TRIANGULAR REPRESENTATIONS OF SPLITTING RINGS

BY
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ABSTRACT. The term "splitting ring" refers to a nonsingular ring $R$ such that for any right $R$-module $M$, the singular submodule of $M$ is a direct summand of $M$. If $R$ has zero socle, then $R$ is shown to be isomorphic to a formal triangular matrix ring $(A, C, B_A)$, where $A$ is a semiprime ring, $C$ is a left and right artinian ring, and $CBA$ is a bimodule. Also, necessary and sufficient conditions are found for such a formal triangular matrix ring to be a splitting ring.

1. Introduction and notation. In [5, Theorem 10], we showed that any right nonsingular ring is an essential product of a ring with essential right socle and a ring with zero right socle. [An essential product of two rings is any subdirect product which contains an essential right ideal of the direct product.] Using this result, [5, Theorem 12] reduces the problem of characterizing splitting rings to characterizing those with either essential socle or zero socle. Since the case of essential socle has been taken care of by [4, Corollary 5.4], only the case of zero socle remains. The purpose of this paper is to study the structure of such a splitting ring with zero socle by representing the ring as a formal triangular matrix ring.

In this paper all rings are associative with identity, and all modules are unital. Unspecified modules are right modules; thus any statement about a bimodule $CBA$ refers to the module $B_A$ unless the module $CB$ is specifically mentioned. The reader is assumed to be familiar with the standard notions of singular and nonsingular modules; [3] or [4] may be consulted for details.

We give here for reference our notation, which coincides with that in [4]. For any ring $R$, we let $\mathcal{S}(R)$ denote the collection of essential right ideals of $R$. The singular submodule of a module $A$ is denoted $Z(A)$, and for a ring $R$, we use $Z_r(R)$ in place of $Z(R_R)$. A submodule $A$ of a module $B$ is said to be $\mathcal{S}$-closed in $B$ provided $B/A$ is nonsingular, and we let $L^*(B)$ denote the collection of $\mathcal{S}$-closed submodules of $B$. Given any submodule $A$ of $B$, there is a smallest $\mathcal{S}$-closed submodule $C$ of $B$ which contains $A$, and $C$ is called the $\mathcal{S}$-closure of $A$ in $B$. We note that the $\mathcal{S}$-closure of a two-sided ideal in a ring $R$ is again a two-sided ideal of $R$. For ease of expression, a two-sided ideal of $R$ which belongs to $L^*(R)$ is referred to as a "two-sided ideal in $L^*(R)$". Finally, we use $S^o$ (or $S^o_2$) $S$ is an exact functor from right $R$-modules to right modules over the ring $S^oR$ (which coincides with the maximal right quotient ring of $R$).

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2. **The representation of a splitting ring.** In this section, we represent a right nonsingular splitting ring $R$ (with zero socle) as a formal triangular matrix ring $(\mathcal{A}_\mathcal{C})$, where $\mathcal{A}$ is a semiprime ring and $\mathcal{C}$ is a left and right artinian ring. In order to accomplish this, some sort of chain condition on $R$ is needed. This is provided by Theorem 1, which shows that the prime radical of $R$ is finite dimensional. We conjecture that in fact $R$ itself must be finite dimensional.

**Theorem 1.** Let $R$ be a right nonsingular splitting ring such that $\text{soc}(R_R) = 0$. If $P$ denotes the prime radical of $R$, then $P_R$ is finite dimensional.

**Proof.** Our general procedure is to show first that all nilpotent ideals of $R$ are finite dimensional. This allows us to prove that $P$ has plenty of finite-dimensional submodules, which we use to show that $P$ itself is finite dimensional. Organizationally, the proof consists of a series of lemmas, some of which are proved in slightly more generality than needed here in order to be used in subsequent theorems. We stipulate that Lemmas C through J include the hypothesis that $R$ is a right nonsingular splitting ring with $\text{soc}(R_R) = 0$.

