INVARIANT DIFFERENTIAL OPERATORS ON A
REAL SEMISIMPLE LIE ALGEBRA AND
THEIR RADIAL COMPONENTS

BY

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ABSTRACT. Let $S(\mathfrak{g}_C)$ be the symmetric algebra over the complexification $\mathfrak{g}_C$ of the real semisimple Lie algebra $\mathfrak{g}$. For $u \in S(\mathfrak{g}_C)$, $\delta(u)$ is the corresponding differential operator on $\mathfrak{g}$, $\mathfrak{D}(\mathfrak{g})$ denotes the algebra generated by $\delta(S(\mathfrak{g}_C))$ and multiplication by polynomials on $\mathfrak{g}_C$. For any open set $U \subset \mathfrak{g}$, $\text{Diff}(U)$ is the algebra of differential operators with $C^\infty$-coefficients on $U$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}'$ the set of its regular points and $\pi = \prod_{\alpha \in P} \alpha$, $P$ some positive system of roots. Let $\mathbb{W} = (\mathfrak{h})^G$, $G$ the connected adjoint group of $\mathfrak{g}$.

Harish-Chandra showed that, for each $D \in \text{Diff}(\mathbb{W})$, there is a unique differential operator $\delta_h(D)$ on $\mathfrak{h}'$ such that $(D|)_{\mathfrak{h}'} = \delta_h(D)|_{\mathfrak{h}'}$ for all $G$-invariant $f \in C^\infty(\mathbb{W})$, and that if $D \in \mathfrak{D}(\mathfrak{h})$, then $\delta_h(D) = \pi^{-1} \circ D \circ \pi$ for some $D \in \mathfrak{D}(\mathfrak{h})$. In particular $\delta(u) = \delta(u|_{\mathfrak{h}'})$, $u \in S(\mathfrak{g}_C)$ and invariant.

We prove these results by different, yet simpler methods. We reduce evaluation of $\delta_h(\delta(u))$ ($u \in S(\mathfrak{g}_C)$, invariant) via Weyl's unitarian trick, to the case of compact $G$. This case is proved using an evaluation of a family of $G$-invariant eigenfunctions on:

$$n(H \exp(H')) \int_{\mathbb{W}} \exp B(H^2, H') \, dx = c \sum_{s \in \mathbb{W}(\mathfrak{g}_C, \mathfrak{h}_C)} (s) \exp B(sH, H'),$$

$H, H' \in \mathfrak{g}_C, c > 0$.

For $G$-invariant $D \in \mathfrak{D}(\mathfrak{g})$, we prove $\pi^{-1} \circ \delta_h(D) \circ \pi \in \mathfrak{D}(\mathfrak{h})$ using properties of derivations $E \rightarrow [\delta(u), E]$ of $\mathfrak{D}(\mathfrak{g})$ induced by $\delta(u)$ ($u \in S(\mathfrak{g}_C)$) and of the algebra of polynomials on $\mathfrak{g}_C$ invariant under the Weyl group.

1. Preliminaries. Our aim here is to give alternative proofs to some results of Harish-Chandra on radial components of invariant differential operators on a real semisimple Lie algebra $[1], [2]$.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$, $\mathfrak{g}_C$ its complexification, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{h}_C$ complexification of $\mathfrak{h}$. Denote by $\mathfrak{g}'$ the set of regular elements of $\mathfrak{g}$ and $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$. We have $H \in \mathfrak{h}'$ if and only if $\pi(H) \neq 0$.

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where $\pi = \Pi_{a \in P} a$, $P$ being a positive system of roots of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_c$.

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $S(\mathfrak{g}_c)$ and $P(\mathfrak{g}_c)$ be the symmetric and the polynomial algebras over $\mathfrak{g}_c$, and $Q(\mathfrak{g}_c)$ quotient field of $P(\mathfrak{g}_c)$. We assume as usual that $\mathfrak{g}_c \subset S(\mathfrak{g}_c)$. Let $U$ be an open subset of $\mathfrak{g}$, and $C^{\infty}(U)$ the algebra of $C^{\infty}$ functions on $U$. An element $X \in \mathfrak{g}$ acts as a derivation of $C^{\infty}(U)$ given by

$$ (Xf)(u) = df(u + tx)/dt|_{t=0} $$

where $f \in C^{\infty}(U)$, $u \in U$, and $t \in \mathbb{R}$. We denote the above derivation by $\partial(X)$ and extend the map $\partial: X \mapsto \partial(X)$ uniquely to a homomorphism of $S(\mathfrak{g}_c)$ into the associative algebra of endomorphisms of $C^{\infty}(U)$. If $\phi \in C^{\infty}(U)$, then $\phi$ is identified with the endomorphism $\phi: f \mapsto \phi(f)$ of $C^{\infty}(U)$. The algebra generated by $\{\phi, \partial(p)\}$ $\phi \in C^{\infty}(U), p \in S(\mathfrak{g}_c)$ is denoted by Diff $(U)$. It is called the algebra of differential operators on $\mathfrak{g}$ with $C^{\infty}$ coefficients. We denote by $\mathbb{D}(\mathfrak{g})$ the algebra generated by $P(\mathfrak{g}_c)$ and denote it as the algebra of differential operators on $\mathfrak{g}$ with polynomial coefficients. We write $f(\partial)$ to mean $(\partial f)(u)$, for $D \in$ Diff $(U)$, $u \in U$, $f \in C^{\infty}(U)$. For any $D \in$ Diff $(U)$, and $u \in U$, there is a unique $p \in S(\mathfrak{g}_c)$, such that $f(\partial) = f(u; \partial(p))$. $\partial(p)$ is called the local expression of $D$ at $u$, and is denoted by $D_u$.

Let $U$ be an open subset of $\mathfrak{g}$ invariant under $G$. $G$ acts naturally on $C^{\infty}(U)$ and Diff $(U)$ if we set

$$ f^X(x) = f(x^{-1}) \quad (x \in G, \, x \in U, \, f \in C^{\infty}(U)) $$

and

$$ D^X f = (D f)^{x^{-1}} \quad (D \in$ Diff $(U), \, x \in G, \, f \in C^{\infty}(U)). $$

$D$ (resp. $f$) is called invariant under $G$ if $D^x = D$ (resp. $f^x = f$) for all $x \in G$.

Let $l^{\infty}(U) = \{ f: f \in C^{\infty}(U), \, f^x = f \ \text{for all} \ x \in G \}$, and Diff $^{inv}(U) = \{ D \in$ Diff $(U): D^x = D \ \text{for all} \ x \in G \}$. We put $\mathfrak{d}(\mathfrak{g}) = \{ D \in \mathbb{D}(\mathfrak{g}) | D^x = D \ \text{for all} \ x \in G \}$.

