cosh^{-1}$ since we are assumming that $s \geq 0$. Then

$$\sinh \alpha s = \sqrt{((t \sin \alpha)/\sin \alpha t)^2 - 1}$$

where again we take the positive square root since $s \geq 0$. Then we can write (3.7) as

$$\frac{1}{\alpha}\cosh^{-1}\left(\pm\frac{t\sin\alpha}{\sin\alpha t}\right)\mp\sqrt{\left(\frac{t\sin\alpha}{\sin\alpha t}\right)^2-1}\cos\alpha t=0 \tag{3.12}$$

where, to be consistent, we take either the upper signs or the lower signs together. Let us first examine the case of the upper signs. We have to solve

$$T(t)=\cosh^{-1}\left(\frac{t\sin\alpha}{\sin\alpha t}\right)-\sqrt{\frac{t\sin\alpha}{\sin\alpha t}-1}\cos\alpha t=0 \tag{3.13}$$

for $2n\pi<\alpha t<(2n+1)\pi$. Now if $\alpha<\alpha_1$, then $(\sin\alpha)/\alpha>(\sin\alpha t)/\alpha t$ (see Figure 3.1) and $(t\sin\alpha)/\sin\alpha t=((\sin\alpha)/\alpha)/((\sin\alpha t)/\alpha t)>1$. More generally, if $\alpha<\alpha_{2k}$, then $(t\sin\alpha)/\sin\alpha t>1$ for $2n\pi<\alpha t<(2n+1)\pi$ with $n\geq k$, so that the right side of (3.13) is continuous with continuous derivative for $t\in(2n\pi/\alpha,(2n+1)\pi/\alpha)$. We now examine $T(t)$ at the points $t=2n\pi/\alpha$, $(2n+\frac{1}{2})\pi/\alpha$ and $(2n+1)\pi/\alpha$. At $2n\pi/\alpha$ both terms tend to $+\infty$ but the $\cosh^{-1}$ term grows only logarithmically so that it is dominated by the square root term. Since $\cos(2n\pi)=1$, $T(t)\to-\infty$ for $t\to 2n\pi/\alpha$. At $t=(2n+\frac{1}{2})\pi/\alpha$, $T(t)>0$, since $\cos\alpha t=0$. At $t=(2n+1)\pi/\alpha$, $\cos\alpha t=-1$, so that $T(t)\to+\infty$ as $t\to(2n+1)\pi/\alpha$. Hence by the intermediate value theorem, we have a root for $2n\pi/\alpha<t<(2n+\frac{1}{2})\pi/\alpha$. To see that this is the only root in that interval, we examine $T'(t)$. To this end we let $f(t)=(t\sin\alpha)/\sin\alpha t$. Then

$$f'(t)=\frac{\sin\alpha\cos\alpha t}{\sin^2\alpha t}(\tan\alpha t-\alpha t)$$

and

$$T'(t)=\frac{1}{\sqrt{f^2-1}}\left[\frac{\sin\alpha}{\alpha}\left(1-\alpha t\frac{\cos\alpha t}{\sin\alpha t}\right)f'+\alpha(f^2-1)\sin\alpha t\right]$$

$$=\frac{1}{\sqrt{f^2-1}}\frac{\sin^2\alpha\cos^2\alpha t}{\sin^3\alpha t}(\tan\alpha t-\alpha t)^2+\alpha(f^2-1)>0.$$

Thus if $\alpha<\alpha_{2n}$ there is only root in the strip $2n\pi<\alpha t<(2n+1)\pi$, and this root actually lies in the strip $2n\pi<\alpha t<(2n+\frac{1}{2})\pi$. Also equation (3.11) shows that $s=\operatorname{Re}z>0$.

By the same reasoning, if we take the lower signs in (3.12), we see that if $\alpha<\alpha_{2n-1}$ there is exactly one root in the strip $(2n-1)\pi<\alpha t<2n\pi$ with $s=\operatorname{Re}z>0$.

3. Now the above argument breaks down only if $f(t)=1$ which implies that $s=0$ by (3.9). Now looking again at the + sign in (3.6), we see that for $s=0$ (3.6) becomes $(\sin\alpha t)/\alpha t=(\sin\alpha)/\alpha$ and we see from Figure 3.1 that this occurs only if $\alpha\geq\alpha_{2n}$ and $2n\pi<\alpha t<(2n+1)\pi$, in which case we get two roots in this interval, unless $\alpha=\alpha_{2n}$ in which case we get one root, namely $\alpha t=\xi_{2n}$. Again

if we take the $-$ sign in (3.6) we get the same behavior in the interval $(2n - 1)\pi < \alpha t < 2n\pi$.

4. To check the multiplicity of the roots we need only check the multiplicity of the roots of the equations (3.6) $z \sin \alpha = \pm \sin \alpha z$ since we have seen that the roots of these equations do not coincide. So looking at the $+$ sign we get a multiple root if and only if

$$d(z \sin \alpha - \sinh \alpha z)/dz = \sin \alpha \cosh \alpha z = 0. \tag{3.14}$$

Now in order that (3.14) hold, $\cosh(\alpha z)$ must be real, which can happen only if $s = 0$ or $\alpha t = n\pi$. The latter possibility we can ignore since we have found no roots with $t = n\pi/\alpha$. Thus the only possible double roots are imaginary. Then plugging (3.6) into (3.14) we get $\alpha t = \tan \alpha t = \xi_{2n}$ so that we get a double root if and only if $\alpha = \alpha_{2n}$, and then only at the point $t = \xi_{2n}/\alpha_{2n}$. If we take the $-$ sign in (3.6) we get a double root at $t = \xi_{2n-1}/\alpha_{2n-1}$ for $\alpha = \alpha_{2n-1}$.

