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BERNSTEIN APPROXIMATION PROBLEM FOR DIFFERENTIABLE FUNCTIONS AND QUASI-ANALYTIC WEIGHTS

BY

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ABSTRACT. The Bernstein problem for differentiable functions is considered. Sufficient conditions in order that a decreasing family of weights be fundamental are given. Some of these conditions are also related to the concept of quasi-analytic weight.

1. **Introduction.** The Bernstein problem for continuous functions is equivalent to finding necessary and sufficient conditions for localizability (see Nachbin [3]). Also, the Bernstein problem has a solution only in a particular case (see Mergelyan [2]). On the other hand, many outstanding results dealing with sufficient conditions for localizability depend on statements about fundamental weights (see Nachbin [3] and Nachbin, Machado and Prolla [4] for results and other references). So it is interesting to give sufficient conditions for a weight to be fundamental. Among those sufficient conditions for localizability, the quasi-analytic criterion proved to be of wide use (see [3] and [4]).

In this paper we continue the research on Bernstein problem for differentiable functions. This was initiated in Zapata [6] and was partly announced in Zapata [7]. This problem has also a solution in a particular case (see Sibony [5]). Here we give only sufficient conditions for a family of weights to be fundamental, these conditions being related to the concept of quasi-analytic weight (namely Theorems 1 and 2). Also, the concept of quasi-analytic weight is behind the quasi-analytic criterion of localizability.

The contents of this paper are, for the most part, either results or generalization of results contained in the author's Ph. D. thesis. This was written under the direction of Professor Leopoldo Nachbin, to whom the author expresses his gratitude.

2. **Preliminaries.** Let $V=(V_\alpha)$, $\alpha\in\mathbf{N}^n_m$, be a family of directed sets of weights on $\mathbf{R}^n$, where $\mathbf{N}^n_m$ denotes the set of multi-indexes $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbf{N}^n$

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of order $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq m$ and $m \in \mathbf{N} \cup \{+\infty\}$, $n \in \mathbf{N}^*$. The vector space $\mathcal{E}^m V_\infty(\mathbf{R}^n; \mathbf{K})$ of all m-times continuously differentiable $\mathbf{K}$-valued functions f on $\mathbf{R}^n$ such that $v_\alpha \cdot \partial^\alpha f$ tends to zero at infinity for every $v_\alpha \in V_\alpha$ and $\alpha \in \mathbf{N}^n_m$ will be given the topology ω_V defined by the family of semi-norms

$$f \mapsto \sup\{v_\alpha(t)|\partial^\alpha f(t)|;\ t \in \mathbf{R}^n\}$$

for all such v_α and α, where $\mathbf{K}$ denotes either $\mathbf{R}$ or $\mathbf{C}$. In particular, when each V_α consists of a single weight v_α and we consider the family $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, we will write $\mathcal{E}^m v_\infty(\mathbf{R}^n; \mathbf{K})$ in place of $\mathcal{E}^m V_\infty(\mathbf{R}^n; \mathbf{K})$ and will denote the corresponding topology by ω_v.

The family $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, of weights on $\mathbf{R}^n$ is said to be *rapidly decreasing at infinity* if $\mathcal{E}^m v_\infty(\mathbf{R}^n; \mathbf{K})$ contains the algebra $\mathcal{P}(\mathbf{R}^n; \mathbf{K})$ of all $\mathbf{K}$-valued polynomials on $\mathbf{R}^n$ or equivalently, if each v_α is rapidly decreasing at infinity in the sense of Definition 1.24 of Nachbin [3]. The family $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, of weights on $\mathbf{R}^n$, rapidly decreasing at infinity, is called *fundamental* when $\mathcal{P}(\mathbf{R}^n; \mathbf{K})$ is dense in $\mathcal{E}^m v_\infty(\mathbf{R}^n; \mathbf{K})$. The *Bernstein problem* consists in asking for necessary and sufficient conditions for a given family v to be fundamental. The Weierstrass theorem on approximation of differentiable functions by polynomials means that every family $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, of weights on $\mathbf{R}^n$, each of them having compact support, is fundamental.

Remark 1. It can be shown that $\mathcal{E}^m v_\infty(\mathbf{R}^n; \mathbf{K})$ is always the closure of the set of analytic functions that it contains. Also, it can be shown that, if $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, is a fundamental family then each v_α is a fundamental weight in the classical sense, that is in the sense of Definition 1.24 (loc. cit.), the reciprocal being an open question.