**Lemma A.** Let $Z_r(R) = 0$, and let $N$ be any nilpotent two-sided ideal of $R$. If $M$ is the $\mathcal{S}$-closure of $N_R$ in $R_R$, then $M$ is a two-sided ideal in $L^*(R)$ whose left annihilator belongs to $\mathcal{S}(R)$.

**Proof.** Letting $H$ denote the left annihilator of $M$, we infer from the nonsingularity of $R$ that $H$ is also the left annihilator of $N$. To get $H \in \mathcal{S}(R)$, it suffices to show that any element $x \in R \setminus H$ has a nonzero right multiple in $H$. Observing that $xN \neq 0$, we infer that there must be a positive integer $k$ for which $xN_k \neq 0$ and $xN_{k+1} = 0$. Choosing some $r \in N_k$ such that $xr \neq 0$, we see that $xr$ is a nonzero element of $H$.

**Lemma B.** Let $A$ be any nonsingular right $R$-module with zero socle. If $B$ is any submodule of $A$, then $B$ is the intersection of those essential submodules of $A$ which contain $B$.

**Proof.** Given $x \in A \setminus B$, let $K$ be maximal among those submodules of $A$ which contain $B$ but not $x$. We claim that $K$ is essential in $A$.

Letting $f: A \to A/K$ denote the natural map, we see from the maximality of $K$ that $fx$ is a nonzero element of $A/K$ which is contained in every nonzero submodule of $A/K$. Thus $(fx)_R$ is a simple, essential submodule of $A/K$. Inasmuch as $\text{soc}(A) = 0$, $(fx)_R$ cannot be projective. Recalling that all simple modules are either singular or projective [4, pp. 55, 56], we see that $(fx)_R$ must be singular. Since $(A/K)/(fx)_R$ is singular also, we infer that $A/K$ is singular. Inasmuch as $A$ is nonsingular, we conclude that $K$ is essential in $A$.

**Lemma C.** Let $M$ be a two-sided ideal in $L^*(R)$ whose left annihilator belongs to $\mathcal{S}(R)$. Then $M_R$ is a direct summand of $R_R$. 
Proof. If $H$ denotes the left annihilator of $M$, then $H$ is a two-sided ideal in $\mathcal{S}(R)$. Inasmuch as $R/M$ is a nonsingular right $R$-module, [4, Lemma 5.2] says that $(R/M)/(R/M)H$ is a projective right $(R/H)$-module. Then $R/(M + H)$ is projective as an $(R/H)$-module, whence $(M + H)/H$ is a direct summand of $R/H$. Thus there exists an element $m \in M$ such that $m^2 - m \in H$ and $mR + H = M + H$.

From the equation $m^2 - m \in H$, we obtain $m^3 - m \in H$. Since $m \in M$, this yields $m^4 - w^2 = 0$; hence the element $e = m^2$ is an idempotent. Observing that $eR + H = M + H$, we multiply this equation on the right by $M$ to obtain $eM = M^2$, whence $M^2 = eR$. Thus $M^2$ is a direct summand of $R_R$; hence it suffices to show that $M = M^2$.

If $M \neq M^2$, then according to Lemma B, $M_R$ must have a proper essential submodule $K$ which contains $M^2$. We infer that $M/K = Z(R/K)$, from which it follows that $M/K$ is a direct summand of $R/K$. Thus $R$ must have a right ideal $J$ such that $M + J = R$ and $M \cap J = K$. Observing that $M = M^2 + JM$, we obtain $M \leq J$, which leads to the contradiction $M = K$.

Lemma D. Let $e$ be an idempotent in the ring $Q = S^oR$ such that $R \cap eQ$ is a two-sided ideal of $R$.

(a) $Re$ is a unital subring of $eQe$.
(b) $Re$ is an essential right $(Re)$-submodule of $eQe$.
(c) $Z_r(Re) = 0$ and $S^o(Re) = eQe$.

Proof. (a) We must show that $Re$ is closed under multiplication, and that $e$ is an identity for $Re$. Given any $x \in R$, we have $x(R \cap eQ) \leq R \cap eQ \leq eQ$. Since $R \cap eQ$ is essential in $eQ$, it follows that $xeQ \leq eQ$. Therefore $Re \subseteq eQ$, from which the required results are immediate.

