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THE REPRESENTATION OF NORM-CONTINUOUS MULTIPLIERS ON L^∞-SPACES(1)

BY

GREGORY A. HIVELY

ABSTRACT. If G is a group and $\mathfrak{L}^\infty(G, \mathcal{S})$ is an appropriate space of bounded measurable functions on G, a representation is obtained for the algebra of norm-continuous multipliers on $\mathfrak{L}^\infty(G, \mathcal{S})$ as an algebra of bounded additive set functions on G. If G is a locally compact group, a representation of the norm-continuous multipliers on the quotient space $L^\infty(G)$ is obtained in terms of a quotient algebra of bounded additive set functions on G.

1. **Introduction.** If G is a group, X is a Banach space, and X is a left G-module [8, Definition 1.1 (b)], then a *multiplier* on X is a norm-continuous operator on X that commutes with the action of G on X. The algebra of multipliers for X will be denoted by $\mathrm{Hom}_G(X)$. In this paper we are concerned with the problem of representing the multipliers on various L^∞-spaces of bounded measurable functions (or equivalence classes of such functions) defined on a group G, where the action of G is given by left translation. Our main result (Theorem 5.3) concerns the Banach space $L^\infty(G)$ associated with a locally compact group G. In this case we show that $\mathrm{Hom}_G(L^\infty(G))$ is isometrically and algebraically isomorphic to a certain quotient algebra of bounded additive set functions on G. The main analytical tool required to establish this result is the lifting theorem of A. and C. Ionescu Tulcea [6, Theorem 5].

It should be noted that, in the case of a locally compact group, there are at least two alternative concepts of what a multiplier on $L^\infty(G)$ should be. With convolutions defined as in [5], we denote by $\mathrm{Hom}_L(L^\infty(G))$ (resp. $\mathrm{Hom}_M(L^\infty(G))$) the algebra of norm-continuous operators on $L^\infty(G)$ that commute with left convolution by $L^1(G)$ (resp. $M(G)$). Since $L^1(G)$ is an ideal in $M(G)$ and admits an approx-

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imate identity, it is easy to see that $\mathrm{Hom}_L(L^\infty(G)) = \mathrm{Hom}_M(L^\infty(G))$. Using the known representation of $\mathrm{Hom}_L(L^\infty(G))$ as the dual of the space $C_{ru}(G)$ of bounded right uniformly continuous functions on G [2, Theorem 3.3], one can show that $\mathrm{Hom}_L(L^\infty(G)) \subseteq \mathrm{Hom}_G(L^\infty(G))$. Rudin [11] has recently shown that this inclusion is generally proper.

2. **Multipliers on $\mathfrak{L}^\infty$-spaces.** Let Φ denote the real or complex field of scalars and let G be a group with unit e. If f is a function on G and $a \in G$ let ${}_af$ (resp. f_a) be the function on G defined by ${}_af(x) = f(a^{-1}x)$ (resp. $f_a(x) = f(xa^{-1})$) for each $x \in G$. We say that ${}_af$ (resp. f_a) is the *left* (resp. *right*) *translate* of f by a. Similarly, the function $\tilde{f}$ defined by $\tilde{f}(x) = f(x^{-1})$ for each $x \in G$ is said to be the *inversion* of f.

If $\mathcal{S}$ is a σ-ring on G we write $\mathfrak{L}^\infty(G, \mathcal{S})$ for the set of bounded, Φ-valued, $\mathcal{S}$-measurable functions defined on G. $\mathfrak{L}^\infty(G, \mathcal{S})$ is a Banach algebra under the supremum norm $\|\cdot\|$ and is a C^*-algebra if $\Phi = C$ and the involution is defined by complex conjugation. In order that $\mathfrak{L}^\infty(G, \mathcal{S})$ be left (or right) translation invariant and inversion invariant, it is necessary and sufficient that $\mathcal{S}$ be *G-invariant* in the sense that whenever $E \in \mathcal{S}$ and $a \in G$, then $aE \in \mathcal{S}$, $E^{-1} \in \mathcal{S}$, and (hence) $Ea \in \mathcal{S}$. For the remainder of this section, it will be assumed that $\mathcal{S}$ is a G-invariant σ-ring on G.