We now take $U = (\xi')^G$. Then, corresponding to any $D \in$ Diff $^{inv}(U)$, Harish-Chandra [1] constructed a differential operator $\delta_U^H(D)$ on $\xi'$ such that

$$ f(H; D) = \tilde{f}(H; \delta_U^H(D)) \quad (f \in l^{\infty}((\xi')^G)) $$

for all $f \in l^{\infty}(U)$, and $H \in \xi'$. $\delta_U^H(D)$ is called the radial component of $D$ on $\xi'$ and we have the following result (Harish-Chandra [1]):

**Theorem 1.1.** Given any $D \in$ Diff $^{inv}((\xi')^G)$, there is exactly one $\delta_U^H(D) \in$ Diff $(\xi')$ such that

$$ f(H; D) = \tilde{f}(H; \delta_U^H(D)) \quad (f \in l^{\infty}((\xi')^G)). $$

The operator $\delta_U^H(D)$ is invariant under the normalizer of $\xi$ in $G$. The map $D \mapsto \delta_U^H(D)$ is a homomorphism of Diff $^{inv}((\xi')^G)$ into Diff $(\xi')$. 

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It turns out to be important to calculate $S(D)$ explicitly for at least the most important of the invariant differential operators $D$. This was first done by Harish-Chandra. We shall now describe two of his results. Let $l_s(\mathfrak{g}_c)$ be the algebra of $G$-invariant elements of $S(\mathfrak{g}_c)$, and $l_s(\mathfrak{h}_c)$ the subalgebra of $S(\mathfrak{h}_c)$ of elements invariant under the Weyl group $\mathbb{W}(\mathfrak{g}_c, \mathfrak{h}_c)$. If $p \in l_s(\mathfrak{g}_c)$, then by a theorem of Chevalley, there is a unique $p_b \in l_s(\mathfrak{h}_c)$ such that $p - p_b$ is in the ideal generated by the root-spaces. Harish-Chandra [1] showed that, for $p \in S(\mathfrak{g}_c)$,

$$D(p) = S(D(p)),$$

Generalizing this, he proved in [2] that, for any $D \in \mathcal{D}(\mathfrak{g}_c)$ which is invariant, $\pi \circ D(D) \circ \pi$ is the restriction to $\mathfrak{h}_c$ of an element of $\mathcal{D}(\mathfrak{h}_c)$ that is invariant with respect to $\mathbb{W}(\mathfrak{g}_c, \mathfrak{h}_c)$.

We shall obtain these two results by a method that is somewhat different from Harish-Chandra's. We shall prove (3) first when $G$ is compact and then extend it for noncompact $G$ by the "unitarian trick". The second result is then deduced from formula (3). The proof of (3) for compact $G$ is quite simple and goes as follows. By invariant integration and the Weyl character formula, we obtain explicit formulae for a class of invariant eigenfunctions on $\mathfrak{g}$, the formula (3) applied to these eigenfunctions now determines $S(D(p))$ uniquely.

2. Case when $G$ is compact. We now assume that $G$ is compact and simply connected. Let $B$ be the analytic subgroup of $G$ corresponding to $\mathfrak{h}_c$. $B$ is a maximal torus of $G$ and $G = B^G$. Exp is a homomorphism of $\mathfrak{h}_c$ onto $B$. Let $\hat{B}$ be the character group of $B$. If $\xi \in \hat{B}$, $\xi \circ \text{exp}$ is a character of $\mathfrak{h}_c$, and so there is a $C$-linear function $\lambda$ on $\mathfrak{h}_c$, which takes pure imaginary values on $\mathfrak{h}_c$, such that $\xi \circ \text{exp} = e^{\lambda \xi}$. $\lambda$ is uniquely determined by $\xi$, and we write $\lambda = \xi$. Thus $\xi(\text{exp} H) = e^{\lambda(H)}$ (H $\in \mathfrak{h}_c$).

Let $\mathcal{L}$ be the set of all linear functions on $\mathfrak{h}_c$ with the property that $e^{\lambda} = \xi \circ \text{exp}$ for some $\xi \in \hat{B}$. $\mathcal{L}$ is an additive subgroup of $\mathfrak{h}_c$ and the correspondence $\lambda \rightarrow \xi$ (l $\in \mathcal{L}$) is an isomorphism of $\mathcal{L}$ onto $\hat{B}$. Since $G$ is simply connected, $\mathcal{L}$ is precisely the set of all integral linear functions on $\mathfrak{h}_c$ [6]. If $\alpha$ is a root of $(\mathfrak{g}_c, \mathfrak{h}_c)$, then $\alpha \in \mathcal{L}$. Now $B$ is the centralizer of $\mathfrak{h}_c$ in $G$, Ad$(B)$ leaves the root-subspaces corresponding to $\alpha$ invariant, and we have $X_{\alpha}^b = \xi_{\alpha}(b)X_{\alpha}$ (b $\in B$, $X_{\alpha} \in \mathfrak{g}_\alpha$). Define $\delta = \frac{1}{2} \sum_{\alpha \in P} \alpha^2$, $\delta$ is integral, and so $\delta \in \mathcal{L}$. Let $\Delta = \xi_{\delta} \prod_{\alpha \in P} \xi_{\alpha}^{-1}$. $\Delta$ is a finite sum of characters of $B$. If we write $\epsilon(s) = \det(s)$ for $s \in W = W(\mathfrak{g}_c, \mathfrak{h}_c)$, it is known that $\Delta = \sum_{s \in W} \epsilon(s) \xi_{\delta}$. Denote by $B^+$ the set of $b \in B$ such that $\Delta(b) \neq 0$, i.e., $\xi_{\alpha}(b) \neq 1$ for all $\alpha \in P$.

Let $\mathcal{D}_p^+$ be the set of all dominant integral linear functions on $\mathfrak{h}_c$. For any $\lambda \in \mathcal{D}_p^+$, let $\pi_\lambda$ be the irreducible representation of $\mathfrak{g}_c$ with highest weight $\lambda$. Since $G$ is simply connected, $\pi_\lambda$ lifts to a representation of $G$, denoted by $\pi_\lambda$ again.
Let $\psi_\lambda(x) = \text{tr} \pi(x)$, $x \in G$. We then have the following formula of Weyl.

$\psi_\lambda(b) = \sum_{s \in \mathbb{W}} \epsilon(s) s_\lambda(b) \Delta(b)$ \quad $(b \in B')$.