5. To show that no roots have multiplicity more than two, we differentiate (3.14) again, getting $-\alpha^2 \sin \alpha z = 0$ which cannot happen for any of the roots we have found. This concludes the proof of the lemma.

4. **The residues.** We now have, from §2, the Fourier transform of the solution we are seeking, so we must now apply an inverse Fourier transform with respect to k to the function $v(k, \zeta)$ defined by equations (2.5–(2.9). To obtain the information we need, we shall want to deform the path of integration in the integral of the inverse Fourier transform and pick up the residues of the function $e^{ik\eta}v(k, \zeta)$, at the poles included in the path of integration in the complex k-plane. We shall assume now that the function $e^{ik\eta}b(k, \zeta)$ is analytic. This assumption will be justified below. Now, looking at the equations (2.5)–(2.9), we see that the only possible poles of $e^{ik\eta}v(k, \zeta)$ are at the zeros of $\sinh k\alpha$, $\sinh(k - 2i)\alpha$, and $p(k)$. (Note, in particular, that $v_0(k, \zeta)$ defined by (2.7) is analytic for all k.) So we need to examine the residues of $e^{ik\eta}v(k, \zeta)$ at the points $k = n\pi i/\alpha$, $(n\pi/\alpha + 2)i$, and at the zeros of $p(k)$ as found in the previous section, i.e. at the points $k = z + i$ where z is a zero of $q(z)$.

Lemma 4.1. *$v(k, \zeta)$ is analytic in k at $k = i$.*

Proof. The only possible pole is due to the double root of $E(k)$ at $k = i$. Now for $k = i$, $H(\zeta)$ and $H'(\zeta)$ are identically zero so the numerator over $E(k)$ has a double zero. Thus the quotient is analytic.

Lemma 4.2. *The residues of $e^{ik\eta}v(k, \zeta)$ at 0, $2i$, are zero. In fact, $v(k, \zeta)$ is analytic in k at these points.*

Proof. We first assume that $\alpha \neq \pi/2$. The first two terms on the right side of (2.5) are clearly analytic at 0 and $2i$. Now at 0 we have

$$E(0) = (-8i(\sin 2\alpha)/\alpha)(\tan \alpha - \alpha) \neq 0,$$

so that that there is no singularity there.

At $k = 2i$, $E(k)$ has a double zero (one from $p(k)$ and one from $\sinh(k-2i)\alpha$) but the numerator also has a double zero which cancels. Perhaps this can be seen most clearly by examining equation (4.1) in the proof of the next lemma.

If $\alpha = \pi/2$, then we have the case already dealt with by Osher in [7]. At $k = 0$ the only possible contributions are from the last two terms of (2.5) and they are easily seen to vanish. At $k = 2i$ there is also a contribution from the second term but it cancels with the contributions from the last two terms. Q.E.D.

Lemma 4.3. *The residues of* $e^{ik\eta}v(k, \zeta)$ *at* $k = n\pi i/\alpha$, *and at* $k = (n\pi/\alpha + 2)i$, $n = \pm 1, \pm 2, \cdots$, *are all zero and, in fact,* $v(k, \zeta)$ *is analytic at these points.*

Proof. We first rewrite the expression $CH(k, \zeta) + DH(k, \zeta + \alpha)$ in a form that groups together all the numerators of the same zero denominators i.e.

$$\begin{aligned}
&CH(k, \zeta) + DH(k, \zeta+\alpha)\\
&= \frac{1}{p(k)}\Big\{(\text{something analytic})\\
&\quad + \frac{1}{\sinh k\alpha}\,[-k(k-2i)v_0(-\alpha)[(\cosh(k-2i)\alpha\cosh k\alpha - 1)\sinh k\zeta\\
&\qquad + (\cosh(k-2i)\alpha - \cosh k\alpha)\sinh k(\zeta+\alpha)]\\
&\qquad + k\sinh(k-2i)\alpha v_0'(-\alpha)\sinh k\alpha\cosh k\zeta]\\
&\quad + \frac{(k-2i)}{\sinh(k-2i)\alpha}\,[[k(\cosh(k-2i)\alpha\cosh k\alpha - 1)v_0(-\alpha)\\
&\qquad + \cosh(k-2i)\alpha\sinh k\alpha v_0'(-\alpha)]\sinh(k-2i)\zeta\\
&\qquad + [k(\cosh(k-2i)\alpha - \cosh k\alpha)v_0(-\alpha) - \sinh k\alpha v_0'(-\alpha)]\\
&\qquad \cdot \sinh(k-2i)(\zeta+\alpha)]\Big\}.
\end{aligned} \tag{4.1}$$

Now at $k = (n\pi/\alpha + 2)i$ we need only consider the last of these three expressions and using the identities

$$\begin{aligned}
&\sinh k\alpha = i(-1)^n \sin 2\alpha, \quad \sinh(k-2i)\alpha = 0,\\
&\cosh k\alpha = (-1)^n \cos 2\alpha, \quad \cosh(k-2i)\alpha = (-1)^n,
\end{aligned} \tag{4.2}$$

we see that the two brackets in this expression cancel each other out.