If G acts on $\mathfrak{L}^\infty(G, \mathcal{S})$ by left translation, then $\mathfrak{L}^\infty(G, \mathcal{S})$ is a left G-module and we write $\mathrm{Hom}_G(\mathfrak{L}^\infty(G, \mathcal{S}))$ for the algebra of multipliers on $\mathfrak{L}^\infty(G, \mathcal{S})$. A *multiplier* on $\mathfrak{L}^\infty(G, \mathcal{S})$ is thus a norm-continuous operator T on $\mathfrak{L}^\infty(G, \mathcal{S})$ such that $T({}_af) = {}_a(Tf)$ for each $f \in \mathfrak{L}^\infty(G, \mathcal{S})$ and $a \in G$. It is clear that $\mathrm{Hom}_G(\mathfrak{L}^\infty(G, \mathcal{S}))$ is a Banach algebra (with unit I) under the usual operator norm.

In the case of a locally compact group G, it is easy to see that the relation

$$T(f)(e) = \phi(f), \qquad f \in C_{ru}(G),$$

for $T \in \mathrm{Hom}_G(C_{ru}(G))$ $(= \mathrm{Hom}_L(C_{ru}(G)))$ and $\phi \in C_{ru}(G)^*$, determines an isometric isomorphism between $\mathrm{Hom}_G(C_{ru}(G))$ and $C_{ru}(G)^*$. This simple fact is at the core of [2, Theorem 3.3]. In the same manner, the following simple observation provides the key to the representation of $\mathrm{Hom}_G(\mathfrak{L}^\infty(G, \mathcal{S}))$. If T is an operator on $\mathfrak{L}^\infty(G, \mathcal{S})$, then the relation

$$l(f) = (Tf)(e), \qquad f \in \mathfrak{L}^\infty(G, \mathcal{S}),$$

defines an element $l \in \mathfrak{L}^\infty(G, \mathcal{S})^*$. Conversely, *if T is a multiplier on $\mathfrak{L}^\infty(G, \mathcal{S})$, then T is determined by l.* We shall begin, therefore, by recalling the canonical representation of $\mathfrak{L}^\infty(G, \mathcal{S})^*$.

Let $ba(G, \mathcal{S})$ denote the set of bounded, additive, Φ-valued set functions defined on $\mathcal{S}$. With the usual operations, $ba(G, \mathcal{S})$ is a Banach space under total variation norm. The following result is a special case of [3, p. 258].

Proposition 2.1. *The bilinear pairing of* $\mathfrak{L}^\infty(G, \mathcal{S})$ *and* $ba(G, \mathcal{S})$ *defined by*

$$(f, \phi) = \int f(x)\, d\phi(x), \qquad f \in \mathfrak{L}^\infty(G, \mathcal{S}),\ \phi \in ba(G, \mathcal{S}),$$

determines an isometric isomorphism between $ba(G, \mathcal{S})$ *and* $\mathfrak{L}^\infty(G, \mathcal{S})^*$.

Definition 2.2. *If* $\phi \in ba(G, \mathcal{S})$ and $f \in \mathfrak{L}^\infty(G, \mathcal{S})$, the function $\phi * f$ is defined by

$$\phi * f(x) = \int f(xt^{-1})\, d\phi(t), \qquad x \in G.$$

It is clear that $\phi * f$ is a bounded Φ-valued function on G that depends linearly on each factor separately. Moreover, for each $\phi \in ba(G, \mathcal{S})$ and $f \in \mathfrak{L}^\infty(G, \mathcal{S})$, we have

$$\phi * {}_af = {}_a(\phi * f), \qquad a \in G, \tag{2.1}$$

and, for each $\phi \in ba(G, \mathcal{S})$,

$$\|\phi\| = \sup_{\|f\| \le 1} \|\phi * f\|. \tag{2.2}$$

Definition 2.3. Let $mba(G, \mathcal{S})$ denote the closed subspace of $ba(G, \mathcal{S})$ consisting of those $\phi \in ba(G, \mathcal{S})$ for which $\phi * f \in \mathfrak{L}^\infty(G, \mathcal{S})$ for each $f \in \mathfrak{L}^\infty(G, \mathcal{S})$. The Banach space $mba(G, \mathcal{S})$ is nontrivial since, for each $a \in G$, it contains the set function $\delta_a \in ba(G, \mathcal{S})$ defined by

$$\delta_a(E) = \begin{cases} 1 & \text{if } a \in E, \\ 0 & \text{if } a \notin E, \end{cases} \qquad E \in \mathcal{S}.$$

The elements of $mba(G, \mathcal{S})$ are easily characterized.

Proposition 2.4. *If* $\phi \in ba(G, \mathcal{S})$, *the following conditions are equivalent:*

(i) $\phi \in mba(G, \mathcal{S})$;

(ii) *for each* $E \in \mathcal{S}$, *the function* $t \mapsto \phi(Et)$ *is in* $\mathfrak{L}^\infty(G, \mathcal{S})$;

(iii) *viewing* ϕ *as an element of* $\mathfrak{L}^\infty(G, \mathcal{S})^*$, *the function* $t \mapsto \phi(f_t)$ *is in* $\mathfrak{L}^\infty(G, \mathcal{S})$ *for each* $f \in \mathfrak{L}^\infty(G, \mathcal{S})$.