A family $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, of weights on $\mathbf{R}^n$ will be called a *decreasing family* if for every $\alpha, \beta \in \mathbf{N}^n_m$ with $\beta \leq \alpha$, there exists a constant $C_{\alpha,\beta}$ such that $v_\alpha \leq C_{\alpha,\beta} \cdot v_\beta$.

In the following, we will write $\mathcal{E}^m v_\infty(\mathbf{R}^n)$ and $\mathcal{P}(\mathbf{R}^n)$ instead of $\mathcal{E}^m v_\infty(\mathbf{R}^n; \mathbf{K})$ and $\mathcal{P}(\mathbf{R}^n; \mathbf{K})$, respectively. Also, we will write $\mathcal{D}^m(\mathbf{R}^n; \mathbf{K})$ or $\mathcal{D}^m(\mathbf{R}^n)$ for the algebra of all m-times continuously differentiable $\mathbf{K}$-valued functions on $\mathbf{R}^n$ having compact support. We put $\mathcal{D}(\mathbf{R}^n) = \bigcap_{m \in \mathbf{N}} \mathcal{D}^m(\mathbf{R}^n)$.

3. Some basic lemmas.

Lemma 1. *Let $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, be a decreasing family of weights on $\mathbf{R}^n$. Then $\mathcal{D}^m(\mathbf{R}^n)$ is dense in $\mathcal{E}^m v_\infty(\mathbf{R}^n)$.*

Proof. Let $f \in \mathcal{E}^m v_\infty(\mathbf{R}^n)$, $m' \in \mathbf{N}$, $m' \leq m$, and $\epsilon > 0$ be given. Since the set $\{t|v_\alpha(t)|\partial^\alpha f(t)| \geq \epsilon/2\}$ is compact, for every α, and since $\mathbf{N}^n_{m'}$ is finite, there exists a compact K such that $t \notin K$ implies $v_\alpha(t)|\partial^\alpha f(t)| < \epsilon/2$ for all $\alpha \in \mathbf{N}^n_{m'}$.

Let $\lambda > 0$ be given. There exists $\theta_\lambda \in \mathfrak{D}^m(\mathbf{R}^n)$, real, such that $0 \le \theta_\lambda \le 1$, $\theta_\lambda \mid K = 1$ and $\sup\{|\partial^\beta \theta_\lambda(t)|;\ t \in \mathbf{R}^n\} \le \lambda$ for all $\beta \in \mathbf{N}^n_m$, such that $\beta \ne 0$. Let $\alpha \in \mathbf{N}^n_m$, and $t \in \mathbf{R}^n$ be given. If $\alpha = 0$ then $v_0(t)|f(t) - (\theta_\lambda \cdot f)(t)| = v_0(t)|f(t)|\,|1 - \theta_\lambda(t)|$. So

$$v_0(t)|f(t) - (\theta_\lambda \cdot f)(t)| < \epsilon/2. \tag{1}$$

If $\alpha \ne 0$, then

$$\begin{aligned} &v_\alpha(t)|\partial^\alpha f(t) - \partial^\alpha(\theta_\lambda \cdot f)(t)| \\ &\quad \le v_\alpha(t)|\partial^\alpha f(t)||1 - \theta_\lambda(t)| + \sum_{\beta \le \alpha, \beta \ne 0} \binom{\alpha}{\beta} |\partial^\beta \theta_\lambda(t)| v_\alpha(t) |\partial^{\alpha-\beta} f(t)|. \end{aligned}$$

Since $v_\alpha \le C_{\alpha,\alpha-\beta} \cdot v_\beta$ and $\beta \ne 0$ we have

$$\begin{aligned} &|\partial^\beta \theta_\lambda(t)| v_\alpha(t) |\partial^{\alpha-\beta} f(t)| \\ &\qquad \le \lambda C_{\alpha,\alpha-\beta} \sup\{v_{\alpha-\beta}(s)|\partial^{\alpha-\beta} f(s)|;\ s \in \mathbf{R}^n\}. \end{aligned}$$

So, we can take λ small enough such that

$$\sum_{\beta \le \alpha, \beta \ne 0} \binom{\alpha}{\beta} |\partial^\beta \theta_\lambda(t)| v_\alpha(t) |\partial^{\alpha-\beta} f(t)| < \frac{\epsilon}{2}.$$