In particular, for $H \in \mathfrak{h}$ for which $\exp H \in B'$,

$\text{tr} \pi_\lambda(\exp H) = \sum_{s \in \mathbb{W}} \epsilon(s) s_\lambda(H) \prod_{\alpha \in \mathcal{P}} \left( e^{\alpha(H)/2} - e^{-\alpha(H)/2} \right)$

and

$\dim \pi_\lambda = \prod_{\alpha \in \mathcal{P}} \langle \lambda + \delta, \alpha \rangle \prod_{\alpha \in \mathcal{P}} \langle \delta, \alpha \rangle.$

Let $\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}_c$ and $\mathcal{Z}$ the center of $\mathfrak{g}$. By Schur's lemma there is a homomorphism $\chi_\lambda$ (called the infinitesimal character of $\pi_\lambda$) of $\mathcal{Z}$ into $\mathbb{C}$, such that $\pi_\lambda(z) = \chi_\lambda(z) \cdot 1$ for all $z \in \mathcal{Z}$. Let $\mathcal{H}$ be the subalgebra of $\mathfrak{g}$ generated by 1 and $\mathfrak{h}_c$, and let $\mathcal{P}$ be the left ideal $\sum_{\alpha \in \mathcal{P}} \mathfrak{g}_\alpha$. Then it can be shown that for each $z \in \mathcal{Z}$, there is a unique element $\beta_p(z) \in \mathcal{H}$ such that $z - \beta_p(z) \in \mathcal{P}$. The map $\beta_p : z \mapsto \beta_p(z)$ is a homomorphism of $\mathcal{Z}$ onto $\mathcal{H}$, and (cf. Harish-Chandra [4])

$\chi_\lambda(z) = \beta_p(z)(\lambda) \quad (z \in \mathcal{Z}).$

Let $U$ be a $G$-invariant open subset of $\mathfrak{g}$ and $f \in C^\infty(U)$. We say $f$ is an invariant eigenfunction on $\mathfrak{g}$, if $f$ is $G$-invariant, and for each $p \in I_\mathcal{S}(\mathfrak{g}_c)$, there is a $\chi(p) \in \mathbb{C}$ such that

$\partial(p)f = \chi(p)f \quad \text{on } U.$

$\chi$ is uniquely determined if $f \neq 0$ and $\chi(p \rightarrow \chi(p))$ is a homomorphism of $I_\mathcal{S}(\mathfrak{g}_c)$ into $\mathbb{C}$.

Lemma 2.1. Let $dx$ be the normalized Haar measure on $G$, and $X' \in \mathfrak{g}_c$.

Define

$f_{X'}(X) = f(X' : X) = \int_G \exp B(X^x, X') dx \quad (X \in \mathfrak{g}).$

Then $f_{X'}$ is an invariant eigenfunction on $\mathfrak{g}$, and

$\partial(p)f_{X'} = \tilde{p}(X')f_{X'} \quad (p \in I_\mathcal{S}(\mathfrak{g}_c))$

where $p \mapsto \tilde{p}$ is the isomorphism of $S(\mathfrak{g}_c)$ onto $P(\mathfrak{g}_c)$ induced by the Cartan-Killing form.
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Proof. Let \( \phi(x, X) = \exp B(X^x, X') \), \( x \in G, \ X \in \mathfrak{g}_c \). Then \( \phi \in C^\infty(G \times \mathfrak{g}) \) and as \( G \) is compact, the integral defined by \( /_{X'}(X) \) exists. The function \( X \rightarrow /_{X'}(X) \) is of class \( C^\infty \) on \( \mathfrak{g} \) and we can differentiate under the integral sign. If \( p \in S(\mathfrak{g}_c) \),

\[
/f_{X'}(X; \partial(p)) = \int_G \phi(x; X; \partial(p)) \, dx \quad (X \in \mathfrak{g}).
\]

Let \( \omega \in \mathfrak{g}_c^* \). Then using the natural identification of \( S(\mathfrak{g}_c) \) with \( P(\mathfrak{g}_c^*) \), we obtain \( \partial(p)e^{\omega} = \rho(\omega)e^{\omega} \). If \( Y \in \mathfrak{g}_c \) and \( \omega \) is the linear function \( X \rightarrow B(X, Y) \), then \( \rho(\omega) = \tilde{\rho}(Y) \). From these and the fact that \( \phi(x, X) = \exp B(X, X^x_1x^{-1}) \), we get, for all \( x \in G, \ X \in \mathfrak{g} \),

\[
\phi(x, X; \partial(p)) = \tilde{\rho}(X^x_1x^{-1})\phi(x, X).
\]

If we assume \( p \in I_s(\mathfrak{g}_c) \), we obtain \( \partial(p)/_{X'} = \tilde{\rho}(X')/_{X'} \). Invariance of \( /_{X'} \) follows from translation invariance of \( dx \).

Our aim in the rest of this section is to evaluate the \( /_{X'} \). This requires some preparation. For any subspace \( \alpha \) of \( \mathfrak{g}_c \), we shall identify \( S(\alpha) \) with the subalgebra of \( S(\mathfrak{g}_c) \) generated by \( 1 \) and \( \alpha \). For any \( n \geq 0 \), write \( S_n(\mathfrak{g}_c) \) for the subspace of \( S(\mathfrak{g}_c) \) spanned by homogeneous elements of degree \( n \). Set \( S^{(n)}(\mathfrak{g}_c) = \Sigma_{0 \leq r \leq n} S_r(\mathfrak{g}_c) \). Let \( \lambda \) be the customary symmetrizer map of \( S(\mathfrak{g}_c) \) onto \( \mathfrak{g}_c \). Since \( \lambda(\rho^x) = \lambda(\rho^x) \) \( (p \in S(\mathfrak{g}_c), x \in G) \), \( \lambda \) is a linear bijection of \( I_s(\mathfrak{g}_c) \) onto \( \mathfrak{g}_c \). It is also known to be a linear bijection of \( S^{(n)}(\mathfrak{g}_c) \) onto the subspace \( \mathfrak{g}_c^{(n)} \) of \( \mathfrak{g}_c \) spanned by \( 1 \) and elements of the form \( X_1 \cdots X_r \) \((1 \leq r \leq n, X_i \in \mathfrak{g}_c)\).

Lemma 2.2. Let \( \mathfrak{h} \) be the subspace of \( \mathfrak{g}_c \) spanned by \( X_{\alpha} \), \( \alpha \in \Delta \). Then \( S(\mathfrak{g}_c) = S(\mathfrak{h}_c) + S(\mathfrak{g}_c)\mathfrak{h} \). For any \( p \in S(\mathfrak{g}_c) \), let \( p_\mathfrak{h} \) denote the unique element of \( S(\mathfrak{h}_c) \) such that \( p - p_\mathfrak{h} \in S(\mathfrak{g}_c)\mathfrak{h} \). Then \( p \rightarrow p_\mathfrak{h} \) is an isomorphism of \( I_s(\mathfrak{g}_c) \) onto \( I_s(\mathfrak{h}_c) \). If \( p \in I_s(\mathfrak{g}_c) \cap S^*(\mathfrak{g}_c) \) then \( p_\mathfrak{h} \in I_s(\mathfrak{h}_c) \cap S^*(\mathfrak{h}_c) \).

