LATTICE POINTS AND LIE GROUPS. I

BY

ROBERT S. CAHN(1)

ABSTRACT. Assume that \( G \) is a compact semisimple Lie group and \( \mathfrak{g} \) its associated Lie algebra. It is shown that the number of irreducible representations of \( G \) of dimension less than or equal to \( n \) is asymptotic to \( kn^{a/b} \), where \( a = \) the rank of \( \mathfrak{g} \) and \( b = \) the number of positive roots of \( \mathfrak{g} \).

Let \( G \) be a simple, compact or complex, simply connected Lie group and \( \mathfrak{g} \) its associated Lie algebra. If \( G \) is compact a representation is a real analytic group homomorphism \( f: G \rightarrow GL(V) \) where \( V \) is a complex vector space. If \( G \) is complex a representation is a complex analytic group homomorphism \( f: G \rightarrow GL(V) \). In either case \( f \) will be called irreducible if \( V \) has no nontrivial invariant subspaces under the action of \( f(G) \). A homomorphism of Lie groups induces a homomorphism of the associated Lie algebras,

\[ f^*: \mathfrak{g} \rightarrow \mathfrak{gl}(V), \]

a Lie algebra representation, and \( f^* \) will be called irreducible if \( V \) has no nontrivial invariant subspaces under the action of \( f^*(\mathfrak{g}) \). It is seen from this definition that \( f \) is irreducible \( \iff \) \( f^* \) is irreducible. If \( G \) is simply connected a Lie algebra representation of \( \mathfrak{g} \) induces a group representation of \( G \) and we thus have a bijection between irreducible representations of \( G \) and \( \mathfrak{g} \). By the dimension of a representation we mean the dimension of \( V \). Identifying conjugate representations we ask, "How many irreducible representations of \( G \) (or equivalently \( \mathfrak{g} \)) are of dimension \( \leq T \)?" The question is simpler when asked of Lie algebras since the structure of the representations is less complex.

\( \mathfrak{g} \) is a complex simple Lie algebra if \( G \) is a complex simple Lie group or a compact real form of a complex simple Lie algebra when \( G \) is a compact simple Lie group. In the latter case there is a bijection between the complex representations of \( \mathfrak{g} \) defined over \( \mathbb{R} \) and the complex representations of its complexifications, \( \mathfrak{g} \otimes \mathbb{C} \), a complex simple Lie algebra so that we need only consider the case of \( \mathfrak{g} \) complex and simple.

The root space decomposition of a simple complex Lie algebra is well known

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and is found in [1] and [2]. We let $\mathfrak{g}$ be a Cartan subalgebra, $\mathfrak{g}^*$ its dual and $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}_\alpha$ be the canonical root space decomposition of $\mathfrak{g}$,

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, H \in \mathfrak{g} \}.$$ 

$R = \{ \alpha \in \mathfrak{g}^* \mid \mathfrak{g}_\alpha \neq 0 \}$ is called the set of roots. A subset of $R$, $\{ \alpha_1, \ldots, \alpha_n \}$, will be called simple if they are linearly independent, span $\mathfrak{g}^*$ and form an integer basis for $R$. The dimension of $\mathfrak{g} = a_{\mathfrak{g}}$ is the rank of $\mathfrak{g}$.

The Killing form is defined by $(X, Y) = \text{Tr}(\text{Ad}X \circ \text{Ad} Y)$. Restricted to $\mathfrak{g}$ it is symmetric and nondegenerate. $(, )$ induces a dual form on $\mathfrak{g}^*$ so we may speak of $(\alpha, \beta)$ when $\alpha$ and $\beta$ are roots. Further, there are unique vectors $H_\alpha$, $H_\beta \in \mathfrak{g}$ such that $(\alpha, \beta) = \alpha(H_\beta) = \beta(H_\alpha) = (H_\alpha, H_\beta)$.