(b) If not, then $eQ$ contains a nonzero element $t$ such that $Re \cap tRe = 0$.

Since $tQ$ is nonzero, it cannot be a singular right $R$-module. Thinking of $t$ as an endomorphism of $Q_R$, it follows that $t$ cannot be essential in $Q_R$. Inasmuch as $R_R$ is essential in $Q_R$, we infer that $R \cap t^{-1}R$ is essential in $Q_R$, and thus that $(R \cap t^{-1}R \cap eQ) \otimes (1 - e)Q$ is essential in $Q_R$. Therefore $(R \cap t^{-1}R \cap eQ) \otimes (1 - e)Q$ cannot be contained in ker $t$. Noting that $(1 - e)Q \leq ker t$ already, we see that $R \cap t^{-1}R \cap eQ \leq R \cap ker t$. In view of Lemma B, it follows that $R \cap t^{-1}R$ must have an essential submodule $F$ which contains $R \cap ker t$ but not $R \cap t^{-1}R \cap eQ$.

Set $I = \{(x,tx)|x \in F\}$ and $J = \{(x,tx)|x \in R \cap t^{-1}R\}$, both of which are submodules of $R^2$. Since $F$ is essential in $R \cap t^{-1}R$, it follows that $I$ is essential in $J$, whence $J/I$ is singular. There exists a map $f: R^2 \rightarrow Q$ given by $f(x,y) = y - tx$, and we check that ker $f = J$. Thus $R^2/J$ is nonsingular; hence $J/I = Z(R^2/I)$. Therefore $R^2$ must contain a submodule $K$ for which $J + K = R^2$ and $J \cap K = I$.

There exists an element $x \in (R \cap t^{-1}R \cap eQ)/F$. Set $m_1 = x$ and $m_2 = tx.$
and note that \( m_1, m_2 \in R \cap eQ \). Also, let \( z_1 \) and \( z_2 \) denote the elements \((1,0)\) and \((0,1)\) in \( R^2 \).

Since \( R^2 = J + K \), we obtain \( z_i + (u_i, tu_i) \in K \) for some \( u_i \in R \cap t^{-1}R \). Note that \( z_i m_i + (u_i m_i, tu_i m_i) \in K \) also. Observing that \( tu_i m_i \in R \cap \ker t \leq F \), so that \( (u_i m_i, tu_i m_i) \in I \leq K \), from which we get \( z_i m_i \in K \).

Now \((x,tx) = z_1 m_1 + z_2 m_2 \in K \). Since \( x \in R \cap t^{-1}R \), we also have \((x,tx) \in J \), whence \( x \in F \), which is a contradiction.

(c) Inasmuch as \( Q \) is a regular, right self-injective ring, \( eQ \) must be a nonsingular injective right \( Q \)-module. According to [4, Proposition 1.17], \( eQe \) is thus a regular, right self-injective ring. In view of (a) and (b), [4, Proposition 1.16] now says that \( Z_r(Re) = 0 \) and \( S^0(Re) = eQe \).

**Lemma E.** Let \( e \) be an idempotent in \( R \) such that \( eR \) is a two-sided ideal. If \( R(1 - e) \subseteq S(R) \), then all nonsingular right \((eRe)\)-modules are projective.

**Proof.** Setting \( C = eRe \), we see from Lemma D that \( Z_r(C) = 0 \) and \( S^0C = eQe \), where \( Q = S^0R \). Lemma D also shows that \( C = Re \), from which we infer that \( r \mapsto re \) is a unital ring map of \( R \) onto \( C \). Therefore \( C \cong R/R(1-e) \).

Since \( R(1-e) \subseteq S(R) \), it follows as in [4, Theorem 5.3] that \( C \) is a right perfect ring. According to [1, Theorem P], this means that all flat right \( C \)-modules are projective; hence it suffices to show that all nonsingular right \( C \)-modules are flat. By [4, Proposition 2.1], this is equivalent to showing that \( (S^0C)_C \) is flat and that \( \text{GWD}(C) \leq 1 \). We shall prove this by showing that all right \( C \)-submodules of \( S^0C \) are flat.