Proof. The statements are all consequences of the theorem of Chevalley mentioned earlier.

Put \( H = \{ a \in H : a^s = a \ \text{for all} \ s \in W \} \). The map \( \lambda \), by restriction, induces an isomorphism of \( S(\mathfrak{h}_c) \) onto \( H \), and \( I_s(\mathfrak{h}_c) \) onto \( H \). Let \( H^\alpha = \lambda(\mathfrak{h}^\alpha(\mathfrak{h}_c)) \). \( H \) is the direct sum of \( H^\alpha \)'s, and for any \( v \in H \), we write \( v^\alpha \) for its component in \( H^\alpha \). Put \( H^{(n)} = \Sigma_{0 \leq r \leq n} H_r \). Then

Lemma 2.3. Let \( p \in I_s(\mathfrak{g}_c) \cap S^*(\mathfrak{g}_c) \). Then \( \lambda(p) \in \mathfrak{g}_c \cap \mathfrak{g}^{(n)} \), \( \beta_\mathfrak{h}(\lambda(p)) \in H^{(n)}, \beta_\mathfrak{h}(\lambda(p)) = \lambda(p_\mathfrak{h}) \).

Proof. Let \( \{ H_1, \cdots, H_d \} \) be a basis for \( \mathfrak{h}_c \) and \( P = \{ \alpha_1, \cdots, \alpha_d \} \). Let \( (q) = (q_1, \cdots, q_d) \), \( (c) = (c_1, \cdots, c_i) \), \( (r) = (r_1, \cdots, r_d) \), where \( q_i, c_j, \text{and} \ r_k \) are integers \( \geq 0 \). Define
Let \( p \in I_s(\mathfrak{g}_c) \) if and only if it is a linear combination of \( M((q), (c), (r)) \) for which \( \sum_{i=1}^d (r_i - q_i) \alpha_i = 0 \). Hence we can find constants \( A((q), (c), (r)) \) (all but finitely many of them zero) such that

\[
p = p_b + \sum_{\Omega} A((q), (c), (r))M((q), (c), (r))
\]

where the sum is over the set \( \Omega \) of all \( ((q), (c), (r)) \) for which \( \sum_{i=1}^d (r_i - q_i) \alpha_i = 0 \), \( \sum_{i} r_i > 0 \), \( \sum_{i} q_i > 0 \) and \( \sum_{i=1}^d (r_i + q_i) = n \).

On the other hand by considering \( M((q), (c), (r)) \) as an element of \( G \), we can show that if \( z \in \mathcal{L} \cap G(n) \), then \( \beta_p(z) \in H(n) \). Let \( z = \tilde{x}(p) \), then using (8) and the result that \( \tilde{x}(M((q), (c), (r))) \in \mathcal{L} + G(n-1) \), and the fact that \( \beta_p \) maps \( G(n-1) \) into \( H(n-1) \) we conclude

\[
\beta_p(z) \in \tilde{x}(p_b) + H(n-1).
\]

But \( \tilde{x}(p_b) \in H_n \). Hence (9) implies the result.

Consider now \( u \in S_n(\mathfrak{g}_c) \), then \( p = \int_G u^* dx \) is a well-defined element of \( l_s(\mathfrak{g}_c) \cap S_n(\mathfrak{g}_c) \). Also forming \( \int_G \tilde{x}(u)^* dx \), we obtain an element \( z \in \mathcal{L} \cap G(n) \).

Since \( \tilde{x}(u^*) = \tilde{x}(u)^* \), \( x \in G \), we have \( z = \tilde{x}(p) \). Lemma 2.3, now shows that \( \beta_p(z) = \tilde{x}(p_b) \). Taking \( u = H^n \) for \( H \in \mathfrak{h}_c \) we get

**Lemma 2.4.** Let \( n \geq 0 \). Then for any \( H \in \mathfrak{h}_c \)

\[
\zeta_{H,n} = \int_G (H^n)^* dx
\]

is a well-defined element of \( l_s(\mathfrak{g}_c) \cap S_n(\mathfrak{g}_c) \) while considering the same integral in \( G \), we obtain an element \( z_{H,n} \in \mathcal{L} \cap G(n) \).

We have \( \tilde{x}(\zeta_{H,n}) = z_{H,n} \) and \( \beta_p(z_{H,n}) = \tilde{x}(\zeta_{H,n}) \).

**Corollary 2.5.** Let \( \mu \rightarrow H'_\mu \) be the isomorphism of \( \mathfrak{h}_c^* \) with \( \mathfrak{h}_c \) induced by the Cartan-Killing form. Then for any \( H \in \mathfrak{h}_c \), \( n \geq 0 \) and \( \mu \in \mathfrak{h}_c^* \),

(9) \[
\int_G B(H^*, H'_\mu)^n dx = \beta_p(z_{H,n})(\mu).
\]

**Proof.** From Lemma 2.4, we have \( \beta_p(z_{H,n})(\mu) = \tilde{x}(\zeta_{H,n})_b(\mu) \) for \( \mu \in \mathfrak{h}_c^* \).

From isomorphism \( p \rightarrow \tilde{p} \) of \( S(\mathfrak{g}_c) \) with \( P(\mathfrak{g}_c) \) and the definition of \( \zeta_{H,n} \), we obtain

\[
\zeta_{H,n}(Y) = \int_G B(H^*, Y)^n dx \quad (H \in \mathfrak{h}, Y \in \mathfrak{g}_c).
\]

So for \( \mu \in \mathfrak{h}_c^* \),

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\[ \int_G B(H^\mu, H'_\mu) \, dx = \zeta_{H,n}(H'_\mu). \]

On the other hand, the map \( p \rightarrow p_{_{\mathfrak{h}}} \) corresponds, via the isomorphism \( p \rightarrow \widetilde{p} \), to the restriction map \( \widetilde{p} \rightarrow \widetilde{p}_{_{\mathfrak{b}}} \) of \( P(\mathfrak{g}) \) onto \( \mathfrak{p}(\mathfrak{b}) \) so \( \zeta_{H,n}_{_{\mathfrak{b}}} = (\zeta_{H,n}_{_{\mathfrak{b}}}_{_{\mathfrak{g}}})^\omega \), from which we get \( \zeta_{H,n}(H'_\mu) = (\zeta_{H,n}_{_{\mathfrak{b}}})(\mu), \mu \in \mathfrak{b}^*. \) Since \( \lambda(\mathfrak{h}^n, H) \), \( H \in \mathfrak{h}_C \),

\( \lambda(\zeta_{H,n}_{_{\mathfrak{b}}})(\mu) = (\zeta_{H,n}_{_{\mathfrak{b}}})(\mu). \)

We are now in a position to obtain a formula for \( f_X \), defined in Lemma 2.1.