Proof. The equivalence of (i) and (iii) is immediate from the definitions and the implication (iii) $\Rightarrow$ (ii) follows at once by setting $f = \chi_E$. A simple computation shows that (ii) is equivalent to the condition that $\phi * \chi_E$ be in $\mathfrak{L}^\infty(G, \mathcal{S})$ for each $E \in \mathcal{S}$. Since each $f \in \mathfrak{L}^\infty(G, \mathcal{S})$ is the uniform limit of a sequence of $\mathcal{S}$-simple functions, it follows that (ii) $\Rightarrow$ (i). Q.E.D.

The equivalence of conditions (i) and (iii) in the above proposition suggests the possibility of defining a convolution on $mba(G, \mathcal{S})$ as in [5, (19.1)]. Rather than pursuing this, we prefer to take the following more direct approach based on the equivalence of (i) and (ii).

Definition 2.5. If $\phi \in ba(G, \mathcal{S})$ and $\psi \in mba(G, \mathcal{S})$, the set function $\phi * \psi \in ba(G, \mathcal{S})$ is defined by

$$(\phi * \psi)(E) = \int \psi(Et^{-1})\, d\phi(t), \qquad E \in \mathcal{S}.$$

It is clear that $\phi * \psi$ depends linearly on each factor separately.

Lemma 2.6. *If $\phi \in ba(G, \mathcal{S})$, $\psi \in mba(G, \mathcal{S})$, and $f \in \mathcal{L}^\infty(G, \mathcal{S})$, then $(\phi * \psi) * f = \phi * (\psi * f)$.*

Proof. By (2.2) it suffices to prove the assertion in the case where f is a characteristic function. But if $E \in \mathcal{S}$, then, for each $x \in G$, we have

$$\begin{aligned}(\phi * \psi) * \chi_E(x) &= \int \chi_E(xt^{-1})\, d(\phi * \psi)(t) = (\phi * \psi)(E^{-1}x)\\ &= \int \psi(E^{-1}xt^{-1})\, d\phi(t) = \int (\psi * \chi_E)(xt^{-1})\, d\phi(t)\\ &= \phi * (\psi * \chi_E)(x). \qquad \text{Q.E.D.}\end{aligned}$$

Theorem 2.7. *The Banach space $mba(G, \mathcal{S})$ is a Banach algebra under the $*$-operation and δ_e is a unit for this algebra.*

Proof. If $\phi_i \in mba(G, \mathcal{S})$, $i = 1, 2, 3$, Lemma 2.6 implies that $\phi_1 * \phi_2 \in mba(G, \mathcal{S})$, that $\phi_1 * (\phi_2 * \phi_3) = (\phi_1 * \phi_2) * \phi_3$, and, in conjunction with (2.2), that $\|\phi_1 * \phi_2\| \le \|\phi_1\| \|\phi_2\|$. Thus $mba(G, \mathcal{S})$ is a Banach algebra. The fact that δ_e is a unit for the algebra $mba(G, \mathcal{S})$ may be verified directly. Q.E.D.

Definition 2.8. If $\phi \in mba(G, \mathcal{S})$, the operator $\tilde{S}_\phi$ on $\mathcal{L}^\infty(G, \mathcal{S})$ is defined by

$$\tilde{S}_\phi f = \phi * f, \qquad f \in \mathcal{L}^\infty(G, \mathcal{S}).$$

It is clear from (2.1) and (2.2) that $\tilde{S}_\phi \in \mathrm{Hom}_G(\mathcal{L}^\infty(G, \mathcal{S}))$ and $\|\tilde{S}_\phi\| = \|\phi\|$.

Theorem 2.9. *Let the mapping $\tilde{\Psi}$ from $mba(G, \mathcal{S})$ into $\mathrm{Hom}_G(\mathcal{L}^\infty(G, \mathcal{S}))$ be defined by $\tilde{\Psi}(\phi) = \tilde{S}_\phi$. Then $\tilde{\Psi}$ is an isometric algebraic isomorphism of $mba(G, \mathcal{S})$ onto $\mathrm{Hom}_G(\mathcal{L}^\infty(G, \mathcal{S}))$.*

Proof. It is clear that $\tilde{\Psi}$ is linear, isometric, and $\tilde{\Psi}(\delta_e) = I$. Lemma 2.6 implies that $\tilde{\Psi}$ is an algebraic homomorphism. Thus, it remains only to show that $\tilde{\Psi}$ is onto.