Thus consider any \( E \subseteq (eQe)_C \). Noting that \( ER \) is a nonsingular right \( R \)-module and that \( H = R(1-e) \) is a two-sided ideal in \( S(R) \), we obtain from [4, Lemma 5.2] that \( ER/EH \) is a projective right \((R/H)\)-module. We have an abelian group epimorphism \( f: ER \to E \) given by \( fx = xe \), and it is easily checked that the kernel of \( f \) is \( EH \). Inasmuch as \( f(xr) = (fx)(re) \) for all \( x \in ER \) and \( r \in R \), we conclude that \( E \) must be a projective right \( C \)-module. Therefore \( E_C \) is certainly flat.

**Lemma F.** If \( N \) is any nilpotent two-sided ideal of \( R \), then \( N_R \) is finite dimensional.

**Proof.** Setting \( M \) equal to the \( S \)-closure of \( N_R \) in \( R_R \), we see from Lemma A that \( M \) is a two-sided ideal in \( L^0(S) \) whose left annihilator belongs to \( S(R) \). In view of Lemma C, there exists an idempotent \( e \in R \) such that \( eR = M \). Then Lemmas D and E say that \( Z_r(eRe) = 0 \) and that all nonsingular right \((eRe)\)-modules are projective. According to [4, Theorem 2.11], \( eRe \) must be finite dimensional as a right module over itself. Letting \( Q = S^0R \), we have \( S^0(eRe) = eQe \) by Lemma D; hence [4, Theorem 1.26] says that \( eQe \) is a semisimple ring.

There must exist orthogonal idempotents \( e_1, \ldots, e_n \in eQe \) such that \( e_1 + \cdots + e_n = e \) and each \( e_iQe_i \) is a division ring. Noting that \( Q \) is a semiprime ring (because it is regular), we see from [6, Proposition 2, p. 63] that the \( e_iQ \) are...
minimal right ideals of $Q$. Therefore $eQ$ is a finitely generated semisimple right $Q$-module. Observing that $S^c(eR) = eQ$, we infer from [4, Theorem 1.24] that $(eR)_R$ is finite dimensional, whence $N_R$ is finite dimensional.

**Lemma G.** Let $P$ denote the prime radical of $R$. Then any nonzero submodule of $P_R$ contains a nonzero finite-dimensional submodule.

**Proof.** Let $T$ denote the union of all nilpotent two-sided ideals of $R$, and note that $T$ is a two-sided ideal. Any nonzero submodule $A$ of $T_R$ must have nonzero intersection with some nilpotent two-sided ideal $N$, and $A \cap N$ is a finite dimensional module by Lemma F. Therefore every nonzero submodule of $T_R$ has a nonzero finite dimensional submodule. To prove that $P_R$ satisfies the same property, it suffices to show that $T_R$ is essential in $P_R$.

Let $H$ be maximal among those two-sided ideals of $R$ containing $T$ for which $T_R$ is essential in $H_R$. We claim that $P \leq H$.

Suppose not. Inasmuch as $P$ is contained in every semiprime ideal of $R$ [7, Theorem 4.20], $H$ cannot be a semiprime ideal. Thus there exists a two-sided ideal $K$, properly containing $H$, such that $K^2 \leq H$. Due to the maximality of $H$, $T_R$ is not essential in $K_R$, from which we infer that $H_R$ is not essential in $K_R$. Therefore there exists a nonzero element $x \in K$ which has no nonzero right multiples in $H$. Letting $J$ denote the left annihilator of $K$, we infer from the equation $K^2 \leq H$ that $x \in J \cap K$. But $J \cap K$ is nilpotent and hence contained in $T$, from which it follows that $x \in H$, which is impossible.

Therefore $P \leq H$; hence $T_R$ is essential in $P_R$.