**Theorem 2.6.** Let \( G \) be a compact and semisimple real Lie group, \( g \) its Lie algebra, \( \mathfrak{b} \subset g \) a Cartan subalgebra, \( \pi \) a positive system of roots, \( \pi = \Pi_{\alpha \in P} \alpha, \)

\( \omega = \Pi_{\alpha \in P} H_{\alpha}^\cdot. \) Then for all \( H, H' \in \mathfrak{h}_C \) we have

\[ \pi(H)n(H') \int_G \exp B(H^\mu, H') \, dx = \left( \prod_{\alpha \in P} \langle \delta, \alpha \rangle \right) \sum_{s \in W} \epsilon(s) e^{B(sH, H')} \cdot \]

**Proof.** We have \( \exp B(X, Y) = \sum_{n \geq 0} B(X, Y)^n/n! . \) Since \( |B(X^\mu, H')| \) is bounded as \( x \) varies in \( G \), we can write

\[ \int_G \exp B(H^\mu, H') \, dx = \sum_{n \geq 0} \frac{1}{n!} \int_G B(H^\mu, H')^n \, dx. \]

Upon replacing \( H' \) by \( H'_\mu (\mu \in \mathfrak{b}^*) \) in above and using Corollary 2.5, we get

\[ \int_G \exp B(H^\mu, H'_\mu) \, dx = \sum_{n \geq 0} \frac{1}{n!} \beta_p (z_{H,n})(\mu). \]

For \( \lambda \in \mathfrak{b}_P^*, \) let \( \pi \) be the irreducible representation of \( \mathfrak{g}_C \) and \( G \) with highest weight \( \lambda \). From (7) we obtain

\[ \beta_p (z_{H,n})(\lambda) = \chi^\lambda (z_{H,n}). \]

Hence, as \( z_{H,n} \in \mathfrak{z} \)

\[ \tr \pi (z_{H,n}) = (\dim \pi) \chi^\lambda (z_{H,n}) = (\dim \pi) \beta_p (z_{H,n})(\lambda). \]

Using (6) in above we get

\[ \omega(\lambda + \delta) \beta_p (z_{H,n})(\lambda) = \prod_{\alpha \in P} \langle \delta, \alpha \rangle \tr \pi (z_{H,n}). \]

But from the definition of \( z_{H,n} \),

\[ \tr \pi (z_{H,n}) = \int_G \tr \pi (H^n) \, dx = \tr \pi (H^n). \]

Now, for \( t \in \mathbb{R} \),

\[ \pi(\exp tH) = e^{t\pi (H)} = \sum_{n \geq 0} \frac{t^n \pi (H^n)}{n!}. \]
So
\[ \text{tr } \pi(\exp tH) = \sum_{n \geq 0} \frac{\text{tr } \pi(H^n)}{n!} t^n. \]

From Weyl's character formula (5)
\[ \prod_{\alpha \in \mathcal{P}} (e^{t\alpha(H)^2} - e^{-t\alpha(H)/2}) \text{tr } \pi(\exp tH) = \sum_{s \in \mathcal{W}} \epsilon(s) e^{t\sigma(\lambda + \delta)(H)}, \quad t \in \mathbb{R}, H \in \mathfrak{g}. \]

Using above formula in (11) we conclude
\[ \prod_{\alpha \in \mathcal{P}} (e^{t\alpha(H)^2} - e^{-t\alpha(H)/2}) \sum_{n \geq 0} \frac{\text{tr } \pi(H^n)}{n!} t^n = \sum_{s \in \mathcal{W}} \epsilon(s) e^{t\sigma(\lambda + \delta)(H)}. \]

Upon expanding \( \prod_{\alpha \in \mathcal{P}} (e^{t\alpha(H)/2} - e^{-t\alpha(H)/2}) \) and letting \( d = [P] \) we get
\[ \prod_{\alpha \in \mathcal{P}} (e^{t\alpha(H)/2} - e^{-t\alpha(H)/2}) = t^d \pi(H) \sum_{m \geq 0} q_m t^m. \]

For sufficiently small \(|t|\) above converges, hence from (12)
\[ t^d \pi(H) \sum_{n \geq 0} \frac{\text{tr } \pi(H^n)}{n!} t^n = \left[ \sum_{m \geq 0} q_m t^m \right]^{-1} \sum_{s \in \mathcal{W}} \epsilon(s) e^{t\sigma(\lambda + \delta)(H)}. \]

Since \( q_0 = 1 \), for small \(|t|\), we have \( \left( \sum_{m \geq 0} q_m t^m \right)^{-1} = \sum_{m \geq 0} q' m^m \). Using this in above formula, expanding the right-hand side, and equating the coefficients of \( t^{n+d} \), we get
\[ \frac{\pi(H)}{n!} \text{tr } \pi(H^n) = \frac{\sum_{s \in \mathcal{W}} \epsilon(s) s(\lambda + \delta)(H)^{n+d}}{(n+d)!} + \frac{\sum_{s \in \mathcal{W}} \epsilon(s) s(\lambda + \delta)(H)^{n+d-2}}{(n+d-2)!} + \cdots. \]

Combining this with (10) we get
\[ \left( \prod_{\alpha \in \mathcal{P}} (\sigma, \alpha) \right)^{-1} \frac{\pi(H)}{n!} \omega(\lambda + \delta) \beta_p(z_H, n)(\lambda) = \frac{\sum_{s \in \mathcal{W}} \epsilon(s) s(\lambda + \delta)(H)^{n+d}}{(n+d)!} + \frac{\sum_{s \in \mathcal{W}} \epsilon(s) s(\lambda + \delta)(H)^{n+d-2}}{(n+d-2)!} + \cdots. \]

\( q_i \)'s are independent of \( \lambda \), and since above is true for \( \lambda \in \mathfrak{g}_+^e \), it is true for \( \lambda \in \mathfrak{g}_c^e \). In above formula we equate components which are homogeneous polynomials of degree \( n+d \) in \( \lambda \):
\[ \frac{\pi(H)}{n!} \omega(\lambda) \beta_p(z_H, n)(\lambda) = \prod_{\lambda \in \mathcal{P}} (\lambda, \alpha) \frac{\sum_{s \in \mathcal{W}} \epsilon(s) s(\lambda)(H)^{n+d}}{(n+d)!}. \]