Since $v_\alpha(t)|\partial^\alpha f(t)|\,|1 - \theta_\lambda(t)| < \epsilon/2$ we get

$$v_\alpha(t)|\partial^\alpha f(t) - \partial^\alpha(\theta_\lambda \cdot f)(t)| < \epsilon. \tag{2}$$

Letting $\phi = \theta_\lambda \cdot f$ we have $\phi \in \mathfrak{D}^m(\mathbf{R}^n)$ and from (1) and (2) we get

$$\sup\{v_\alpha(t)|\partial^\alpha f(t) - \partial^\alpha \phi(t)|;\ t \in \mathbf{R}^n\} < \epsilon \quad \text{for all } \alpha \in \mathbf{N}^n_m.$$

Lemma 2. *Let Γ be a nonempty set of fundamental weights on $\mathbf{R}$ in the classical sense of Bernstein. Assume that for every $u \in \Gamma$ there exists $u' \in \Gamma$ such that*

(1) $(1 + |t|) \cdot u(t) \le u'(t)$, *for all* $t \in \mathbf{R}$;

(2) $u'(t_2) \le u'(t_1)$ *for all* $t_1, t_2 \in \mathbf{R}$, $|t_1| \le |t_2|$.

Let $v = (v_i)$, $0 \le i \le m$, be a decreasing family of weights on $\mathbf{R}$ such that $v_0 \le u$ for some $u \in \Gamma$. Then v is fundamental.

Proof. It is enough to assume m finite and $\mathbf{K} = \mathbf{R}$. We will use induction on m. In fact, when $m = 0$ the conclusion is trivially verified. Letting m be an integer ≥ 1, assume the lemma to be true for $m - 1$. From the hypothesis on Γ there exists $u' \in \Gamma$ such that (1) and (2) are true. Let w be the family of weights given by

(3) $w_0(t) = u'(t)/(1 + |t|)$ $(t \in \mathbf{R})$, $w_i = u'$ for $i \ne 0$.

We claim that w is fundamental. In fact, let $f \in \mathfrak{E}^m w_\infty(\mathbf{R})$ and $\epsilon > 0$ be

given. If v' is the family of weights given by $v'_j = u'$ for all j, $0 \le j \le m-1$, then v' is fundamental by the induction hypothesis. Since $w_{j+1} = v'_j$ for $0 \le j \le m-1$ we have $f' \in \mathcal{E}^{m-1}v'_\infty(\mathbf{R})$ so there exists $P_0 \in \mathcal{P}(\mathbf{R})$ such that

(4) $\sup\{u'(s)|(f')^{(j)}(s) - P_0^{(j)}(s)|;\ s \in \mathbf{R}\} < \epsilon$ for $0 \le j \le m-1$.

Let $P \in \mathcal{P}(\mathbf{R})$ be such that

(5) $P' = P_0$, $P(0) = f(0)$.

Let $t \in \mathbf{R}$ be given. The mean value theorem and (5) give $|f(t) - P(t)| = |t|\,|f'(s) - P_0(s)|$ for some s such that $|s| \le |t|$. Then from (3) and (2), we get $w_0(t)|f(t) - P(t)| \le u'(s)|f'(s) - P_0(s)|$. Whence, from (4) it follows that

(6) $w_0(t)|f(t) - P(t)| < \epsilon$.

Also, from (5) and (4), it follows that

(7) $w_i(t)|f^{(i)}(t) - P^{(i)}(t)| < \epsilon$, $1 \le i < m$.

Since t is arbitrary, it follows from (6) and (7) that w is fundamental. Notice that, for every i, $0 \le i \le m$, we have $v_i \le (\text{constant}) \cdot w_i$ so the inclusion $\mathcal{E}^m w_\infty(\mathbf{R}) \subset \mathcal{E}^m v_\infty(\mathbf{R})$ is defined and continuous. From this it follows that $\mathcal{P}(\mathbf{R})$ is ω_v-dense in $\mathcal{D}^m(\mathbf{R})$. From Lemma 1, $\mathcal{D}^m(\mathbf{R})$ is dense in $\mathcal{E}^m v_\infty(\mathbf{R})$, hence $\mathcal{P}(\mathbf{R})$ is dense in $\mathcal{E}^m v_\infty(\mathbf{R})$.