Let $\tilde{S} \in \mathrm{Hom}_G(\mathcal{L}^\infty(G, \mathcal{S}))$. Define a linear functional $l \in \mathcal{L}^\infty(G, \mathcal{S})^*$ by setting $l(f) = \widetilde{Sf}(e)$ for each $f \in \mathcal{L}^\infty(G, \mathcal{S})$. By Proposition 2.1 there is a unique $\phi \in ba(G, \mathcal{S})$ such that

$$l(f) = \int f(t)\, d\phi(t), \qquad f \in \mathfrak{L}^\infty(G, \mathfrak{S}).$$

For each $f \in \mathfrak{L}^\infty(G, \mathfrak{S})$ we have $\tilde{S}f(e) = l(\tilde{f}) = \phi * f(e)$. By (2.1) it follows that $\tilde{S}f = \phi * f$. Thus $\phi \in mba(G, \mathfrak{S})$ and $\tilde{S} = \tilde{S}_\phi = \Psi(\phi)$. Q.E.D.

3. **Multipliers for pairs of $\mathfrak{L}^\infty$-spaces.** In this section we extend the results of §2 to the slightly more general situation of multipliers for a pair of $\mathfrak{L}^\infty$-spaces. Since the proofs are merely slight modifications of those already given, we shall content ourselves with a simple statement of definitions and results. Throughout this section it is assumed that G is a group and that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are G-invariant σ-rings on G.

A norm-continuous operator T from $\mathfrak{L}^\infty(G, \mathfrak{S}_1)$ into $\mathfrak{L}^\infty(G, \mathfrak{S}_2)$ is said to be a *multiplier* from $\mathfrak{L}^\infty(G, \mathfrak{S}_1)$ into $\mathfrak{L}^\infty(G, \mathfrak{S}_2)$ if $T({}_af) = {}_a(Tf)$ for each $f \in \mathfrak{L}^\infty(G, \mathfrak{S}_1)$ and $a \in G$. The set of such operators, denoted by $\mathrm{Hom}_G(\mathfrak{L}^\infty(G, \mathfrak{S}_1), \mathfrak{L}^\infty(G, \mathfrak{S}_2))$, is a Banach space under the usual operator norm.

Definition 3.1. Let $mba(G, \mathfrak{S}_1, \mathfrak{S}_2)$ denote the closed subspace of $ba(G, \mathfrak{S}_1)$ consisting of those $\phi \in ba(G, \mathfrak{S}_1)$ for which $\phi * f \in \mathfrak{L}^\infty(G, \mathfrak{S}_2)$ for each $f \in \mathfrak{L}^\infty(G, \mathfrak{S}_1)$.

Theorem 3.2. *If $\phi \in mba(G, \mathfrak{S}_1, \mathfrak{S}_2)$ and the operator T is defined by*

$$(*) \qquad Tf = \phi * f, \qquad f \in \mathfrak{L}^\infty(G, \mathfrak{S}_1),$$

then T is a multiplier from $\mathfrak{L}^\infty(G, \mathfrak{S}_1)$ into $\mathfrak{L}^\infty(G, \mathfrak{S}_2)$. Conversely, each multiplier from $\mathfrak{L}^\infty(G, \mathfrak{S}_1)$ into $\mathfrak{L}^\infty(G, \mathfrak{S}_2)$ is of the form (*) for some unique $\phi \in mba(G, \mathfrak{S}_1, \mathfrak{S}_2)$. The mapping thus defined between $mba(G, \mathfrak{S}_1, \mathfrak{S}_2)$ and $\mathrm{Hom}_G(\mathfrak{L}^\infty(G, \mathfrak{S}_1), \mathfrak{L}^\infty(G, \mathfrak{S}_2))$ is an isometric isomorphism.

4. **The lifting theorem.** From this point on, G will denote a locally compact group and the following notation will be employed: λ will denote a left Haar measure on G, $\mathfrak{S}_\lambda$ will denote the σ-algebra of λ-measurable subsets of G, and $\mathfrak{N}_\lambda$ will denote the hereditary σ-ring of locally λ-null sets in G. The σ-rings $\mathfrak{S}_\lambda$ and $\mathfrak{N}_\lambda$ are G-invariant. (With regard to their inversion invariance, see [5, (20.2)].) With these notations, the closed ideal in $\mathfrak{L}^\infty(G, \mathfrak{S}_\lambda)$ consisting of the functions in $\mathfrak{L}^\infty(G, \mathfrak{S}_\lambda)$ that are locally λ-null is precisely the space $\mathfrak{L}^\infty(G, \mathfrak{N}_\lambda)$. Let $L^\infty(G)$ be the quotient Banach algebra $\mathfrak{L}^\infty(G, \mathfrak{S}_\lambda)/\mathfrak{L}^\infty(G, \mathfrak{N}_\lambda)$ with quotient mapping π and quotient norm $\|\cdot\|_\infty$.