If $f^*: \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation it has a weight space decomposition, $V = \bigoplus \lambda V_\lambda$, where

$$V_\lambda = \{ v \neq 0 \mid f^*(H)v = \lambda(H)v, \text{any } H \in \mathfrak{g} \}.$$ 

If $f^*$ is finite dimensional it is necessary that

$$\lambda(H_i) = \lambda(2H_i/(\alpha_i, \alpha_i)) = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbb{Z}$$

for any $\alpha_i$, $i = 1, \ldots, n$. If $f^*$ is irreducible there exists a weight $\lambda$, called the dominant weight, such that $\lambda \geq \lambda'$ for any other $\lambda'$ in $f^*$ and $\lambda(H_i) \in \mathbb{Z}^+$, $i = 1, \ldots, n$. Furthermore, if $f^{**}$ is another irreducible representation with $\lambda$ as dominant weight then $f^*$ is conjugate to $f^{**}$. Thus we may identify $f^*$ with its dominant weight and we will write $\pi_\lambda$ for $f^*$. The lattice of dominant weights is $\mathbb{Z}^+ \lambda_1 \oplus \cdots \oplus \mathbb{Z}^+ \lambda_n$ where $\lambda_i(H_i) = \delta_{ij}$. The interest of this is that the dimension of $\pi_\lambda$ is a polynomial in $\lambda$. By the Weyl character formula

$$f_{\mathfrak{g}^*}(\lambda) = \dim \pi_\lambda = \prod_{\alpha > 0} (\lambda + \delta, \alpha) / \prod_{\alpha > 0} (\delta, \alpha)$$

where $\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha \cdot \delta = \sum \lambda_i$ [1, p. 257], so if $\lambda$ belongs to the lattice of dominant weights then $\lambda + \delta$ belongs to the lattice of dominant weights. If we change coordinates to $\Lambda = \lambda + \delta = \sum \Lambda_i \lambda_i$ where $\Lambda_i \in \mathbb{R}$, then

$$\dim \pi_\Lambda = f_{\mathfrak{g}^*}(\Lambda) = \prod_{\alpha > 0} (\Lambda, \alpha) / \prod_{\alpha > 0} (\delta, \alpha).$$

The number of irreducible representations of $\mathfrak{g}$ of dimension $\leq n$ is then equal to the number of lattice points, $\Lambda$, such that $\Lambda_i > 0$ and $f_{\mathfrak{g}^*}(\Lambda) \leq n$. We now state

**Theorem.** Let $G$ be a simply connected, simple, complex or compact Lie group. The number of irreducible representations of $G$ of dimension $\leq n$ is asymptotic to $kn^{a_\mathfrak{g}/b_\mathfrak{g}}$, $b_\mathfrak{g}$ the number of positive roots of $\mathfrak{g}$.

**Proof.** We first note that $(\Lambda, \alpha)$ is a linear homogeneous polynomial in the coefficients of $\Lambda$ since
If \( e_1, \ldots, e_a \) is an orthonormal basis of \( \mathbb{Q}^* \) and if \( M: \lambda_i \rightarrow e_i \), then if \( M' \) is the transpose of \( M \) with respect to \((, )\)

\[(\Lambda, \alpha) = (M^{-1}M \Lambda, \alpha) = (M \Lambda, (M^{-1})^t \alpha)\]

and \( M \Lambda \) lies in the regular integer lattice in \( \mathbb{R}^a \). Thus if \( L = \sum_{i=1}^{a} X_i e_i, \ X_i > 0, \)

\[\chi(D = \sum \frac{(M^{-1})^t \alpha}{\chi(M \delta, M^{-1})^t \alpha})\]

then \( \chi_0(\sum_{i=1}^{a} X_i e_i) = \chi_0(\sum_{i=1}^{a} X_i \lambda_i) \) so we may regard \( \chi \) as having asymptotes \( e_i = 0 \) and the lattice of weights as the ordinary integer lattice. We now prove a lemma on homogeneous functions.

**Lemma 1.** Let \( f \) be a homogeneous function on \( \mathbb{R}^a \) of degree \( b \) which is the product of linear forms \( \sum m_i x_i, m_i \geq 0 \). If \( f = 0 \) on the planes \( x_i = 0, \ i = 1, \ldots, a, \) and if

\[S(1) = \{x \in \mathbb{R}^a | f(x) \leq 1, x_i \leq 0 \}\]

has finite volume then the number of lattice points in

\[S(r) = \{x \in \mathbb{R}^a | f(x) \leq r, x_i \leq 0 \}\]

is asymptotic to \( \text{Vol}(S(1)) r^{a/b} \).