If $g_1, \cdots, g_n$ are $\mathbf{K}$-valued functions defined on $\mathbf{R}$, we denote by $g_1 \times \cdots \times g_n$ the function

$$(t_1, \cdots, t_n) \in \mathbf{R}^n \to g_1(t_1) \cdot \ldots \cdot g_n(t_n).$$

Lemma 3. *Let Γ be as in Lemma 2. Let $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, be a decreasing family of weights on $\mathbf{R}^n$ such that there exist $u_1, \cdots, u_n \in \Gamma$ verifying $v_0 \le u_1 \times \cdots \times u_n$. Then v is fundamental.*

Proof. Let w be the family of weights on $\mathbf{R}^n$ given by $w_\alpha = u_1 \times \cdots \times u_n$, for all $\alpha \in \mathbf{N}^n_m$. It is clear that w is rapidly decreasing at infinity. We claim that $\mathcal{P}(\mathbf{R}^n)$ is ω_w-dense in $\mathcal{D}^m(\mathbf{R}^n)$. Moreover, w is fundamental by Lemma 1. For $j = 1, \cdots, n$ let w^j be the family of weights on $\mathbf{R}$ given by $w^j_i = u_j$ for all i, $0 \le i \le m$. Letting $f_j \in \mathcal{E}^m(w^j)_\infty(\mathbf{R})$, $j = 1, \cdots, n$, then $f_1 \times \cdots \times f_n$ is in $\mathcal{E}^m w_\infty(\mathbf{R}^n)$. In fact, for every $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{N}^n_m$ we have $w_\alpha \cdot \partial^\alpha(f_1 \times \cdots \times f_n) = (u_1 \cdot \partial^{\alpha_1} f_1) \times \cdots \times (u_n \cdot \partial^{\alpha_n} f_n)$. Hence $w_\alpha \cdot \partial^\alpha(f_1 \times \cdots \times f_n)$ tends to zero at infinity. Also, it follows that the mapping π from $\mathcal{E}^m(w^1)_\infty(\mathbf{R}) \times \cdots \times \mathcal{E}^m(w^n)_\infty(\mathbf{R})$ into $\mathcal{E}^m w_\infty(\mathbf{R}^n)$, defined by $\pi(f_1, \cdots, f_n) = f_1 \times \cdots \times f_n$, is n-linear and continuous. Since w^j $(j = 1, \cdots, n)$ is fundamental by Lemma 2, it follows that $\mathcal{D}(\mathbf{R}) \times \cdots \times \mathcal{D}(\mathbf{R})$ (n-times) is contained in the closure of $\mathcal{P}(\mathbf{R}) \times \cdots \times \mathcal{P}(\mathbf{R})$ in the product topology of $\mathcal{E}^m(w^1)_\infty(\mathbf{R}) \times \cdots \times \mathcal{E}^m(w^n)_\infty(\mathbf{R})$. By continuity of π it follows that $\pi[\mathcal{P}(\mathbf{R}) \times \cdots \times \mathcal{P}(\mathbf{R})]$ is ω_w-dense in $\pi[\mathcal{D}(\mathbf{R}) \times \cdots \times \mathcal{D}(\mathbf{R})]$. Let $\mathcal{D}(\mathbf{R}) \otimes \cdots \otimes \mathcal{D}(\mathbf{R})$ denote the vector space generated by $\pi[\mathcal{D}(\mathbf{R}) \times \cdots \times \mathcal{D}(\mathbf{R})]$.

Since $\pi[\mathcal{P}(\mathbf{R}) \times \cdots \times \mathcal{P}(\mathbf{R})] \subset \mathcal{P}(\mathbf{R}^n)$ and since the closure of $\mathcal{P}(\mathbf{R}^n)$ is also a vector subspace of $\mathcal{E}^m w_\infty(\mathbf{R}^n)$ we conclude that $\mathcal{P}(\mathbf{R}^n)$ is ω_w-dense in $\mathcal{D}(\mathbf{R}) \otimes \cdots \otimes \mathcal{D}(\mathbf{R})$. Let $\alpha \in \mathbf{N}_m^n$ be given. Since w_α is bounded, there exists a constant $C > 0$ such that

$$\sup\{w_\alpha(t)|\partial^\alpha f(t)|;\ t \in \mathbf{R}^n\} \leq C \sup\{|\partial^\alpha f(t)|;\ t \in \mathbf{R}^n\}$$