Substituting above in
\[
\int_G \exp B(H^x, H'_\mu) \, dx = \sum_{n \geq 0} \frac{1}{n!} B_p(z_{H,n})(\mu)
\]

obtained earlier we conclude

\[
\pi(H) \sim (\mu) \int_G \exp B(H^x, H'_\mu) \, dx
\]

(13)

\[
= \left( \prod_{a \in \mathbb{P}} \langle \delta, a \rangle \right) \sum_{m \geq d} \frac{1}{m!} \sum_{s \in W} \epsilon(s)((s\mu)(H))^m = \left( \prod_{a \in \mathbb{P}} \langle \delta, a \rangle \right) \sum_{s \in W} \epsilon(s)e^{(s\mu)(H)}.
\]

For \( p \geq 0 \) define \( f_p(\lambda) = \sum_{s \in W} \epsilon(s)((s\lambda)(H))^p, \lambda \in \mathbb{C}^*. \) \( f_p \) is homogeneous of degree \( p \) and is skew symmetric. It follows [1] that \( f_p = 0, \ p < d. \) In view of above, we can rewrite (13) as

\[
\pi(H) \sim (\mu) \int_G \exp B(H^x, H'_\mu) \, dx
\]

(13)

\[
= \left( \prod_{a \in \mathbb{P}} \langle \delta, a \rangle \right) \sum_{m \geq d} \frac{1}{m!} \sum_{s \in W} \epsilon(s)((s\mu)(H))^m = \left( \prod_{a \in \mathbb{P}} \langle \delta, a \rangle \right) \sum_{s \in W} \epsilon(s)e^{(s\mu)(H)}.
\]

Replacing \( H'_\mu \) by \( H' \), and noting that \( \sim (\mu) = \pi(H'_\mu) \) and \( (s\mu)(H) = B(sH, H'_\mu) \) we obtain the result.

From this theorem we obtain immediately the sought-for formulae for the eigenfunctions \( f_{X'} \).

**Theorem 2.7.** Let \( f_{X'} \) be defined as in Lemma 2.1, then

\[
f_{H'}(H) = \left( \prod_{a \in \mathbb{P}} \langle \delta, a \rangle \right) \sum_{s \in W} \epsilon(s)e^{B(sH,H')} \frac{\pi(H)\pi(H')}{\pi(H)\pi(H')} \quad (H, H' \in \mathbb{C}).
\]

(14)

3. The explicit calculation of \( \delta'_b(\partial(p)) \) (\( p \in \mathbb{C}(\mathbb{C}) \)). We shall now prove (3), for the case \( G \) is compact. Let \( g \) be of compact type and define

\[
\delta'_b(D) = \pi \circ \delta'_b(D) \circ \pi^{-1} \quad (D \in \text{Diff}(U)).
\]

(15)

**Lemma 3.1.** Let \( E \in \text{Diff}(\mathbb{C}) \) such that \( E_{\delta_{H'}^t} = 0 \) on \( \mathbb{C} \) where \( g_{H}(H) = \sum_{s \in W} \epsilon(s)e^{B(sH,H')} \) \( (H, H' \in \mathbb{C}). \) Then \( E = 0. \)

**Proof.** Choose \( q_1, \ldots, q_k \) linearly independent in \( \mathbb{C}(\mathbb{C}) \); then \( E = \sum_{i=1}^k g_i \delta(q_i) \in C^\infty(\mathbb{C}). \) \( E_{\delta_{H'}^t} = 0 \) implies

\[
\sum_{i=1}^k \sum_{s \in W} \epsilon(s)g_i(H) \delta'(sH')e^{B(sH,H')} = 0 \quad (H, H' \in \mathbb{C}).
\]

(16)
Fix $H \in \mathfrak{g}'$, then the points $sH$, $s \in W$, are distinct (Chevalley). Thus $H' \rightarrow B(sH, H')$ are distinct linear forms. From a result of [3] and (16) we conclude

$$
\sum_{i=1}^{k} \epsilon(s)g^i_s(H)q^i_s(H') = 0 \quad (H' \in \mathfrak{g}).
$$

Taking $s = 1$ and using the linear independence of $q_i$'s we obtain $g^i_s(H) = 0$, $1 \leq i \leq k$. Therefore $E \equiv 0$.

Lemma 3.2. For $p \in l_s(\mathfrak{g}_c)$, $\delta_\mathfrak{g}(\partial(p)) = \partial(\mathfrak{p}_b)$.

Proof. If $g$ is a function on $(\mathfrak{g}')^G$, let $\overline{g} = g|_{\mathfrak{g}'}$. From (14) we get

$$
\overline{f}_{H'}(H; n^{-1} \circ \partial(\mathfrak{p}_b) \circ n) = \overline{P}(H') \overline{f}_{H'}(H).
$$

But Lemma 2.1 implies $\partial(p)|_{\mathfrak{g}'} = \overline{P}(H')|_{\mathfrak{g}'}$ on $\mathfrak{g}$. From the definition of radial component it follows that

$$
\overline{f}_{H'}(H; \delta_\mathfrak{g}(\partial(p))) = \overline{P}(H') \overline{f}_{H'}(H) \quad (H \in \mathfrak{g}).
$$

Above formulae along with (15) imply $(\partial(\mathfrak{p}_b) - \delta_\mathfrak{g}(\partial(p)))g_{H'} = 0$ on $\mathfrak{g}'$ where $g_{H'}(H) = \sum_{s \in W} \epsilon(s)B(sH, H')$ $(H, H' \in \mathfrak{g})$. The result follows from Lemma 3.1.

We are now in a position to prove (3) for an arbitrary semisimple Lie algebra over $\mathbb{R}$. Let $\mathfrak{g}$ be one such and $G$ a connected Lie group with Lie algebra.

Lemma 3.3. Let $p \in l_s(\mathfrak{g}_c)$, then $\delta_\mathfrak{g}(\partial(p)) = n^{-1} \circ \partial(\mathfrak{p}_b) \circ n$ on $\mathfrak{g}'$ if and only if $\partial(\mathfrak{p}_b)(\mathbf{q}_c) = n(\partial(p)\mathbf{q})_b$ on $\mathfrak{g}_c$ for all $q \in l_s(\mathfrak{g}_c)$.