Definition 4.1. A mapping $\rho: \mathfrak{L}^\infty(G, \mathfrak{S}_\lambda) \to \mathfrak{L}^\infty(G, \mathfrak{S}_\lambda)$ is said to be a *lifting of $\mathfrak{L}^\infty(G, \mathfrak{S}_\lambda)$ commuting with left translation* if ρ satisfies the following conditions:

(i) ρ is linear,
(ii) $\rho(fg) = \rho(f)\rho(g)$,

(iii) $\rho(1) = 1$,

(iv) $\rho(\bar{f}) = \overline{\rho(f)}$ (void if $\Phi = R$),

(v) $\pi(\rho(f)) = \pi(f)$,

(vi) $\rho(f) = \rho(g)$ if $\pi(f) = \pi(g)$,

(vii) $\|\rho(f)\| = \|\pi(f)\|_\infty$,

(viii) $\rho({}_a f) = {}_a\rho(f)$, $a \in G$.

Note that if ρ is a lifting of $\mathcal{L}^\infty(G, \mathcal{S}_\lambda)$ commuting with left translation, then, in particular, ρ is a multiplier on $\mathcal{L}^\infty(G, \mathcal{S}_\lambda)$.

Theorem 4.2. *There exists a lifting of* $\mathcal{L}^\infty(G, \mathcal{S}_\lambda)$ *commuting with left translation.*

Proof. In the case $\Phi = R$, it is easily seen that Definition 4.1 is equivalent to the definition given in [6, p. 64]. Hence, in this case, the assertion is equivalent to [6, Theorem 5].

If $\Phi = C$, let ρ_r be a lifting of *real*-$\mathcal{L}^\infty(G, \mathcal{S}_\lambda)$ commuting with left translation. Define ρ on $\mathcal{L}^\infty(G, \mathcal{S}_\lambda)$ by

$$\rho(f) = \rho_r(\mathrm{Re}\, f) + i\rho_r(\mathrm{Im}\, f), \qquad f \in \mathcal{L}^\infty(G, \mathcal{S}_\lambda).$$

All the properties required of ρ are trivially satisfied except condition (vii), which may be established using the fact that $\mathcal{L}^\infty(G, \mathcal{S}_\lambda)$ and $L^\infty(G)$ are C^*-algebras. Q.E.D.

We remark that, in general, a lifting of $\mathcal{L}^\infty(G, \mathcal{S}_\lambda)$ commuting with left translation is not unique. This is the case even for $G = R$ [6, p. 90]. It follows that $\mathrm{Hom}_G(\mathcal{L}^\infty(G, \mathcal{S}_\lambda))$ may fail to be commutative even if G is Abelian. Indeed, if ρ_1 and ρ_2 are distinct liftings of $\mathcal{L}^\infty(G, \mathcal{S}_\lambda)$ commuting with left translation and $f \in \mathcal{L}^\infty(G, \mathcal{S}_\lambda)$ is such that $\rho_1(f) \neq \rho_2(f)$, then we see that $\rho_1(\rho_2(f)) = \rho_1(f) \neq \rho_2(f) = \rho_2(\rho_1(f))$.

5. **Multipliers on $L^\infty(G)$.** If $f \in \mathcal{L}^\infty(G, \mathcal{S}_\lambda)$ and $a \in G$ we define the *left translate* ${}_a\pi(f)$ of the coset $\pi(f)$ to be the coset $\pi({}_a f)$. This definition is permissible since $\mathcal{N}_\lambda$, and hence $\mathcal{L}^\infty(G, \mathcal{N}_\lambda)$, is left translation invariant. If G acts on $L^\infty(G)$ by left translation, then $L^\infty(G)$ is a left G-module and we write $\mathrm{Hom}_G(L^\infty(G))$ for the algebra of multipliers on $L^\infty(G)$. A *multiplier* on $L^\infty(G)$ is thus a norm-continuous operator T on $L^\infty(G)$ such that $T({}_a\pi(f)) = {}_a(T(\pi(f)))$ for each $f \in \mathcal{L}^\infty(G, \mathcal{S}_\lambda)$ and $a \in G$. It is clear that $\mathrm{Hom}_G(L^\infty(G))$ is a Banach algebra (with unit) under the usual operator norm.