**Proof.** It is clear that the volume of \( S(r) = \text{Vol}(S(1)) r^{a/b} \). If \( x \in S(r) \) then

\[f(x/(r^{1/b})) = (r^{-1/b}) b f(x) = r^{-1} f(x) \leq 1.\]

Since we are in \( \mathbb{R}^a \) the Jacobian of the coordinate change \( x \rightarrow ax \) is \( \alpha^a \) so \( \text{Vol}(S(r)) = r^{a/b} \text{Vol}(S(1)). \) We will be done if the number of lattice points in \( S(r) \sim \text{Vol}(S(r)) \). To see this, draw a unit \( a \)-cube at every lattice point of \( S(r) \), \( w \), with vertices at \( w, w + e_i \) any \( i \). Call the union of these cubes \( \mathcal{L}(r) \); a set which will contain \( S(r) \cap \{x_i \geq 1 \text{ all } i \} \) since \( f \) will be increasing in each coordinate. Now at each lattice point, \( w \), draw a unit cube with vertices \( w, w - e_i \) any \( i \). Call the union of these cubes \( \mathcal{L}(r) \). \( \mathcal{L}(r) \subset S(r) \) and \( \text{Vol} \mathcal{L}(r) = \text{Vol} \mathcal{L}(r) \). Call \( E(r) = S(r) \cap \{x_i \leq 1 \text{ some } i \} \). Then

\[\mathcal{L}(r) \subset S(r) \subset \mathcal{L}(r) \cup E(r)\]

which implies \( |\text{Vol} S(r) - \text{the number of lattice points}| \leq \text{Vol} E(r) \). However

\[\text{Vol} E(r) = r^{a/b} \text{Vol} \{x \in S(1) | x_i \leq r^{-1/b} \text{ some } i\} \]
and since $\text{Vol } S(1) < \infty$ the volume of this latter set $\to 0$ by dominated convergence. Thus $\text{Vol } E(r)$ is $o(\text{Vol } S(r))$ and the number of lattice points in $S(r)$ is asymptotic to $\text{Vol } S(r)$. □

We now have a criterion we would like to apply to the polynomials $f_A$. A canonical example is the algebra $A_2$. The positive roots of $A_2$ are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$ and the polynomial $f_{A_2}^0(x, y) = kxy(x + y)$. We wish to show

$$\text{Vol } \{x, y | x > 0, y > 0, kxy(x + y) \leq 1\} < \infty$$

or equivalently $\text{Vol } A < \infty$ where

$$A = \{x, y | x > 0, y > 0, xy(x + y) \leq 1\}.$$

We divide $A$ into two subsets, $A_x = A \cap \{x \geq y\}$, $A_y = A \cap \{x \leq y\}$. If $(x, y) \in A_x$, $xy(x + y) \leq 1$ which implies $x^2y \leq 1$.

$$A_x \subset \{(x, y) | x > y > 0, x^2y \leq 1\}.$$

Vol $A_x \cap \{x \in [0, 1]\} \leq \frac{1}{2}$ so Vol $A_x$ is finite if

$$\text{Vol } \{(x, y) | x > y, x > 1, x^2y \leq 1\} < \infty.$$

The volume of this set is $\int_1^\infty x^{-2} dx = 1$ so Vol $A_x \leq 3/2$. Similarly, Vol $A_y \leq 3/2$ so Vol $A < 3$ and the theorem is true for the algebra $A_2$. We now extend this method to higher dimensions.

**Lemma 2.** In $\mathbb{R}^a$ let $f(x)$ be a sum of monomials of degree $b$. If for every permutation $i$ of $\{1, \ldots, a\}$ there exists in $f(x)$ a monomial $X_{i(1)}^{s_1} \cdots X_{i(a)}^{s_a}$ where $s_1 > \cdots > s_a > 0$, then the volume of the set $S(1) = \{x | f(x) \leq 1, x_i \geq 0\}$ is finite.