Proof. Let $E = \delta_\mathfrak{g}(\partial(p)) - n^{-1} \circ \partial(\mathfrak{p}_b) \circ n$ and $\mathfrak{f}$ the algebra of all real-valued $G$-invariant polynomials on $\mathfrak{g}$. It can be shown that $E \equiv 0$ on $\mathfrak{g}'$ if and only if $E|_{\mathfrak{g}} = 0$ for all $g \in \mathfrak{f}$. Recall that $\overline{g} = g|_{\mathfrak{g}'}$. This is equivalent to $n^{-1} \circ \partial(\mathfrak{p}_b)(\mathbf{q}_c) = \delta_\mathfrak{g}(\partial(p))\mathbf{q}_b$ on $\mathfrak{g}'$ for all $q \in l_s(\mathfrak{g}_c)$. Using the definition of $\delta_\mathfrak{g}(\partial(p))$ and above, we get $\partial(\mathfrak{p}_b)(\mathbf{q}_c) = n(\partial(p)\mathbf{q})_b$ on $\mathfrak{g}'$. Since we are dealing with polynomials this implies the result.

Theorem 3.4. Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{g}_c$ its complexification. Let $p \in l_s(\mathfrak{g}_c)$; then $\delta_\mathfrak{g}(\partial(p)) = \partial(\mathfrak{p}_b)$.

Proof. Let $U$ be a compact real form of $\mathfrak{g}_c$, and $\mathfrak{b}$ a Cartan subalgebra of it. Then $\mathfrak{g}_c = \mathbb{C} \cdot \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}_c$ and hence $\mathfrak{b}_c = \mathfrak{g}_c$ for some $x \in G_c$.

(2) It suffices to show $E|_{\mathfrak{g}} = 0$ for $H_0 \in \mathfrak{g}'$. $l_s(\mathfrak{g}_c)$ is finitely generated by algebraically independent homogeneous elements of positive degree. It can be shown that we can choose these generators so that their restrictions to $\mathfrak{g}$ are real-valued. Let $\mathbf{q}_1, \ldots, \mathbf{q}_l$ be one such set. Then by the Chevalley theorem $\{\mathbf{q}_1, \cdots, \mathbf{q}_l\}$ generates $l_s(\mathfrak{b}_c)$. A theorem of Shephard and Todd [5] implies $\{\mathbf{q}_1, \cdots, \mathbf{q}_l\}$ forms a $C^\infty$ coordinate system around $H_0$. A straightforward argument now implies the claim.
Let \( \pi \) be the product of positive roots of \((\mathfrak{g}_c, \mathfrak{b}_c)\). We arrange matters so that \( \pi = \pi_\mathfrak{g} \). By Lemma 3.2, \( \delta_\mathfrak{g}(\partial(p)) = \pi^{-1} \circ \partial(p) \circ \pi \) on \( \mathfrak{b}_c \). By Lemma 3.3, 
\[
\partial(p_\mathfrak{b})(\pi_\mathfrak{g} q_\mathfrak{b}) = \pi_\mathfrak{b}(\partial(p)q)_\mathfrak{b} \quad \text{on } \mathfrak{b}_c. 
\]
We then have
\[
\partial(p_\mathfrak{b})(\pi_\mathfrak{g} q_\mathfrak{b}) = \pi_\mathfrak{b}((\partial(p)q)_\mathfrak{b})^\pi.
\]
But \( p_\mathfrak{g} = p_\mathfrak{b} \), \( q_\mathfrak{g} = q_\mathfrak{b} \). So we have \( \partial(p_\mathfrak{b})(\pi q_\mathfrak{b}) = \pi(\partial(p)q)_\mathfrak{b} \). This proves the theorem in view of Lemma 3.3.

4. A theorem on \( \delta_\mathfrak{g}(D) \) when \( D \in \mathfrak{D}(\mathfrak{g}_c) \). We now use the previous results to obtain the main theorem concerning \( \delta_\mathfrak{g}(D) \), \( D \in \mathfrak{D}(\mathfrak{g}_c) \) (cf. Harish-Chandra [2]). For any \( p \in \mathfrak{S}(\mathfrak{g}_c) \) define \( \mu_\mathfrak{D}(D) = \partial(p) \circ D - D \circ \partial(p) \) \((D \in \text{Diff } ((\mathfrak{g}_c)^G)) \). \( \mu_\mathfrak{D} \) is a derivation of \( \mathfrak{D}(\mathfrak{g}_c) \). Define \( \mu_\mathfrak{m}(D) = \mu(\mu_\mathfrak{m}^{-1}(D)) \), \( m \geq 2 \).

Lemma 4.1. Let \( D \in \mathfrak{D}(\mathfrak{g}_c) \), \( p \in \mathfrak{S}(\mathfrak{g}_c) \). Then there is an integer \( m = m(p, D) \geq 0 \) such that \( \mu_\mathfrak{m}(D) = 0 \).

Proof. Since we can write \( D = \sum_{i=1}^l p_i \partial(q_i) \), \( p_i \in \mathfrak{P}(\mathfrak{g}_c) \), \( q_i \in \mathfrak{S}(\mathfrak{g}_c) \) we may assume \( D = p_1 \partial(q_1) \). We assert \( \mu(p_1)(D) \) can be written as \( \sum_{i=1}^k p_i \partial(q_i) \) with \( \deg p_i < \deg p_1 \) for all \( i \). Write \( E = F(q_1) \) where \( F = [\partial(p), p_1] \). Direct calculations show that \( F \) can be written as \( \sum_{j=1}^N f_j \partial(q_j) \) where each \( f_j \) is of the form \( \partial(b_j) p_1 \) for some \( b_j \in \mathfrak{P}(\mathfrak{g}_c) \) which is homogeneous of positive degree. So \( \deg f_j < \deg p_1 \) for all \( j \). Since \( E = \sum_{j} \partial(q_j) q_j \) the result follows by induction on \( \max_i \deg p_i \).

Lemma 4.2. Let \( U \subset \mathfrak{g}_c \) be a connected and open set, \( f \in C^\infty(U) \). Suppose for every \( p \in \mathfrak{I}_S(\mathfrak{g}_c) \), there is an integer \( m = m(p) > 0 \) such that \( \partial(p)f = 0 \) on \( U \). Then \( f \) is the restriction to \( U \) of a polynomial on \( \mathfrak{g}_c \).

Proof. Let \( p_1, \ldots, p_l \) be homogeneous generators of \( \mathfrak{I}_S(\mathfrak{g}_c) \). Then it can be seen that the assumption implies \( \partial(p_i)^m = 0 \) for some \( m \geq 0, 1 \leq i \leq l \). Let \( d_i = \deg(p_i) \) and \( k = m(d_1 + \cdots + d_l) \). For \( H \in \mathfrak{g}_c \) consider \( Q(\zeta) = \Pi_{s \in \mathfrak{W}}(\zeta^k - (sH)^k) \). \( Q(H) = 0 \), so \( \partial(Q(H))f = 0 \). Let \( \omega = [W] \). Then
\[
Q(\zeta) = (\zeta^k)^{\omega} + (\zeta^k)^{\omega-1}Q_{1,H} + \cdots + Q_{\omega,H}
\]
where \( Q_{j,H} \) is a homogeneous element of \( \mathfrak{I}_S(\mathfrak{g}_c) \) of degree \( jk \). For fixed \( j \), \( Q_{j,H} \) is a linear combination of monomials \( p_1^{a_1} \cdots p_l^{a_l} \) with \( d_1 a_1 + \cdots + d_l a_l = jk \)
\[
k = m(d_1 + \cdots + d_l). \]
Hence there is at least one \( a_i > m \), so that \( \partial(p_1^{a_1} \cdots p_l^{a_l})f = 0 \). Thus \( \partial(Q_{j,H})f = 0 \). Hence \( \partial(H^k \omega)f = 0 \). As \( U \) is connected, this implies \( f \) is a polynomial of degree \( \leq lk \omega \).