At $k = n\pi i/\alpha$ we consider the second expression in the sum and use the identities

$$\begin{aligned} \sinh k\alpha &= 0, & \sinh (k-2i)\alpha &= -i(-1)^n \sin 2\alpha, \\ \cosh k\alpha &= (-1)^n, & \cosh (k-2i)\alpha &= (-1)^n \cos 2\alpha, \end{aligned} \tag{4.3}$$

to compute the residue of (4.1). We get

$$\frac{1}{p(n\pi i/\alpha)}(-4)\frac{(-1)^n}{\alpha}\frac{n\pi}{\alpha}\left(\frac{n\pi}{\alpha}-2\right)\sin^2\alpha \sinh\left(\frac{n\pi i\zeta}{\alpha}\right)v_0(-\alpha)$$

$$= -\frac{(-1)^n}{\alpha}\sinh\left(\frac{n\pi i\zeta}{\alpha}\right)v_0(-\alpha).$$

To this we add the residue at $k = n\pi i/\alpha$ of the term $[(\sinh k\zeta)/(\sinh k\alpha)]v_0(-\alpha)$ and get zero. Q.E.D.

In the next lemma we shall need the following identities which are immediate from equations (2.6) and (2.7).

$$H'(0) = -\frac{k \sinh (k-2i)\alpha}{\sinh k\alpha} + (k-2i), \tag{4.4}$$

$$v_0(k,-\alpha) = \frac{i}{4(k-i)}\int_0^{-\alpha}\left[\frac{\sinh k(t-\zeta)}{k} - \frac{\sinh (k-2i)(t-\zeta)}{k-2i}\right]h(k,t)\,dt \tag{4.5}$$

$$v_0'(k,-\alpha) = \frac{i}{4(k-i)}\int_0^{-\alpha}[-\cosh k(t-\zeta) + \cosh (k-2i)(t-\zeta)]h(k,t)\,dt. \tag{4.6}$$

Lemma 4.4. *Let $\hat{k}$ be a simple root of $p(k) = 0$ such that* Re $\hat{k} \geq 0$ *and $\hat{k} \neq 0, 2i$. Then the residue of $e^{ik\eta}v(k,\zeta)$ at $\hat{k}$ is given by*

$$\begin{aligned} \operatorname*{res}_{k=\hat{k}} e^{ik\eta}v(k,\zeta) &= \frac{H'(0)}{P'(\hat{k})}\left(-\frac{1}{4}\frac{\sinh \hat{k}\alpha}{\sinh (\hat{k}-2i)\alpha}[H(\zeta) \pm H(\zeta+\alpha)]\right) \\ &\quad \cdot \frac{i}{4(\hat{k}-i)}\int_0^{-\alpha}\left[\frac{\hat{k}(-1 \mp \cosh \hat{k}\alpha)}{\sinh \hat{k}\alpha}\left(\frac{\sinh \hat{k}(t+\alpha)}{\hat{k}} - \frac{\sinh (\hat{k}-2i)(t+\alpha)}{\hat{k}-2i}\right)\right. \\ &\qquad \left. \mp (-\cosh \hat{k}(t+\alpha) + \cosh (\hat{k}-2i)(t+\alpha))\right] \cdot e^{i\hat{k}\eta} \\ &\quad \cdot \int_{-\infty}^{\infty} e^{-i\hat{k}v - 4v} g(e^{-v}\cos t, -e^{-v}\sin t)\,dv\,dt. \end{aligned} \tag{4.7}$$

Proof. We first note that

$$\operatorname*{res}_{k=\hat{k}} v(k,\zeta) = \operatorname*{res}_{k=\hat{k}} [C(k)H(k,\zeta) + D(k)H(k,\zeta+\alpha)] \tag{4.8}$$

since the other terms are analytic at such a zero. Now let $\hat{C}$ and $\hat{D}$ respectively, be the numerators of C and D in equation (2.9), i.e.

$$\begin{aligned} \hat{C} &= \frac{k}{\sinh k\alpha}(-H'(0) + \cosh k\alpha H'(\alpha))v_0(-\alpha) + H'(\alpha)v_0'(-\alpha), \\ \hat{D} &= \frac{k}{\sinh k\alpha}(H'(\alpha) - H'(0)\cosh k\alpha)v_0(-\alpha) - H'(0)v_0'(-\alpha). \end{aligned} \tag{4.9}$$

Now (4.8) yields

$$\operatorname*{res}_{k=\hat{k}} v(k, \zeta) = \frac{\hat{C}(\hat{k})H(\hat{k}, \zeta) + \hat{D}(\hat{k})H(\hat{k}, \zeta + \alpha)}{[(-4 \sinh(\hat{k} - 2i)\alpha)/\sinh \hat{k}\alpha]p'(\hat{k})}. \tag{4.10}$$

Now if $p(k) = 0$, then $E(k) = 0$ and then from (2.8) we see that

$$H'(\alpha) = \mp H'(0). \tag{4.11}$$

(This equation is really the same as (3.6) with the same consistency of upper and lower signs.) Thus (4.9) becomes

$$\begin{aligned} \hat{C} &= \frac{kH'(0)}{\sinh k\alpha} (-1 \mp \cosh k\alpha)v_0(-\alpha) \mp v_0'(-\alpha), \\ \hat{D} &= \frac{kH'(0)}{\sinh ka} (\mp 1 - \cosh k\alpha)v_0(-\alpha) - v_0'(-\alpha) = \pm \hat{C}, \end{aligned} \tag{4.12}$$

so that (4.10) now becomes

$$\operatorname*{res}_{k=\hat{k}} v(k, \zeta) = \frac{C(\hat{k})(H(\hat{k}, \zeta) \pm H(\hat{k}, \zeta + \alpha))}{[(-4 \sinh(\hat{k} - 2i)\alpha)/\sinh \hat{k}\alpha]p'(\hat{k})}. \tag{4.13}$$