**Lemma H.** The ring $Q = S^c R$ is a splitting ring.

**Proof.** We first show that $Q \otimes_R Q$ is a nonsingular right $R$-module. According to [2, Theorem 1.6], it suffices to show that for any $a \in Q$, the right ideal $I = \{x \in R | ax \in R\}$ has a finitely generated essential submodule. Using [4, Theorem 4.6 and Proposition 4.8], we see that $I$ is $AFG$, i.e., that $I/soc(I)$ is finitely generated. Inasmuch as $soc(R_R) = 0$, it follows that $I$ is finitely generated. Therefore $(Q \otimes_R Q)_R$ is nonsingular. According to [4, Lemma 1.25], it follows that the natural map $Q \otimes_R Q \to Q$ is an isomorphism.

We must show that any short exact sequence $E: 0 \to C \to B \to A \to 0$ splits, where $A$, $B$, and $C$ are right $Q$-modules such that $C$ is singular and $A$ is nonsingular. According to [4, Proposition 1.10], $C_R$ is singular and $A_R$ is nonsingular, whence $E$ splits as a sequence of $R$-modules. Thus we obtain a split exact sequence $E^*: 0 \to C \otimes_R Q \to B \otimes_R Q \to A \otimes_R Q \to 0$ of right $Q$-modules. Inasmuch as the natural map $Q \otimes_R Q \to Q$ is an isomorphism, we infer that $E^*$ is naturally isomorphic to $E$, hence $E$ must split.

**Lemma I.** If $Q = S^c R$, then $soc(Q_Q)$ is finitely generated.

**Proof.** According to [5, Theorem 10], $Q$ is an essential product of a ring with essential right socle and ring with zero right socle. Inasmuch as $Q$ is its own maximal right quotient ring, it follows from [5, Proposition 2] that $Q$ is actually
a direct product $H \times K$, where $H_H$ has essential socle and $K_K$ has zero socle. If $J$ denotes the socle of $Q_Q$, then we see that also $J = \text{soc}(H_H)$, and $J_H$ is essential in $H_H$.

It follows from Lemma H that $H$ is a splitting ring; hence [4, Corollary 5.4] says that $H/J$ is a semiprimary ring. Since $Q$ is a regular ring, we infer that $H/J$ is also a regular ring, whence $H/J$ is actually a semisimple ring.

Now suppose that $J_Q$ is not finitely generated. Then we can write $J = \bigoplus_{n=1}^{\infty} J_n$, where each $J_n$ is an infinite direct sum of simple modules. Observing that $H_H$ is injective (since $Q_Q$ is injective), we infer that for each positive integer $s$, there must exist an idempotent $e_s \in H$ such that $e_s H$ is an injective hull for $J_s$ and $(1 - e_s) H$ is an injective hull for $\bigoplus_{n \neq s} J_n$. Note that the idempotents $e_s$ are mutually orthogonal. Inasmuch as $e_s H$ contains the infinite direct sum $J_s$, it follows that $e_s H$ cannot be semisimple. Thus $e_s H \leq J$, i.e., $e_s \notin J$. Therefore the images of the elements $e_s$ in $H/J$ form an infinite orthogonal sequence of nonzero idempotents, which contradicts the fact that $H/J$ is a semisimple ring.

**Lemma J.** If $P$ denotes the prime radical of $R$, then $P_R$ is finite dimensional.

**Proof.** Set $Q = S^o R$ and $J = \text{soc}(Q_Q)$, and let $F$ denote the sum of all the finite-dimensional submodules of $P_R$. It follows from Lemma G that $F$ is essential in $P_R$, from which we infer that $S^o F = S^o P$. If $A$ is any finite-dimensional submodule of $P_R$, then [4, Theorem 1.24] says that $S^o A$ is a finitely generated semisimple right $Q$-module; hence $A \leq S^o A \leq J$. Thus $F \leq J$, and so $S^o P = S^o F \leq S^o J$.

According to Lemma I, $J_Q$ is finitely generated. Inasmuch as $Q$ is a regular ring, it follows that $J$ is a direct summand of $Q$, from which we infer that $S^o J = J$. Therefore $S^o P \leq J$, whence $S^o P$ is a finitely generated semisimple right $Q$-module. By [4, Theorem 1.24], $P_R$ must be finite dimensional.