**Remark.** From Lemma 1 this implies $\text{Vol } S(r) = \text{Vol } S(1) r^{a/b}$.

**Proof of Lemma 2.** We proceed by induction. If $a = 2$ we have monomials $X_1^{s_1} X_2^{s_2}$ and $X_1^{s_1'} X_2^{s_2'}$, $s_1 > s_2$, $s_1' > s_2'$. Again partitioning $S(1)$ into $A_x$ and $A_y$ we see

$$\text{Vol } A_x \leq \frac{1}{2} + \int_1^\infty x^{-s_1 \wedge s_2} \cdots \int_1^\infty x^{-s_1' \wedge s_2'} = \frac{1}{2} + (s_1 / s_2 - 1)^{-1} < \infty \text{ since } s_1 > s_2.$$  

Similarly $\text{Vol } A_y \leq \frac{1}{2} + (s_1' / s_2' - 1)^{-1}$.

Now assume the lemma true for $a - 1$. Partition $S(1)$ into the sets

$$A_{i_1, \ldots, i_a} = S(1) \cap \{x_{i_1} \geq \cdots \geq x_{i_a}\}.$$

We wish to show $\text{Vol } A_{i_1, \ldots, i_a} < \infty$ for any $i$. As before
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\[ A_{i_1}, \ldots, i_a \subseteq \{ x \mid x_{i_1} \geq \cdots \geq x_{i_a}, x_{i_1} x_{i_2} \cdots x_{i_a} \leq 1 \}. \]

If \( x_{i_1} \geq 1 \) a cross-section of this set at \( x_{i_1} \) is the set
\[ \{ (x_{i_2}, \ldots, x_{i_a}) \mid x_{i_2} \geq \cdots \geq x_{i_a} \geq 0, x_{i_2} \cdots x_{i_a} \leq 1/x_{i_1} \}. \]

By induction and the previous remark the volume of the cross-section = \( k x_{i_1}^{-\gamma} \)
where \( \gamma = s_1 (a - 1)/(\sum a_i - 1) \). The volume of
\[ A_{i_1}, \ldots, i_a \leq \text{Vol}(A_{i_1}, \ldots, i_a \cap \{ x_{i_1} \in [0, 1] \}) + \int_1^\infty y^{-\gamma} dy. \]

The first set is contained in the unit cube so it has volume \( \leq 1 \) and the integral is finite as long as \( \gamma > 1 \). But \( s_1 > s_i \forall i > 1 \) so \((a - 1)s_1 > \sum s_{j=2} s_j \Rightarrow \gamma > 1 \). \( \square \)

The proof of Theorem 1 will be complete if we show the criterion of Lemma 2 applies to the polynomials \( f_A \) for all simple complex Lie algebras.

If \( A = \sum a_i X_i \lambda_i \) then for each \( \alpha = \sum a_i m_i \alpha_i \)
\[ (A, \alpha) = \sum_{i=1}^a m_i (\lambda_i, \alpha_i) x_i. \]

Thus to determine \( f \) we must list all the positive roots of \( \mathfrak{g} \) in terms of the simple roots. We begin with the \( A_n \) algebras.

**Lemma 3.** The monomial \( X_1^{s(1)} \cdots X_n^{s(n)} \) is found in the expansion of \( f_{A_n} \)
for every permutation \( s \) of \( (1, \ldots, n) \).

**Proof.** By referring to Serre [2] the positive roots of \( A_n \) are \( \alpha_1, \ldots, \alpha_n; \alpha_1 + \alpha_2, \ldots, \alpha_{n-1} + \alpha_n; \ldots; \alpha_1 + \cdots + \alpha_n \). Since \((\lambda_1, \alpha_i) = \alpha_i, f_{A_n} =\kappa X_1 \cdots X_n (X_1 + X_2) \cdots (X_{n-1} + X_n) \cdots (X_1 + \cdots + X_n). \) We now apply induction. If \( n = 2, f_{A_2} = X_1^2 X_2 + X_1 X_2^2 \). Now assume the lemma for \( n - 1 \). We write
\[ f_{A_n} = X_n (X_n + X_{n-1}) \cdots (X_1 + \cdots + X_n). \]

Pick an arbitrary permutation \( s \). Then \( s(n) = j \). By induction \( X_1^{s(1)} \cdots X_{n-1}^{s(n-1)} \) occurs in \( f_{A_n} \) where
\[ s(i)' = \begin{cases} s(i) & \text{if } s(i) < j, \\ s(i) - 1 & \text{if } s(i) > j. \end{cases} \]