We now use (4.5) and (4.6) to get

$$\begin{aligned} \hat{C}(\hat{k}) = H'(0) \frac{i}{4(\hat{k} - i)} \int_0^{-\alpha} \Bigg\{ & \frac{\hat{k}(-1 \mp \cosh \hat{k}\alpha)}{\sinh \hat{k}\alpha} \left[\frac{\sinh \hat{k}(t + \alpha)}{\hat{k}} - \frac{\sinh(\hat{k} - 2i)(t + \alpha)}{\hat{k} - 2i} \right] \\ & \mp [-\cosh \hat{k}(t + \alpha) + \cosh(\hat{k} - 2i)(t + \alpha)] \Bigg\} h(\hat{k}, t)\,dt \end{aligned} \tag{4.14}$$

and together with the definition of $h(k, t)$ from (2.4a), i.e.

$$h(k, t) = \int_{-\infty}^{+\infty} e^{-ik\nu - 4\nu} g(e^{-\nu} \cos t, -e^{-\nu} \sin t)\,dt$$

we obtain the result (4.7). Q.E.D.

We shall now show that if k_+ and k_- are roots of $p(k)$ with $k_+ = k_1 + ik_2$, and $k_- = -k_1 + ik_2$, where $k_1 \geq 0$, then the sum of the residues of $e^{ik\eta}v(k, \zeta)$ at k_+ and at k_- is pure imaginary for real f. Moreover, for all but a finite number of ζ for which it vanishes, this sum oscillates sinusoidally in $\log r$ as $k_2\eta \to \infty$ for fixed ζ. However, if k is a purely imaginary simple root of $p(k)$, then the corresponding residue decays without change of sign as $k_2\eta \to \infty$.

Lemma 4.5. *Let k_2 be such that $e^{(k_2-4)\eta}g(e^{-\eta} \cos t, e^{-\eta} \sin t)$ is smooth, real and belongs to $L_2(-\infty, \infty)$ for each t, $0 \leq t \leq \alpha$, and does not vanish identically.*

(a) *If $k_+ = k_1 + ik_2$, $k_1 > 0$, is a root of $p(k)$, then the sum of the residues $e^{ik\eta}v(k, \zeta)$ at k_+ and $k_- = -k_1 + ik_2$ is pure imaginary and has the oscillating* *property mentioned above. More precisely, we have*

$$\sum_{k=k\pm} \operatorname{res} e^{ik\eta} v(k, \zeta) = 2i \operatorname{Re}[e^{ik\eta} F(k_+, \zeta)]. \tag{4.15}$$

(b) *If* $k = ik_2$ *is an imaginary simple root of* $p(k)$, $k \neq 0, 2i$, *then the residue at* k *is pure imaginary and has the decay property mentioned above. More precisely, we have*

$$\operatorname{res} e^{ik\eta} v(k, \zeta) = ie^{ik\eta} F(k, \zeta). \tag{4.16}$$

In either case $F(k, \zeta)$ *is given by*

$$\begin{aligned} F(k, \zeta) = {} & \frac{-H'(0)\sinh k\alpha}{16(k-i)\sinh(k-2i)\alpha} [H(\zeta) \pm H(\zeta+\alpha)] \cdot \frac{1}{p'(k)} \\ & \cdot \int_0^{-\alpha} \left[\frac{k(-1 \mp \cosh k\alpha)}{\sinh k\alpha} \left(\frac{\sinh k(t+\alpha)}{k} - \frac{\sinh(k-2i)(t+\alpha)}{k-2i} \right) \right. \\ & \left. \mp (-\cosh k(t+\alpha) + \cosh(k-2i)(t+\alpha)) \right] \\ & \cdot \int_{-\infty}^{\infty} e^{-ik\nu - 4\nu} g(e^{-\nu}\cos t, -e^{-\nu}\sin t) d\nu dt. \end{aligned} \tag{4.17}$$

Proof. In part (a) it first has to be shown that the sum of the residues is imaginary. We examine the expression in equation (4.7) with k replaced by $-\bar{k}$. We notice that

$$\begin{aligned} \sinh(-\bar{k})\alpha &= \overline{-\sinh k\alpha}, & \sinh(-\bar{k}-2i)\alpha &= \overline{-\sinh(k-2i)\alpha}, \\ \cosh(-\bar{k})\alpha &= \overline{\cosh k\alpha}, & \cosh(-\bar{k}-2i)\alpha &= \overline{-\cosh(k-2i)\alpha}, \\ H(-\bar{k}, \zeta) &= \overline{-H(k,\zeta)}, & H'(-\bar{k}, \zeta) &= \overline{-H'(k,\zeta)}, \\ e^{i(-\bar{k})\eta} &= \overline{e^{ik\eta}}, & p'(-\bar{k}) &= \overline{-p(k)}. \end{aligned} \tag{4.18}$$

Using these relations we see that the residue in (4.7) satisfies

$$\operatorname*{res}_{k=k_+} = \overline{-\operatorname*{res}_{k=k_-}} \tag{4.19}$$

and thus the sum is imaginary. Also, we can easily see that if k is imaginary then the right side of (4.7) is imaginary.