**Theorem 2.** Let $R$ be a right nonsingular splitting ring with $\text{soc}(R_R) = 0$. Then $R$ is isomorphic to a formal triangular matrix ring $(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix})$, where $A$ is a semiprime ring, $C$ is a left and right artinian ring, and $C_B$ is faithful.

**Proof.** We first need a maximal element in the collection $\mathcal{A}$ of those two-sided ideals in $L^*(R)$ whose left annihilators belong to $\mathcal{S}(R)$. Let $P$ denote the prime radical of $R$, and set $\mathcal{B} = \{ A \cap P \mid A \in \mathcal{A} \}$. Inasmuch as $P_R$ is finite dimensional by Theorem 1, [4, Theorem 1.24] says that $L^*(P)$ has ACC. Thus $\mathcal{B}$ must have a maximal element, which is of the form $M \cap P$ for some $M \in \mathcal{A}$. We claim that $M$ is maximal in $\mathcal{A}$.

Consider any ideal $N \in \mathcal{A}$ which contains $M$. Letting $K$ denote the left annihilator of $N$, we see that $N \cap K$ is nilpotent and hence contained in $P$, whence $N \cap K \subseteq N \cap P$. The maximality of $M \cap P$ implies that $N \cap P = M \cap P$, from which we obtain $N \cap K \subseteq M$. Observing that $K \in \mathcal{S}(R)$, we see that $(N \cap K)_R$ is essential in $N_R$. Inasmuch as $M \in L^*(R)$, it follows that $N \subseteq M$. Therefore $M$ is maximal in $\mathcal{A}$.
We next claim that $R/M$ is a semiprime ring. If not, then it must have a nonzero nilpotent two-sided ideal $N/M$. Letting $T$ denote the $\mathcal{S}$-closure of $N_R$ in $R_R$, it follows just as in Lemma A that the ideal $H = \{ r \in R \mid rT \subseteq M \}$ must belong to $\mathcal{S}(R)$. Inasmuch as the left annihilator $K$ of $M$ also belongs to $\mathcal{S}(R)$, we see that $R/K$ and $K/KH$ are both singular right $R$-modules, whence $R/KH$ is singular and $KH \in \mathcal{S}(R)$. Noting that $KHT = 0$, we infer that the left annihilator of $T$ belongs to $\mathcal{S}(R)$. But then $T \in \mathcal{A}$, which contradicts the maximality of $M$.

According to Lemma C, there exists an idempotent $e \in R$ such that $eR = M$. Since $eR$ is a two-sided ideal, we obtain $(1 - e)Re = 0$. Therefore $R$ is isomorphic to the ring $(\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix})$, where $A = (1 - e)R(1 - e)$, $B = eR(1 - e)$, and $C = eRe$. Observing that $R/M \cong A$, we see that $A$ is a semiprime ring.

Since $M \in \mathcal{A}$, its left annihilator $R(1 - e)$ must belong to $\mathcal{S}(R)$. According to Lemmas D and E, $\mathcal{Z}(C) = 0$ and all nonsingular right $C$-modules are projective; hence [4, Theorem 2.12] shows that $C$ is left and right artinian. Any $x \in C$ satisfying $xB = 0$ must also satisfy $xR(1 - e) = 0$, and then $x = 0$ [because $R(1 - e) \in \mathcal{S}(R)$]. Therefore $cB$ is faithful.

3. Formal triangular matrix rings. The purpose of this section is to derive a few basic properties of a formal triangular matrix ring $(\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix})$. We are mainly interested in when such a ring can be nonsingular, and in finding the maximal quotient ring of such a ring.

Throughout this section, we assume that $A$ and $C$ are rings, that $CBA$ is a bimodule, and that $R$ is the ring $(\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix})$. In order to avoid some unnecessary complications, we also make the stipulation that $cB$ is faithful.