for every $f \in \mathcal{D}^m(\mathbf{R}^n)$ so that the topology induced by $\mathcal{E}^m w_\infty(\mathbf{R}^n)$ on $\mathcal{D}^m(\mathbf{R}^n)$ is coarser than the topology of uniform convergence of order m. From Proposition 1.4, §8 of Horvath [1], it follows that $\mathcal{D}(\mathbf{R}) \otimes \cdots \otimes \mathcal{D}(\mathbf{R})$ is dense in $\mathcal{D}^m(\mathbf{R}^n)$ in the inductive limit topology, which is finer than the topology of uniform convergence of order m. So it follows that $\mathcal{D}(\mathbf{R}) \otimes \cdots \otimes \mathcal{D}(\mathbf{R})$ is ω_w-dense in $\mathcal{D}^m(\mathbf{R}^n)$. Hence $\mathcal{P}(\mathbf{R}^n)$ is ω_w-dense in $\mathcal{D}^m(\mathbf{R}^n)$, and our claim is proved. To finish the proof, notice that for every $\alpha \in \mathbf{N}_m^n$ we have $v_\alpha \leq (\text{constant}) \cdot w_\alpha$. So the inclusion $\mathcal{E}^m w_\infty(\mathbf{R}^n) \subset \mathcal{E}^m v_\infty(\mathbf{R}^n)$ is defined and continuous. In particular, v is rapidly decreasing at infinity and $\mathcal{P}(\mathbf{R}^n)$ is ω_v-dense in $\mathcal{D}^m(\mathbf{R}^n)$. From Lemma 1 it follows that $\mathcal{D}^m(\mathbf{R}^n)$ is dense in $\mathcal{E}^m v_\infty(\mathbf{R}^n)$ and we conclude that v is fundamental.

4. Sufficient conditions for a family to be fundamental. Let u be a weight on $\mathbf{R}$. For $k = 0, 1, \cdots$, put

$$M(u)_k = \sup\{u(t)|t|^k;\ t \in \mathbf{R}\}.$$

Assume $M(u)_k$ to be finite for all k. If

$$\mu(u)_k = \inf\{(M(u)_j)^{1/j};\ j \geq k\}$$

then it is easy to prove the following:

$$\sum_{k\geq 1} [\mu(u)_k]^{-1} = +\infty \text{ if and only if } \sum_{k\geq 1} [M(u)_k]^{-1/k} = +\infty.$$

Because of this, a weight that verifies one of these conditions is called *quasi-analytic.*

Remark 2. Every sequence (M_k), $k \in \mathbf{N}$, of positive numbers that verifies the Denjoy-Carleman condition that is $\Sigma_{k\geq 1} (\mu_k)^{-1} = +\infty$, where $\mu_k = \inf\{(M_j)^{1/j};\ j \geq k\}$, gives rise to a quasi-analytic weight through

$$u(t) = \inf\{M_k/|t|^k;\ k = 0, 1, \cdots\}, \qquad t \in \mathbf{R},\ t \neq 0,$$

and $u(0) = 0$ in case some $M_k = 0$, $u(0) = M_0$ otherwise.

Let u be a weight on $\mathbf{R}^n$. Let us consider the following property:

(*) There exist quasi-analytic weights $u_1, \cdots, u_n$ on $\mathbf{R}$ such that $u \leq u_1 \times \cdots \times u_n$.

Remark 3. Let u be a weight on $\mathbf{R}^n$. Assume that u is rapidly decreasing at infinity, that is, for all $\gamma \in \mathbf{N}^n$ the function $t \longmapsto u(t)|t|^\gamma$ vanishes at infinity, where $|t|^\gamma = |t_1|^{\gamma_1} \cdots |t_n|^{\gamma_n}$ and $0^0 = 1$ by courtesy. Define

$$\tilde{u}(t) = \inf\{M(u)_\gamma/|t|^\gamma;\ \gamma \in \mathbf{N}^n\}$$

for all $t \in \mathbf{R}^n$, where $M(u)_\gamma = \sup\{u(s)|s|^\gamma;\ s \in \mathbf{R}^n\}$. Then u satisfies property (*) if and only if the restriction of $\tilde{u}$ to each coordinate axis is a quasi-analytic weight. It can be shown that this is no longer true if we just ask the latter condition on u instead of $\tilde{u}$.