Lemma 4.3. Let \( U \subset \mathfrak{g}_c \) be a connected open set, \( E \in \text{Diff } (U) \). Suppose for each \( p \in \mathfrak{I}_S(\mathfrak{g}_c) \) there is an integer \( m = m(p) > 0 \) such that \( \mu_\mathfrak{m}(E) = 0 \). Then there
is an $F \in D(\mathcal{S})$ such that $E = F$ on $U$, i.e. $E$ has polynomial coefficients.

Proof. Write $E = \sum_{i=1}^{N} f_i \partial(g_i)$, $f_i \in C^\infty(U)$, $g_i \in S(\mathcal{S}_c)$. We will show $f_i \in P(\mathcal{S}_c)$. We may assume $g_i$ are linearly independent, homogeneous and $\deg g_1 \leq \deg g_2 \leq \cdots \leq \deg g_N$. Also we may assume $f_i \notin P(\mathcal{S}_c)$. Let $d = \min_i \deg g_i$ and $s \geq 1$ such that $\deg g_i = d$, $1 \leq i \leq s$, $\deg g_i > d$, $s < i \leq N$. Let $p \in I_s(\mathcal{S}_c)$. It can be shown

$$\mu^m_p(E) = \sum_{i=1}^{N} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} m \\ r \end{pmatrix} (\partial(p))^{m-r} (\partial(g_i)) \partial(p)^r.$$  

Let $M_f$ be the operator of multiplication by $f$, then $\partial(p)^{m-r} M_f = \sum_U U_{i,t} \partial(V_{i,t})$ where $U_{i,t} \in C^\infty(U)$ and $V_{i,t}$ homogeneous. We then get

$$\partial(p)^{m-r} (f_i \partial(g_i)) \partial(p)^r = \sum_U U_{i,t} \partial(V_{i,t} g_i).$$

If $r = 0$ we can write $\partial(p)^{m} M_f = M + \sum_U U_{i,t} \partial(V_{i,t})$ where the $U_{i,t}$ are homogeneous elements of strictly positive degree. We then have

$$\partial(p)^{m} (f_i \partial(g_i)) = \sum_U U_{i,t} \partial(V_{i,t} g_i).$$

It follows from the above two equations that

$$\mu^m_p(E) = \sum_{i=1}^{s} (\partial(p)^m f_i) \partial(g_i) + \sum_{j=1}^{N} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} m \\ r \end{pmatrix} (\partial(p))^{m-r} (\partial(g_m)) \partial(p)^r.$$  

where $f_m \in C^\infty(U)$ and $g_m$ are homogeneous elements of $S(\mathcal{S}_c)$ of degree $> d$. If we choose $m$ so that $\mu^m_p(E) = 0$, then we conclude from above formula that $\partial(p)^{m} f_i = 0$, $1 \leq i \leq s$. Lemma 4.2 now implies that $f_i \in P(\mathcal{S}_c)$, a contradiction.

Theorem 4.4. Let $D \in \mathcal{S}(\mathcal{G})$ and $\delta(\mathcal{G}) = \pi \circ \delta(\mathcal{G}) \circ \pi^{-1}$. There is a unique element of $\mathcal{D}(\mathcal{S})$ which coincides with $\delta(\mathcal{G})$ on $\mathcal{S}'$.

Proof. We need only to prove the existence. By Theorem 2.1, $\delta(\mathcal{G}) : D' \rightarrow \pi \circ \delta(\mathcal{G}) \circ \pi^{-1}$ is a homomorphism of $\mathcal{S}(\mathcal{G})$ into $\text{Diff}(\mathcal{S}')$. Suppose now $p \in I(\mathcal{G}_c)$. By Lemma 4.1, there is an integer $m \geq 0$ such that $\mu^m_p(D) = 0$. So

$$\sum_{r=0}^{m} (-1)^r \begin{pmatrix} m \\ r \end{pmatrix} (\partial(p))^{m-r} D \partial(p)^r = 0.$$  

Apply homomorphism $\delta(\mathcal{G})$ to above and note that by Theorem 3.4, $\delta(\partial(q)) = \partial(q_c)$ ($q \in I_s(\mathcal{G}_c)$) we will get

$$\sum_{r=0}^{m} (-1)^r \begin{pmatrix} m \\ r \end{pmatrix} (\partial(p))^{m-r} \delta(\partial(p)^r) = 0 \text{ on } \mathcal{S}'.$$  

Comparing above with (17) we get
\[
\mu^m_p(\delta_{\mathfrak{g}}(D)) = 0 \quad \text{on } \mathfrak{g}'.
\]

Let \( U \) be a connected open set in \( \mathfrak{g}' \). Since \( p \to p_\mathfrak{b} \) is an isomorphism of \( L_s(\mathfrak{g}_c) \) onto \( L_s(\mathfrak{g}_c) \), we conclude from above that given any \( q \in L_s(\mathfrak{g}_c) \) which is homogeneous of positive degree, there is an integer \( m = m(p) > 0 \) such that \( \mu^m q(\delta_{\mathfrak{b}}(D)) = 0 \) on \( U \). By Lemma 4.3, we can find an \( L \in \mathcal{D}(\mathfrak{b}) \) such that

\[
L = \delta_{\mathfrak{b}}(D) \quad \text{on } U.
\]

It can be shown \cite{1} that \( \delta_{\mathfrak{b}}(D) \) can be written as \( \pi^{-N'} \circ F \) for some integer \( N' \geq 0 \) and \( F \in \mathcal{D}(\mathfrak{b}) \). So we can write \( \delta_{\mathfrak{b}}(D) \) as \( \pi^{-N} \circ F \) for some \( N \geq 0 \) and \( F \in \mathcal{D}(\mathfrak{b}) \).

Comparing this with (18) and noting a rational function on \( \mathfrak{g}_c \) is determined by its restriction on \( U \), we conclude that \( L = \delta_{\mathfrak{b}}(D) \) on \( \mathfrak{g}' \).

REFERENCES


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