It now remains to be shown that $F(k, \zeta)$ vanishes at most for a finite number of ζ. Suppose this were false. Then either $H'(k, 0) = 0$, or at least one of the functions on the right side of (4.17) enclosed in the square brackets must be identically zero. Now if $H'(0) = 0$, then we see from (4.11) that

$$H'(\alpha) = \frac{k\sinh(k-2i)\alpha}{\sinh k\alpha} [-\cosh k\alpha + \cosh(k-2i)\alpha] = 0,$$

but this is impossible for the k's we are considering. On the other hand, the functions in the square brackets are both linear combinations of the four functions $\sinh kt$, $\cosh kt$, $\sinh(k-2i)t$, $\cosh(k-2i)t$ (in the case of the second of these two functions; the first is a function of ζ). The Wronskian of these functions is computed to be $16(k-i)^2 k(k-2i)$ so that they are linearly independent unless $k = 0, i, 2i$. Since this is not the case they cannot be identically zero. Hence $F(k, \zeta)$ can only vanish for a finite number of ζ.

Remark. $F(k, \zeta)$ can be written in terms of θ and $f(r, \theta)$ as follows:

$$
\begin{aligned}
F(k, -\theta) = \int_0^\infty dr_1 \int_0^\alpha d\theta_1 \Bigg[& \frac{H'(k, 0)\sinh k\alpha}{16(k-i)\sinh(k-2i)\alpha} (H(k, -\theta) + (-1)^n H(k, \zeta + \alpha)) \\
& \cdot \frac{1}{p'(k)} \Bigg[\frac{k(-1-(-1)^n \cosh k\alpha}{\sinh k\alpha} \left(\frac{\sinh k(\alpha - \theta_1)}{k} - \frac{\sinh(k-2i)(\alpha-\theta_1)}{k-2i} \right) \\
& - (-1)^n(-\cosh k(\alpha - \theta_1) + \cosh(k-2i)(\alpha-\theta_1)) \Bigg] r_1^{3+ik} f(r_1, \theta_1) \Bigg].
\end{aligned} \tag{4.17'}
$$

The 'n' appearing in this equation refers to the strip in which k lies, i.e. $n\pi/\alpha + 1 < \operatorname{Im} k < (n+1)\pi/\alpha + 1$ for $\operatorname{Im} k > 0$, and $-(n+1)\pi/\alpha + 1 < \operatorname{Im} k < -n\pi/\alpha + 1$ for $\operatorname{Im} k < 0$.

5. **The asymptotic behavior of $u(r, \theta)$.** We have now completed the calculations necessary to find an asymptotic expansion for the inverse Fourier transform of $v(k, \zeta)$. However, since our formulae for the function $v(k, \zeta)$ involve terms which grow exponentially with large k, we must show that the inversion may actually be performed. To this end we state the following lemma but leave the nasty details of the proof to the appendix.

Lemma 5.1. *$v(k, \zeta)$ can be written as a sum of a finite number of terms, each of which looks like*

$$
\int_0^{-\alpha} \frac{B(k, \zeta, t)}{Q(k)} h(k, t)\, dt \tag{5.1}
$$

where $B(k, \zeta, t)$ is bounded whenever Im k *fixed and k is not a root of $p(k)$. $Q(k)$ grows like k^2 for large k.*

Lemma 5.2. *Let $f(r, \theta)$ be such that $r^{3-k_2} f(r, \theta)$ is in L_2 of the α-wedge for all k_2 with $\theta \le k_2 \le k_2'$, i.e.*

$$
\int_0^\alpha \int_0^\infty [r^{3-k_2} f(r, \theta)]^2 r\, dr\, d\theta, \tag{5.2}
$$

then the function $e^{(k_2-4)\eta} g(e^{-\eta}\cos\zeta, -e^{-\eta}\sin\zeta)$, is square integrable in η for almost all θ. (Here, $g(X, Y) = f(r, \theta)$ according to the transformation (2.2).) *Moreover, the Fourier transform*

$$(5.3) \qquad h(k, \zeta) = \int_{-\infty}^{\infty} e^{-ik\eta} e^{-4\eta} g(e^{-\eta} \cos \zeta, -e^{-\eta} \sin \zeta) d\eta$$

has an analytic extension in $k = k_1 + ik_2$ *for* $k_2 < k_2'$.

The proof is obvious.

Lemma 5.3. *Let* $f(r, \theta)$ *be as in Lemma* 5.2, *and let* $v(k, \zeta)$ *be defined by* (2.5)–(2.9). *Then the inverse Fourier transform of* $v(k, \zeta)$ *exists*

$$(5.4) \qquad \hat{u}(\eta, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\eta} v(k, \zeta) dk$$

and is a solution of the differential equation (2.3) *with its boundary conditions. Moreover, we have*

$$(5.5) \quad \hat{u}(\eta, \zeta) = i \sum_{0<k_2<k_2'} \operatorname{res} e^{ik\eta} v(k, \zeta) + O((e^{-\eta})^{k_2^{(n)}+\epsilon}) \qquad (e^{-\eta} \to 0),$$

where the sum is taken over all roots $k = k_1 + ik_2$ *of* $p(k) = 0$, *such that* $0 < k_2 \le k_2^{(n)} < k_2'$ *and* $\epsilon > 0$.