Theorem 1. *Let $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, be a decreasing family of weights on $\mathbf{R}^n$ such that v_0 satisfies property (*). Then v is fundamental.*

Proof. Let u be a quasi-analytic weight on $\mathbf{R}$. From Lemma 2.29 of Nachbin [3] it follows that u is fundamental. Also, from the proof of that lemma it follows that the weight Cu^c is quasi-analytic for all C, $c \in \mathbf{R}$ such that $C \geq 0$, $c > 0$. Define $\tilde{u}$ as in Remark 3. Then $\tilde{u}$ is a quasi-analytic weight. Also $u \leq \tilde{u}$ and $\tilde{u}(t_2) \leq \tilde{u}(t_1)$ for all $t_1, t_2 \in \mathbf{R}$ such that $|t_1| \leq |t_2|$. Let $u' = C\tilde{u}^{1/2}$ where $C = \sup\{(1 + |s|)\tilde{u}^{1/2}(s);\ s \in \mathbf{R}\}$. Then u' is a quasi-analytic weight such that

(1) $(1 + |t|)u(t) \leq u'(t)$ for all $t \in \mathbf{R}$.

Also

(2) $u'(t_2) \leq u'(t_1)$ for all $t_1, t_2 \in \mathbf{R}$ such that $|t_1| \leq |t_2|$.

Hence the set of quasi-analytic weights on $\mathbf{R}$ verifies the hypotheses of the set Γ as in Lemma 2. From Lemma 3 it follows that v is fundamental.

Theorem 2. *Let $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, be a decreasing family of weights on $\mathbf{R}^n$ such that*

$$\sum_{k \geq 1} [\bar{M}(v_0)_k]^{-1/k} = +\infty$$

where

$$\bar{M}(v_0)_k = \sum_{\gamma \in \mathbf{N}^n, |\gamma| = k} \sup\{v_0(t)|t|^\gamma;\ t \in \mathbf{R}^n\}.$$

Then v is fundamental.

Proof. Let $\|(t_1, \cdots, t_n)\| = \max(|t_1|, \cdots, |t_n|)$ and let $N_k = \sup\{v_0(t)\|t\|^k;\ t \in \mathbf{R}^n\}$, $k = 0, 1, \cdots$. Then $N_k \leq \bar{M}(v_0)_k$ for all k. Define, for each $s \in \mathbf{R}$,

$$u(s) = \inf\{N_k/|s|^k;\ k = 0, 1, \cdots\}.$$

Then u is a quasi-analytic weight and $v_0(t) \leq u(\|t\|)$, $t \in \mathbf{R}^n$. This inequality proves that v is rapidly decreasing at infinity. Also, $u(s_2) \leq u(s_1)$ for all s_1, $s_2 \in \mathbf{R}$, such that $|s_1| \leq |s_2|$. Let $t = (t_1, \cdots, t_n) \in \mathbf{R}^n$ be given. Since $|t_j| \leq \|t\|$, $j = 1, \cdots, n$, it follows that $u(\|t\|) \leq u(t_j)$, for all j. So, $v_0 \leq u^{1/n} \times \cdots \times u^{1/n}$. Since u is a quasi-analytic weight, $u^{1/n}$ is too. Hence, v is fundamental by Theorem 1.

Corollary. *Let* $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, *be a decreasing family of weights on* $\mathbf{R}^n$. *Assume that there exist* $c > 0$ *and* $p \in \mathbf{N}$ *such that*

$$\sup\{v_0(t)|t|^\gamma;\ t \in \mathbf{R}^n\} \leq (c \log_0 |\gamma| \cdots \log_p |\gamma|)^{|\gamma|}$$

for all $\gamma \in \mathbf{N}^n$ *such that* $|\gamma|$ *is large enough, where as usual* $\log_0 k = k$ *and* $\log_p k = \log(\log_{p-1} k)$ *if* $p \geq 1$. *Then* v *is fundamental.*

Proof. In the notation of Theorem 2 we have, for $k \in \mathbf{N}$ large enough,

$$\overline{M}(v_0)_k \leq (k+1)^{n-1}(c \log_0 k \cdots \log_p k)^k,$$

hence the series $\Sigma_{k\geq 1} [\overline{M}(v_0)_k]^{-1/k}$ diverges. The conclusion follows from that theorem.

Remark 4. An important problem that remains unsolved is the following:

Let a family $v = (v_\alpha)$, $\alpha \in \mathbf{N}^n_m$, of weights on $\mathbf{R}^n$ be given such that each v_α satisfies property $(*)$. Is it then true that v is fundamental?

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