Multiply this monomial by \( X_n \) in the first \( j \) factors \( X_1, \ldots, (X_n + \cdots + X_{n+i-1}) \). Now pick the least \( i \) such that \( s(i)' < s(i) \). Multiply the monomial by \( X_i \) in \( (X_1 + \cdots + X_n) \). Then pick the next \( i' \) such that \( s(i') < s(i')' \) and multiply by \( X_{i'} \) in \( (X_2 + \cdots + X_n) \). Since \( i' > i \Rightarrow i' \geq 2, X_i \) is found in \( (X_2 + \cdots + X_n) \). We may thus continue until we have \( X_1^{s(1)} \cdots X_n^{s(n)} \). \( \square \)

**Remark.** The degree of \( f_{A_n} \) is minimal such that we may find monomials
Lemma 4. The monomial \( X_{s(1)}^{2s(1)-1} \cdots X_{s(n)}^{2s(n)-1} \) is found in the polynomials \( f_{B_n} \) and \( f_{C_n} \) for any permutation \( s \).

Proof. The positive roots of \( B_n \) are \( a_1, \ldots, a_n; a_1 + a_2, \ldots, a_{n-1} + a_n; \ldots; a_1 + \cdots + a_n \) and \( a_i + \cdots + a_{i-1} + 2a_j + \cdots + 2a_n \) where \( i < j \leq n \) [2]. \( f_{B_n} = k A_n \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (X_i + \cdots + X_{j-1} + 2X_j + \cdots + 2X_n) \). From Lemma 3 we know the monomial \( X_1^{s(1)} \cdots X_n^{s(n)} \) is in \( f_{A_n} \). We wish then to show that \( X_{s(1)}^{2s(1)-1} \cdots X_{s(n)}^{2s(n)-1} \) where \( s(j) = 1 \) lies in \( f_{A_n} \). We proceed as follows. There are \( n-1 \) factors containing \( X_1 \), \( s(1) - 1 \leq n - 1 \) so we may choose \( X_1 \) in \( s(1) - 1 \) of these factors. There are \( (n-1) + (n-2) \) factors containing \( X_2 \) and \( (s(1) - 1) + (s(2) - 1) \leq (n-1) + (n-2) \) so choose \( X_2 \) in the next \( s(2) - 1 \) factors. Thus we may proceed at each stage being able to choose \( s(i) - 1 \) for \( i < n \), \( i \leq j \leq n - 1 \). The roots are different from \( B_n \) but contain the same \( a_i \) so the argument is the same. \( \square \)

Lemma 5. \( f_{D_n} \) contains monomials of descending degrees for \( n \geq 6 \).

Proof. Referring to Serre the positive roots of \( D_n \) are \( a_1, \ldots, a_n; a_1 + a_2, \ldots, a_{n-1} + a_n; \ldots; a_1 + \cdots + a_n \) and \( a_i + \cdots + a_{i-1} + 2a_j + \cdots + 2a_n \). We may write

\[
 f_{D_n} = k A_n \prod_{i=1}^{n-3} \prod_{j=i+1}^{n-2} (X_i + \cdots + 2X_j + \cdots + 2X_{n-2} + X_{n-1}) \cdot \prod_{i=1}^{n-3} \prod_{j=i+1}^{n-2} (X_i + \cdots + 2X_j + \cdots + 2X_{n-2} + X_{n-1}) 
\]
The bracketed expression is what is needed along with $/A_{n-1}$ to create $/A_n$ except for the missing factor $(X_{n-1} + X_n)$. We compensate by adding the term $(X_{n-3} + 2X_{n-2} + X_{n-1} + X_n)$ to create a function containing every monomial of $/A_n$. The remaining terms we write as

$$
\prod_{i=1}^{n-2} (X_i + \cdots + X_{n-2} + X_n)
\prod_{i=1}^{n-4} \prod_{j=1+1}^{n-2} (X_i + \cdots + 2X_j + \cdots + 2X_{n-2} + X_{n-1} + X_n)
$$