Definition 5.1. Let A denote the subset of $mba(G, \mathcal{S}_\lambda)$ consisting of those $\phi \in mba(G, \mathcal{S}_\lambda)$ for which $\phi * f \in \mathcal{L}^\infty(G, \mathcal{N}_\lambda)$ for each $f \in \mathcal{L}^\infty(G, \mathcal{N}_\lambda)$. Let $A_0 = mba(G, \mathcal{S}_\lambda, \mathcal{N}_\lambda)$. The results of §2 imply that A is a closed subalgebra of

$mba(G, \mathcal{S}_\lambda)$ and that A_0 is a closed 2-sided ideal in A. Since the lifting of Theorem 4.2 is not generally unique, it follows that A_0 will not generally be trivial.

Definition 5.2. If $\phi \in A$, the operator S_ϕ on $L^\infty(G)$ is defined by

$$S_\phi(\pi(f)) = \pi(\phi * f), \qquad f \in \mathfrak{L}^\infty(G, \mathcal{S}_\lambda).$$

It is clear that

$$S_\phi \circ \pi = \pi \circ \tilde{S}_\phi, \tag{5.1}$$

so that $S_\phi \in \mathrm{Hom}_G(L^\infty(G))$, and

$$\|S_\phi\| \leq \|\tilde{S}_\phi\| = \|\phi\|. \tag{5.2}$$

Theorem 5.3. *Let the mapping* Ψ_0 *from* A *into* $\mathrm{Hom}_G(L^\infty(G))$ *be defined by* $\Psi_0(\phi) = S_\phi$. *Then* Ψ_0 *is a norm-decreasing algebra homomorphism of* A *onto* $\mathrm{Hom}_G(L^\infty(G))$ *with* $\ker \Psi_0 = A_0$. *If* Ψ *denotes the induced quotient mapping of* A/A_0 *onto* $\mathrm{Hom}_G(L^\infty(G))$, *then* Ψ *is an isometric algebraic isomorphism.*

Proof. It is clear that Ψ_0 is linear and (5.2) implies that Ψ_0 is norm-decreasing. The fact that $\ker \Psi_0 = A_0$ is immediate from the definitions involved. Using (5.1) and Theorem 2.9 we see that Ψ_0 is an algebra homomorphism.

To complete the proof it remains to be shown that Ψ_0 is onto and that Ψ is isometric. We shall establish these facts simultaneously using Theorem 4.2. Let ρ be a lifting of $\mathfrak{L}^\infty(G, \mathcal{S}_\lambda)$ commuting with left translation. Define $L: L^\infty(G) \to \mathfrak{L}^\infty(G, \mathcal{S}_\lambda)$ by setting $L(\pi(f)) = \rho(f)$ for each $f \in \mathfrak{L}^\infty(G, \mathcal{S}_\lambda)$. The properties of ρ imply that L is well defined, linear, isometric, and is an *intertwinement* for left translation on $L^\infty(G)$ and $\mathfrak{L}^\infty(G, \mathcal{S}_\lambda)$.

Let $S \in \mathrm{Hom}_G(L^\infty(G))$. Define $\tilde{S} \in \mathrm{Hom}_G(\mathfrak{L}^\infty(G, \mathcal{S}_\lambda))$ by setting $\tilde{S} = L \circ S \circ \pi$. Thus $\tilde{S}$ is a *lifting* of S and $S \circ \pi = \pi \circ \tilde{S}$. By Theorem 2.9, $\tilde{S} = \tilde{S}_\phi$ for some unique $\phi \in mba(G, \mathcal{S}_\lambda)$. Since $\tilde{S}f = 0$ for each $f \in \mathfrak{L}^\infty(G, \mathfrak{N}_\lambda)$, $\phi \in A$. For each $f \in \mathfrak{L}^\infty(G, \mathfrak{N}_\lambda)$, we have

$$S(\pi(f)) = \pi \circ \tilde{S}(f) = \pi \circ \tilde{S}_\phi(f) = S_\phi(\pi(f)).$$

Thus $S = S_\phi$ and Ψ_0 is onto. Finally, since

$$\|\phi\| = \|\tilde{S}_\phi\| \leq \|L\| \; \|S\| \; \|\pi\| = \|S_\phi\|$$

and Ψ_0 is norm-decreasing, it follows that $\|S_\phi\| = \|\phi\|$ and that Ψ is isometric. Q.E.D.

We remark that the proof just given shows slightly more than was stated. The proof shows that the coset norm in A/A_0 is always attained. As an application of Theorem 5.3 we have the following.