Proof. The existence of the integral is guaranteed by Lemma 5.1 and the analyticity of the integrand by Lemma 5.2. Thus we can deform the path of integration into the upper half plane to include the all the poles of the integrand with imaginary part less than k_2'. Again, the bounds from Lemma 5.1 guarantee the estimate on the growth as $e^{-\eta} = r \to 0$. Q.E.D.

Theorem 5.4. (a) *If* $r^{3-k_2} f(r, \theta)$ *is in* L_2 *of the* α*-wedge for all* k_2 *with* $0 \le k_2 \le k_2'$, *then the following asymptotic behavior for the solution* $u(r, \theta)$ *of problem* (2.1) *is valid*:

$$(5.6) \qquad \lim_{r \to 0} r^{-k_2^{(n)}} \Big[u(r, \theta) + \sum_i r^{-ik^{(i)}} F(k^{(i)}, -\theta) + 2 \sum_j \operatorname{Re} [r^{-ik_+^{(j)}} F(k_+^{(j)}, -\theta)] \Big] = 0 \quad a.e.$$

where the first sum is taken over the purely imaginary roots of $p(k)$ *which exist if* $\alpha \ge \alpha_1$ *(see Lemma* 3.2), *and the second sum is taken over the complex roots with positive real part. Both sums are taken over those roots whose imaginary part satisfies* $2 < k_2^{(i)} \le k_2^{(n)} < k_2'$. *(We note that the imaginary part of the roots in the first sum are less than the imaginary parts of each root in the second sum so that their behavior dominates as* $r \to 0$.) $F(k, \zeta)$ *is defined in* (4.17) *and the determination of the* $\pm$ *signs is* $(-1)^m$, *where the root lies in the horizontal strip* $m\pi/\alpha + 1 < \operatorname{Im} k < (m+1)\pi/\alpha + 1$.

(b) *If* $r^{3+k_2} f(r, \theta)$ *is in* L_2 *of the* α*-wedge for all* k_2 *with* $0 < k_2 < k_2''$, *then we have the following asymptotic behavior of* $u(r, \theta)$ *for large* r:

$$\lim_{r\to\infty} r^{-k_2^{(-n)}} \Big[u(r, \theta) + \sum r^{-ik^{(-i)}} F(k^{(-i)}, -\theta) + 2 \sum \operatorname{Re} [r^{-ik_+^{(-j)}} F(k_+^{(-j)}, -\theta)] \Big] = 0 \quad a.e. \tag{5.7}$$

where the same discussion as in part (a) *holds except that now the sums are over the roots with negative imaginary part and we sum over the roots which satisfy* $0 < -k^{(-i)} < -k_2^{(-j)} \leq -k_2^{(-n)} < k_2''$.

Proof. The proof of part (a) follows directly from Lemmas 5.3 and 4.5. To prove part (b) we use a lemma exactly like Lemma 5.3 except the contour is now deformed to the lower half of the complex plane. The details are omitted. Q.E.D.

Theorem 5.5. (a) *If* $\alpha < \alpha_1$ *as defined in Definition* 3.2, *then the Green's function for Problem* (2.1) $G(r, \theta, r_0, \theta_0)$, *changes sign an infinite number of times as* $r/r_0 \to 0$ *and* $r/r_0 \to \infty$, *except for perhaps a finite number of* θ *and* θ_0.

(b) *If* $\alpha > \alpha_1$ *then the Green's function does not change sign if* r/r_0 *is sufficiently large or sufficiently small while* θ *and* θ_0 *remain fixed.*

Proof. Choose the function $f(r_1, \theta_1)$ to be equal to 1 on some set such as $r - \epsilon < r_1 < r + \epsilon$, $\theta - \epsilon < \theta_1 < \theta + \epsilon$, where ϵ is chosen so small that the kernel multiplying f in (4.17′) does not change sign in that set, the k we choose is the first zero, i.e. the one with smallest imaginary part in the interval $\pi/\alpha + 1 < \operatorname{Im} k < 2\pi/\alpha + 1$, and use only the term corresponding to this zero in the expression (5.6). We see that if $\alpha < \alpha_1$ then k has a positive real part and so G oscillates with decaying amplitude as $r \to 0$. If, on the other hand, $\alpha > \alpha_1$, then the only term appearing has r with a real exponent and thus no change of sign takes place as $r \to 0$.

The same argument, using (5.7) works as $r \to \infty$. Q.E.D.

APPENDIX

A1. **Computation of equation (2.8) for** $E(k)$. We have from the previous system of linear equations the determinant

$$E(k) = [H'(0)]^2 + [H'(\alpha)]^2.$$

Now from (2.6) we see that

$$H'(k, \zeta) = \frac{d}{d\zeta} H(k, \zeta) = -\frac{k \sinh (k - 2i)\alpha}{\sinh k\alpha} \cosh k\zeta + (k - 2i) \cosh (k - 2i)\zeta$$

so that if we set $z = k - i$, then $k = z + i$ and $k - 2i = z - i$, we get

$$H'(0) = \frac{2i}{\sinh(z+i)\alpha}(z \cosh \alpha z \sin \alpha + \sinh \alpha z \cos \alpha),$$

$$H'(\alpha) = \frac{2i}{\sinh(z+i)\alpha}(z \sin \alpha \cos \alpha - \sinh \alpha z \cosh \alpha z).$$