We know from Lemma 3 that $X_{s(1)}^1 \cdots X_{s(n)}^n$ is found in $/A_n$ for any permutation $s$. We wish to produce a monomial with descending degrees in the $X_{s(i)}$ in $g_{D_n}$ for any permutation $s$. There are two cases. First assume that $s(1) \neq n - 1$. Then we will be done if the monomial

$$X_{s(n)}^{n-2} \cdots X_{s(6)}^4 X_{s(5)}^2 X_{s(4)} X_{s(3)} X_{s(2)}$$

is in $g_{D_n}$. First choose $n - 2$ different $X_i$ from

$$\prod_{i=1}^{n-2} (X_i + \cdots + X_{n-2} + X_n), \quad i \neq n - 1, s(1).$$

We then proceed to the second factor. There are $n - 3$ terms containing $X_1$ so if $s(j) = 1$ we may pick $X_1$ in $j - 3$ terms. Mimicking Lemma 4 we may continue by picking $j' - 3$ $X_{s(j)}$'s; where $s(j') = 2$ and so on to $X_{n-2}$. The sole difference in the procedure will be that if $j \in (1, 2, 3, 4)$ we choose no $X_{s(j)}$'s. After $X_{n-2}$ every term contains $X_{n-1}$ and $X_n$ so we may arbitrarily choose $k - 2 X_{n-1}$'s and $k' - 3 X_n$'s; $s(k) = n - 1, s(k') = n$. We have thus produced the desired monomial belonging to $g_{D_n}$ and multiplying by $X_{s(1)}^1 \cdots X_{s(n)}^n$ we have a monomial with strictly decreasing degrees.

If $n - 1 = s(1)$ we will be done if

$$X_{s(n)}^{n-3} X_{s(n-1)}^{n-3} \cdots X_{s(6)}^4 X_{s(5)}^2 X_{s(4)} X_{s(3)} X_{s(2)} X_{s(1)}$$

is in $g_{D_n}$. First pick $\{X_{s(n-1)}, \ldots, X_{s(2)}\}$ in $\prod_{i=1}^{n-2}(X_i + \cdots + X_{n-2} + X_n)$. Then proceed as before choosing $j - 3 X_1$'s, $j' - 3 X_2$'s and so on again skipping $X_{s(1)}, \ldots, X_{s(4)}$. Proceed to $X_{n-2}$ and then to $X_n$. There will be one remaining term which a priori contains $X_{n-1}$. Multiplying by $X_{n-1}$ from this factor we produce our monomial.

We have proved Theorem 1 for $A_n, B_n, C_n$ and $D_n$ for $n \geq 6$. These are all the complex simple Lie algebras except for the algebras $G_2, F_4, D_4, D_5, E_6, E_7$ and $E_8$. In these cases the conditions of Lemma 2 may be verified directly.

We now summarize the results:
We now extend our results to semisimple Lie algebras.

Corollary. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$, with $\mathfrak{g}_i$ the simple components. If $c_{\mathfrak{g}_1} = \cdots = c_{\mathfrak{g}_s} > c_{\mathfrak{g}_{s+1}} \geq \cdots \geq c_{\mathfrak{g}_n}$, then the number of irreducible representations of $\mathfrak{g}$ of dimension less than or equal to $T$ is asymptotic to $kT^c \Theta_1 \log^S T$.

Proof. We first assume that $\mathfrak{g}$ has two simple factors, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The irreducible representations of $\mathfrak{g}$ are tensor products of irreducible representations of the simple factors and the dimension of the tensor representation is a product of the dimensions of the factor representations. The number of irreducible representations of $\mathfrak{g}$ of dimension $\leq r$ is $b(r) = \sum_{m,n \in \mathbb{Z}^+}^{m \leq r} M_1(m)M_2(n)$, where $M_i(x)$ is the number of irreducible representations of $\mathfrak{g}_i$ of dimension $x$.

We partition $\mathfrak{g} = \{x, y | xy \leq r, x, y \geq 0\}$ into $S_x = S \cap \{x \in [0, r^{1/2}]\}$, $S_y = S \cap \{y \in [0, r^{1/2}]\}$. $S = S_x \cup S_y$ so if we estimate both $b_x(r) = \sum_{x \in \mathbb{Z}^+}^{x \leq r} M_1(x)M_2(n)$ and $b_y(r) = \sum_{y \in \mathbb{Z}^+}^{y \leq r} M_1(m)M_2(y)$ asymptotically, then $b(r) \sim \max(b_x(r), b_y(r))$.