Theorem 5.4. *If G is Abelian and noncompact, then there exists a norm-continuous multiplier on $L^\infty(G)$ that is not w^*-continuous.*

Proof. Let $\mathcal{C}$ denote the space of constant functions in $\mathfrak{L}^\infty(G, \mathcal{S}_\lambda)$. Define the linear functional l_0 on the subspace $\mathfrak{L}^\infty(G, \mathfrak{N}_\lambda) + \mathcal{C}$ by setting $l_0(f + c1) = c$ for each $f \in \mathfrak{L}^\infty(G, \mathfrak{N}_\lambda)$ and $c \in \Phi$. Then l_0 is translation invariant, $\|l_0\| = 1$, and l_0 annihilates $\mathfrak{L}^\infty(G, \mathfrak{N}_\lambda)$. Since G is Abelian, l_0 has an extension to a linear functional l on $\mathfrak{L}^\infty(G, \mathcal{S}_\lambda)$ such that l is translation invariant and $\|l\| = 1$ [1]. By Proposition 2.1, l is represented by some unique $\phi \in ba(G, \mathcal{S}_\lambda)$. Clearly ϕ is translation invariant, $\|\phi\| = 1$, and $\phi * f = 0$ for each $f \in \mathfrak{L}^\infty(G, \mathfrak{N}_\lambda)$. Since ϕ is translation invariant, Proposition 2.4 implies that $\phi \in mba(G, \mathcal{S}_\lambda)$ (indeed, $\phi * f \in \mathcal{C}$ for each $f \in \mathfrak{L}^\infty(G, \mathcal{S}_\lambda)$) so that $\phi \in A$. Since $\phi * 1 = 1$, $\phi \notin A_0$. By Theorem 5.3, S_ϕ is a nonzero norm-continuous multiplier on $L^\infty(G)$. Let $C_c(G)$—the space of continuous functions on G having compact support—be regarded as a subspace of $L^\infty(G)$. The translation invariance of ϕ implies that S_ϕ is identically zero on $C_c(G)$. Since $S_\phi \neq 0$ and $C_c(G)$ is w^*-dense in $L^\infty(G)$, it follows that S_ϕ is not w^*-continuous. Q.E.D.

By an argument analogous to [5, (19.24)] it is easy to see that $\mathrm{Hom}_G(L^\infty(G))$ may fail to be commutative even if G is Abelian. Indeed, if G is Abelian and there exist distinct $\phi_1, \phi_2 \in A$ such that, for each $i = 1, 2$, ϕ_i is translation invariant and $\phi_i * 1 = 1$ (which is the case when $G = \mathbf{Z}$ or $G = \mathbf{R}$), then $S_{\phi_1}S_{\phi_2} \neq S_{\phi_2}S_{\phi_1}$. For since $\phi_1 \neq \phi_2$, there exists $f \in \mathfrak{L}^\infty(G, \mathcal{S}_\lambda)$ and distinct $c_1, c_2 \in \Phi$ such that $\phi_i * f = c_i 1$, $i = 1, 2$. Thus $S_{\phi_1}S_{\phi_2}(\pi(f)) = c_2\pi(1) \neq c_1\pi(1) = S_{\phi_2}S_{\phi_1}(\pi(f))$.

Before concluding this section we shall comment briefly on a paper of James D. Stafney [9] which is closely related to much that we have done and in particular to Theorem 5.3. (We wish to thank Professor Garth Gaudry and the referee for directing our attention to the results of [9].) Specifically, Stafney shows [9, Theorem 5.4] that if G is a locally compact Abelian group and if, moreover, G is assumed to have a certain *Property* A, then there is a topological algebraic isomorphism between $\mathrm{Hom}_G(L^\infty(G))$ and a certain quotient algebra $\mathfrak{B}/\mathfrak{Q}$ of bounded additive set functions on G. Loosely speaking, the space $\mathfrak{B}$ is the analogue within $L^\infty(G)^*$ of the space $mba(G, \mathcal{S}_\lambda)$. More precisely, the space $\mathfrak{B}$ (resp. $\mathfrak{Q}$) is that part of A (resp. A_0) which is contained in the subspace of $ba(G, \mathcal{S}_\lambda)$ corresponding to the dual of $L^\infty(G)$. Finally, it is easy to see that Stafney's Property A is equivalent to a slightly weakened form of the lifting property that we have employed. In particular, Property A always holds.

6. **The w^*-continuous multipliers on $L^\infty(G)$.** In this section we consider the multipliers on $L^\infty(G)$ that are w^*-continuous. It is a well-known consequence of

Wendel's theorem [10] (and [8, Theorem 3.18]) that each such operator is induced by convolution with a measure. In order to state this result in a suitably precise form we shall need some additional terms and facts.