Squaring these two expressions gives us

$$[H'(0)]^2 = -\frac{4}{(\sinh(z+i)\alpha)^2}[z^2 \cosh^2 \alpha z \sin^2 \alpha$$
$$- 2z \sinh \alpha z \cosh \alpha z \sin \alpha \cos \alpha + \sinh^2 \alpha z \ \cos^2 \alpha],$$

$$[H'(\alpha)]^2 = -\frac{4}{(\sinh(z+i)\alpha)^2}[z^2 \sin^2 \alpha \cos^2 \alpha$$
$$- 2z \sinh \alpha z \cosh \alpha z \sin \alpha \cos \alpha + \sinh^2 \alpha z \ \cosh^2 \alpha z]$$

and putting these two together we get

$$[H'(0)]^2 - [H'(\alpha)]^2 = -\frac{4}{(\sinh(z+i)\alpha)^2}[z^2 \sin \alpha(\cosh^2 \alpha z - \cos^2 \alpha) +$$
$$+ \sinh^2 \alpha z(\cos^2 \alpha - \cosh^2 \alpha z)]$$
$$= -\frac{4}{(\sinh(z+i)\alpha)^2}(z^2 \sin^2 \alpha - \sinh^2 \alpha z)(\cosh^2 \alpha z - \cos^2 \alpha)$$
$$= -4\frac{\sinh(z-i)\alpha}{\sinh(z+i)\alpha}(z^2 \sin^2 \alpha - \sinh^2 \alpha z)$$

as was to be shown.

A2. **Proof of Lemma 5.1.** We have the function

$$v(k, \zeta) = v_0(k, \zeta) + v_0(k, -\alpha)\frac{\sinh k\zeta}{\sinh k\alpha} + CH(k, \zeta) + DH(k, \zeta + \alpha).$$

We first rewrite the first two terms of this expression

$$v_0(\zeta) + v_0(-\alpha)\frac{\sinh k\zeta}{\sin k\alpha}$$

$$(1) \quad = \frac{i}{8}\int_0^{\zeta} \frac{\cosh k(t+\zeta+\alpha) - \cosh k(t-\zeta-\alpha)}{k(k-i)\sinh k\alpha} h(t)\,dt$$

$$(2) \quad + \frac{i}{4}\frac{\sinh k\zeta}{\sinh k\alpha}\int_{\zeta}^{-\alpha}\left[\frac{\sinh k(t+\alpha)}{k(k-i)} - \frac{\sinh(k-2i)(t+\alpha)}{(k-i)(k-2i)}\right]h(t)\,dt$$

$$(3) \quad - \frac{i}{4}\int_0^{\zeta}\left[\frac{\sinh(k-2i)(t-\zeta)\sinh k\alpha + \sinh(k-2i)(t+\alpha)\sinh k\zeta}{(k-i)(k-2i)\sinh k\alpha}\right]h(t)\,dt.$$

Now we observe that the terms labeled (1) and (2) already satisfy the conclusion of Lemma 5.1. The quadratic term is in the denominator and to show that the rest of the kernel of this integral is bounded we use the trigonometric identity

$$\cosh k(t+\zeta+\alpha) - \cosh k(t-\zeta-\alpha) = 2\sinh kt \sinh k(\zeta+\alpha)$$

and when we write this out in terms of exponentials we see that this grows (for positive k) at most like $\exp k(-t+\zeta+\alpha)$ and when we divide by $\sinh k\alpha$ we are left with a growth of at most $\exp k(-t+\zeta)$ and since $0 \le -t \le -\zeta$, this term must be bounded. For negative k, the same argument applies. The same kind of argument also shows that each term in (2) is bounded. In (3) the terms do not remain bounded until we get some cancellation from some other terms. Now we can easily see that the term $DH(\zeta+\alpha)$ satisfies the conclusion of the lemma so we now need worry only about the term (3) and the term $CH(\zeta)$. We now look at the growth of this latter term. We see that $v_0(-\alpha)$ grows like $\exp k(t+\alpha)/k^2$ and $v_0'(-\alpha)$ grows like $\exp k(t+\alpha)/k$ so that since $E(k) \sim \exp 2k$ we have $C \sim \exp k(t-2\alpha) + (\exp k(t))/k$ and since $H(\zeta) \sim \exp(-k\zeta)$, we get for the term in question

$$CH(\zeta) \sim \exp k(t-\zeta-2\alpha) + (\exp k(t-\zeta))/k$$

for $k \ge 0$. Now since $0 \le -t,\ -\zeta \le \alpha$, we see that all the expressions here are bounded except for the terms giving rise to expressions with order of magnitude $(\exp k(t-\zeta))/k$ and then only the part of the integral where $\zeta < t < 0$. This leaves us to examine the expression

$$\frac{H(\zeta)iH'(\alpha)}{4(k-i)\sinh k\alpha}\int_0^\zeta h(t)\left[k\cosh k\alpha\ \frac{\sinh k(t+\alpha)}{k} - \frac{\sinh(k-2i)(t+\alpha)}{k-2i}\right.$$
$$\left. + \sinh k\alpha(-\cosh k(t+\alpha) + \cosh(k-2i)(t+\alpha))\right] dt \Big/ E(k)$$
$$= \frac{iH'(\alpha)H(\zeta)}{4\sinh k\alpha(k-i)}\int_0^\zeta \frac{h(t)}{E(k)}\left[(\cosh k\alpha \sinh k(t+\alpha) - \sinh k\alpha \cosh k(t+\alpha))\right.$$
$$\left. - \frac{k}{k-2i}\cosh k\alpha \sinh(k-2i)(t+\alpha) + \sinh k\alpha \cosh(ki-2i)(t+\alpha)\right] dt$$
$$= \frac{iH'(\alpha)H(\zeta)}{4(k-i)E(k)}\int_0^\zeta h(t)\left[\frac{\sinh kt}{\sinh k\alpha} - \frac{k}{k-2i}\ \frac{\sinh[kt-2i(t-\alpha)]}{\sinh k\alpha} - 2i\cosh(k-2i)(t+\alpha)\right] dt.$$