Assume $c_{\mathfrak{g}_1} > c_{\mathfrak{g}_2}$ (for brevity); we will deal with $c_1 = c_2$ later. Theorem 1 states $\sum_{j=1}^{n} M_i(j) \sim \mu_i n c_i$. Thus

$$b_x(r) = \sum_{i=1}^{[r/2]} M_1(i) \sum_{j=1}^{[r/2]} M_2(j).$$

For any $\epsilon$ there exists $r_2$ such that

$$\left| \left( \sum_{j=1}^{L} M_2(j) - \mu_2 L^{c_2} \right) \left/ \sum_{j=1}^{L} M_2(j) \right| \right| < \epsilon \quad \text{any } L \geq r_2.$$
Then

\[ b_x(r) = \mu_2 r^2 \sum_{i=1}^{[r^2]} M_1(i)/i^2 + \epsilon' b_x(r) \]

where \(|\epsilon'| < \epsilon\) if \(r > r_2^2\). Thus

\[ b_x(r) \sim \mu_2 r^2 \sum_{i=1}^{[r^2]} M_1(i)/i^2. \]

By the Abel summation formula

\[
\sum_{i=1}^{[r^2]} M_1(i)/i^2 = \sum_{i=1}^{[r^2]-1} \left( \sum_{j=1}^{i} M_1(j) \right) \left( 1/i^2 - 1/(i+1)^2 \right) + \sum_{i=1}^{[r^2]} M_1(i) \cdot r^{-c_2/2}.
\]

Now \(c_2/i^{c_2+1} > 1/i^{c_2} - 1/(i+1)^2 > c_2/(i+1)^{c_2+1}\), so

\[
\sum_{i=1}^{[r^2]-1} \left( \sum_{j=1}^{i} M_1(j) \right) \left( 1/i^{c_2} - 1/(i+1)^{c_2} \right) > \sum_{i=1}^{[r^2]-1} \left( \sum_{j=1}^{i} M_1(j) \right) \cdot c_2/(i+1)^{c_2+1}.
\]

For any \(\epsilon > 0\) there exists \(r_1\) such that

\[
\left| \left( \sum_{j=1}^{L} M_1(j) - \mu_1 L^{c_1} \right) / \sum_{j=1}^{L} M_1(j) \right| < \epsilon \quad \text{any} \quad L \geq r_1.
\]

If \(r \gg r_1^2, r_2^2\)

\[
\sum_{i=1}^{[r^2]} \left( \sum_{j=1}^{i} M_1(j) \right) \cdot c_2/i^{c_2+1} = \sum_{i=1}^{[r^2]-1} \left( \sum_{j=1}^{i} M_1(j) \right) \cdot c_2/(i+1)^{c_2+1} + E + A
\]

where

\[
|E| < \epsilon \sum_{i=r_1}^{[r^2]-1} \left( \sum_{j=1}^{i} M_1(j) \right) \cdot c_2/i^{c_2+1}
\]

and

\[
A = \sum_{i=1}^{r_1} \left( \sum_{j=1}^{i} M_1(j) - \mu_1 i^{c_1} \right) \cdot c_2/i^{c_2+1}.
\]

\[
\sum_{i=1}^{[r^2]-1} \mu_1 c_2 i^{c_1-1} - c_2 - 1 \sim \mu_1 c_2 \int_{1}^{[r^2]} x^{c_1-1} c_2 - 1 \, dx
\]