Let $\mathcal{S}_b$ denote the σ-algebra of *Borel* subsets of G (the σ-ring generated by the topology of G). We write $M(G)$ for the Banach space of Φ-valued, regular, Borel measures on G under total variation norm. For the following result it need only be assumed that G is a locally compact Hausdorff space.

Proposition 6.1. *If $\mu \in M(G)$ and f is a bounded, Φ-valued, Borel measurable function on $G \times G$, then the formula*

$$h(x) = \int f(x, t)\, d\mu(t), \qquad x \in G,$$

defines (everywhere) a function $h \in \mathfrak{L}^\infty(G, \mathcal{S}_b)$.

Proof. It is enough to consider the case where $\Phi = \mathbf{R}$, μ is a positive (finite) regular Borel measure on G, and f is the characteristic function of a Borel set in $G \times G$.

Let $\mathfrak{M}$ be the class of all Borel sets $E \subseteq G \times G$ for which the formula

$$h(x) = \int \chi_E(x, t)\, d\mu(t), \qquad x \in G, \tag{6.1}$$

defines a function $h \in \mathfrak{L}^\infty(G, \mathcal{S}_b)$. Let $\mathfrak{L}$ be the lattice of open sets in $G \times G$. If $E \in \mathfrak{L}$, then (6.1) defines h as a bounded nonnegative function on G. The inner regularity of μ implies that h is lower semicontinuous and hence Borel measurable. Thus $\mathfrak{L} \subseteq \mathfrak{M}$.

Since $\mathfrak{L}$ is a lattice, the ring $\mathfrak{R}$ generated by $\mathfrak{L}$ consists of finite disjoint unions of proper differences of sets in $\mathfrak{L}$. Since $\mathfrak{M}$ is closed under the formation of disjoint unions and proper differences, it follows that $\mathfrak{R} \subseteq \mathfrak{M}$. Since μ is finite, the monotone convergence theorem implies that $\mathfrak{M}$ is a monotone class. By the monotone class theorem [4, p. 27], $\mathfrak{M}$ contains the σ-ring generated by $\mathfrak{R}$. Thus $\mathfrak{M}$ contains the Borel sets of $G \times G$. Q.E.D.

The proof just given is patterned after an argument of R. A. Johnson [7, p. 117 ff.].

Corollary 6.2. $M(G) \subseteq mba(G, \mathcal{S}_b)$.

We may now state, in precise form, the result previously alluded to.

Theorem 6.3. *The relation*

$$S(\pi(f)) = \pi(\mu * f), \qquad f \in \mathfrak{L}^\infty(G, \mathcal{S}_b),$$

for $S \in \mathrm{Hom}_G(L^\infty(G))$ *and* $\mu \in M(G)$, *determines an isometric isomorphism between the Banach space* $M(G)$ *and the space of* w^**-continuous multipliers on* $L^\infty(G)$.

This last result immediately suggests the following question. If S is a w^*-continuous multiplier on $L^\infty(G)$, $\phi + A_0$ is the coset of A/A_0 that represents S as in Theorem 5.3, and $\mu \in M(G)$ represents S as in Theorem 6.3, then what is the relation between the coset $\phi + A_0$ and the measure μ? Note that the domain of μ is $\mathcal{S}_b$ whereas the domain of each member of $\phi + A_0$ is $\mathcal{S}_\lambda$. We do not know an answer to this question but we venture to suggest the following possibility: μ has an *extension* to an element of the coset $\phi + A_0$. We conclude by showing that the problem of establishing such a relationship between $\phi + A_0$ and μ can be reduced to a multiplier extension problem.

Let ϕ_b denote the restriction of ϕ to $\mathcal{S}_b$. Since

$$\pi(\mu * f) = S(\pi(f)) = \pi(\phi_b * f), \qquad f \in \mathfrak{L}^\infty(G, \mathcal{S}_b),$$

it follows that $\mu - \phi_b \in mba(G, \mathcal{S}_b, \mathfrak{N}_\lambda)$. By Theorem 3.2, $\mu - \phi_b$ induces a multiplier T from $\mathfrak{L}^\infty(G, \mathcal{S}_b)$ into $\mathfrak{L}^\infty(G, \mathfrak{N}_\lambda)$. Since $\mathfrak{N}_\lambda$ is hereditary, it is reasonable to hope that T admits an extension to a multiplier from $\mathfrak{L}^\infty(G, \mathcal{S}_\lambda)$ into $\mathfrak{L}^\infty(G, \mathfrak{N}_\lambda)$. Assuming the existence of such an extension, Theorem 3.2 implies that it is induced by some unique $\psi \in A_0$. Since ψ must be an extension of $\mu - \phi_b$, it follows that μ has an extension to an element of the coset $\phi + A_0$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TEXAS 78712