Now all these terms in this expression are bounded except for the last one and so we now examine

$$\frac{iH'(+\alpha)H(\zeta)}{4(k-i)E(k)}\int_0^\zeta h(t)\left[-\frac{2i}{k-2i}\cosh(k-2i)(t+\alpha)\,dt\right].$$

Now

$$H'(\alpha) = \frac{ki\sin 2\alpha}{\sinh k\alpha} - 2i\cosh(k-2i)\alpha$$

and since the product of the first term with the rest is bounded, we can forget about it. So we are now left with

$$-\frac{iH(\zeta)}{(k-i)(k-2i)}\int_0^\zeta [\cosh(k-2i)\alpha\, \cosh(k-2i)(t+\alpha)/E(k)]h(t)\,dt$$

$$=\frac{-iH(\zeta)}{(k-i)(k-2i)}\int_0^\zeta \frac{h(t)}{2}\,[\cosh(k-2i)(2\alpha+t)+\cosh(k-2i)t]/E(k)\,dt.$$

The second term in this expression is bounded so we are now left with

$$\frac{-iH(\zeta)}{(k-i)(k-2i)E(k)}\int_0^\zeta \frac{1}{2}\cosh(k-2i)(2\alpha+t)h(t)\,dt.$$

Now we finally combine this expression with (3) from A1 and after placing everything under a common denominator and ignoring some terms which are obviously bounded we finally get as the function multiplying $h(k, t)$ under the integral sign

$$\begin{aligned}(4)\qquad &\frac{-i}{2(k-i)(k-2i)\sinh^2 k\alpha E(k)}\\ &\cdot\{\cosh(k-2i)(2\alpha+t)\sinh k\alpha\\ &\qquad[\sinh k\alpha\sinh(k-2i)\zeta-\sinh k\zeta\sinh(k-2i)\alpha]\\ &+\sinh(k-2i)\alpha\cosh 2(k-i)\alpha[\sinh(k-2i)(t-\zeta)\sinh k\alpha\\ &\qquad+\sinh(k-2i)(t+\alpha)\sinh k\zeta]\}.\end{aligned}$$

Now combining the first and third terms in the numerator of this last expression we get

$$\begin{aligned}&\sinh k\alpha[\cosh(k-2i)(2\alpha+t)\sinh k\alpha\sinh(k-2i)\zeta\\ &\qquad+\sinh(k-2i)\alpha\cosh(2k-2i)\alpha\sinh(k-2i)(t-\zeta)].\end{aligned}$$

We now can see how the growth terms in this last expression cancel each other out as we write out

$$\sinh k\alpha[-e^{(k-2i)(2\alpha+t)}e^{k\alpha}e^{-(k-2i)\zeta}+e^{(k-2i)\alpha}e^{(2k-2i)\alpha}e^{(k-2i)(t-\zeta)}]$$

$$=\sinh k\alpha[-e^{k(2\alpha+t+\alpha-\zeta)}e^{-2i(2\alpha+t-\zeta+\alpha)}+e^{k(\alpha+2\alpha+t-\zeta)}e^{-2i(\alpha+2\alpha-t+\zeta)}]=0.$$

Similarly, if we look at the second and fourth terms of expression (4), they also cancel each other out. This whole analysis, which has been done for positive k, can be done in exactly the same way for negative k and thus the proof of the lemma is completed.

BIBLIOGRAPHY

1. L. Bauer and E. Reiss, *Block five diagonal metrics and the fast numerical computation of the biharmonic equation*, Math. Comp. 26 (1972), 311–326.

2. R. J. Duffin, *The maximum principle and biharmonic functions*, J. Math. Anal. Appl. 3 (1961), 399–405. MR 26 #1617.

3. R. J. Duffin and D. H. Shaffer, *On the modes of vibration of a ring-shaped plate*, Bull. Amer. Math. Soc. 58 (1952), 652.

4. P. R. Garabedian, *A partial differential equation arising in conformal mapping*, Pacific J. Math. 1 (1951), 485–524. MR 13, 735.

5. J. Hadamard, *Sur un problème d'analyse rélatif a l'équilibre des plaquea élastiques encastrées*, Mém. Acad. Sci. 33 (1908), no. 4.

6. V. A. Kondrat'ev, *Boundary-value problems for elliptic equations in conical regions*, Dokl. Akad. Nauk SSSR 153 (1963), 27–29 = Soviet Math. Dokl. 4 (1963), 1600–1602. MR 28 #1383.

7. S. Osher, *On the Green's function for the biharmonic equation in a right angle wedge*, J. Math. Anal. Appl. (to appear).

8. G. Szegö, *On membranes and plates*, Proc. Nat. Acad. Sci. U. S. A. 36 (1950), 210–216. MR 11, 757.

Current address: 356 Crown Street, Brooklyn, New York 11225