\[
= \mu_1 c_2/(c_1 - c_2)x^{c_1-1} - c_2 \left[ \frac{x^{c_1-1}}{1} = k \left( c_1 - c_2 \right)^{1/2} + k'.
\]
Also
\[
\sum_{i=1}^{[\sqrt{2}]} M_1(i) \cdot r^{-c_2/2} = k_0 r^{(c_1-c_2)/2} + E'
\]
where \( |E'| < \epsilon \sum_{i=1}^{[\sqrt{2}]} M_1(i) \cdot r^{-c_2/2} \). Thus
\[
\sum_{i=1}^{[\sqrt{2}]} M_1(i)/i^{c_2} = (k + k_0) r^{(c_1-c_2)/2} + (k' + A) + (E + E').
\]
From this
\[(1 + 2e)b_x (r) > (k + k') r^{c_1+c_2/2} + (k' + A) r^{c_2} > (1 - 2e) b_x (r).
\]
Thus \( b_x (r) \sim c_1 r^{c_2} + c' r^{c_1+c_2} \). Similarly \( b_y (r) \sim \mu_1 r^{c_1} \sum_{i=1}^{[\sqrt{2}]} M_2(i)/i^{c_1} \). But
\[2=1 \sum_{i=1}^{[\sqrt{2}]} M_2(i)/i^{c_1} \]
is asymptotic to a constant. To see this
\[
\sum_{i=1}^{[\sqrt{2}]} M_2(i)/i^{c_1} = \sum_{i=1}^{[\sqrt{2}]} M_2(j) \left( 1/i^{c_1} - 1/(i + 1)^{c_1} \right) + \sum_{i=1}^{[\sqrt{2}]} M_2(j) r^{-c_1/2}.
\]
\[\Sigma_{i=1}^{[\sqrt{2}]} M_2(j) \]
is \( O(x^{c_2}) \) and \( (1/i^{c_1} - 1/(i + 1)^{c_1}) < c_1/i^{c_1+1} \), so
\[
\sum_{i=1}^{[\sqrt{2}]} M_2(i)/i^{c_2} \leq k \int_1^{[\sqrt{2}]} x^{c_2-c_1-1} dx + k_0 r^{c_2-c_1/2}
\]
\[= k/(c_1 - c_2) (1 - r^{2-c_1/2}) + k_0 r^{c_2-c_1/2}.
\]
But \( c_2 - c_1 < 0 \) so the above sum is \( \leq 2k/(c_1 - c_2) \) if \( r \) is sufficiently large
and \( \lim_{r \to \infty} \Sigma_{i=1}^{[\sqrt{2}]} M_2(i)/i^{c_1} \) exists and is equal to \( k' \). Thus \( b_y (r) \sim k' r^{c_1} \) and \( b(r) \sim b_y (r) \).

This settles the case of \( k = k_1 = k_2 \) where \( c_1 > c_2 \). By the above argument \( k = k_1 \oplus k_2 \) has asymptotically \( k' r^{c_1} \) irreducible representations of dimension \( \leq n \). By iteration \( (k_1 \oplus k_2) \oplus k_3 \) still has \( \sim k'' r^{c_1} \) irreducible representations and so on. This leaves the case of \( c_1 = \ldots = c_s \). Let \( k = k_1 \oplus k_2 \). Tracing the argument for \( c_1 \neq c_2 \) nothing is changed until we arrive at
\[
\int_1^{[\sqrt{2}]} x^{c_1-c_2-1} dx. \]
This integral now equals \( \int_1^{[\sqrt{2}]} x^{-1} dx = \frac{1}{2} \log r \) so that
\[b_x (r) \sim k r^{c_1} \log r \text{ and } b_y (r) \sim k' r^{c_1} \log r. \]
Now
\[
b(r) = b_x (r) + b_y (r) - \sum_{i,j \in S_x \cap S_y} M_1(i) M_2(j)
\]
and the latter sum equals
\[
\sum_{i=1}^{[\sqrt{2}]} M_1(i) M_2(j) = \sum_{i=1}^{[\sqrt{2}]} M_1(i) \cdot \sum_{j=1}^{[\sqrt{2}]} M_2(j)
\]
which is $O(r^{-1})$ so that $b(r) \sim kr^{-1} \log r$. Taking $\mathfrak{G} = (\mathfrak{G}_1 \oplus \mathfrak{G}_2) \oplus \mathfrak{G}_3$ we arrive at the integral

$$\int_{1}^{r^{1/2}} (\log x)/x \, dx = \frac{1}{8} \log^2 r.$$ 

So $b_x(r) \sim kr^{-1} \log^2 r$, $b_y(r) \sim k' r^{-1} \log^2 r$, $\sum_{i,j \in S \cap S_y} M_1(i) M_2(j)$ is $O(r^{-1} \log r)$ and $b(r) \sim k_0 r^{-1} \log^2 r$. Continuing to the case $\mathfrak{G} = \bigoplus_{i=1}^{s} \mathfrak{G}_i$ we have $b(r) \sim kr^{-1} \log^{s-1} r$ and our corollary is proven. □

BIBLIOGRAPHY


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FLORIDA 33124