Proposition 3. (a) A right ideal $I$ of $R$ belongs to $\mathcal{S}(R)$ if and only if it contains a right ideal of the form $(\begin{smallmatrix} J & 0 \\ 0 & 0 \end{smallmatrix})$, where $J \in \mathcal{S}(A)$ and $K_A$ is essential in $B_A$.

(b) $RR$ is nonsingular if and only if $AA$ and $BA$ are both nonsingular.

(c) $\mathcal{S}(R_R) = \begin{pmatrix} \mathcal{S}(A_A) & 0 \\ \mathcal{S}(B_A) & 0 \end{pmatrix}$.

Proof. (a) If $I \in \mathcal{S}(R)$, then it is easily seen to contain such a right ideal. Conversely, if $I$ contains a right ideal $(\begin{smallmatrix} J & 0 \\ 0 & 0 \end{smallmatrix})$ of the form described, then we easily infer that $(\begin{smallmatrix} J & 0 \\ 0 & 0 \end{smallmatrix})$ is essential in $(\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix})$. Inasmuch as $cB$ is faithful, it follows that $(\begin{smallmatrix} J & 0 \\ 0 & 0 \end{smallmatrix}) \in \mathcal{S}(R)$, from which we infer that $(\begin{smallmatrix} J & 0 \\ 0 & 0 \end{smallmatrix}) \in \mathcal{S}(R)$, and then that $I \in \mathcal{S}(R)$.

(b) If $R_R$ is nonsingular, then it is immediate from (a) that $A_A$ and $B_A$ are nonsingular.

Conversely, assume that $A_A$ and $B_A$ are nonsingular, and consider any element $(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}) \in Z_3(R)$. In view of (a), we obtain $(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}) = 0$ for some $J \in \mathcal{S}(A)$ and some essential submodule $K$ of $B$. We have $aJ = 0$ and $bJ = 0$; hence $a = 0$ and $b = 0$. We also have $cK = 0$, whence $cB$ is an epimorphic image of the singular module $B/K$. It follows that $cB = 0$, and then the faithfulness of $cB$ implies that $c = 0$. Therefore $Z_3(R) = 0$. 

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(c) According to [4, Corollary 1.3], the socle of any module is the intersection of its essential submodules; hence (c) follows immediately from (a).

Let us now assume that $A$ and $B$ are nonsingular, so that $R$ is nonsingular. Set $T = \text{End}_A(S^\circ B)$ and $X = \text{Hom}_A(S^\circ B, S^\circ A)$. Since $CB$ is faithful, we may think of $C$ as a subring of the endomorphism ring of $B_A$. Then each element $c \in C$ induces a unique endomorphism $S^\circ c$ of $S^\circ B$; hence we obtain an embedding $c \mapsto S^\circ c$ of $C \hookrightarrow T$. For ease of notation, we may thus assume that $C$ is a unital subring of $T$ satisfying $CB \leq B$. We note that the faithfulness of $CB$ is now a consequence of the assumption that $C$ is a subring of $T$.

**Proposition 4.** $S^\circ R = (\frac{S^\circ X}{S^\circ T})$.

*Note. To multiply an element $b \in S^\circ B$ by an element $f \in X$, we just let $bf$ stand for the map $x \mapsto bf(x).$*

**Proof.** Set $Q = (\frac{S^\circ X}{S^\circ T})$. Recalling that the $S^\circ A$-homomorphisms from $S^\circ A$ to $S^\circ A$ or to $S^\circ B$ are the same as the $A$-homomorphisms, we see that $S^\circ A$ and $S^\circ B$ may be identified with $\text{End}_A(S^\circ A)$ and $\text{Hom}_A(S^\circ A, S^\circ B)$. With these identifications, $Q$ is naturally isomorphic to the ring $\text{End}_A(S^\circ A \otimes S^\circ B)$. Inasmuch as $S^\circ A \otimes S^\circ B$ is a nonsingular injective right $A$-module, it follows from [4, Proposition 1.17] that $Q$ is regular and right self-injective.

Inasmuch as $A$ is essential in $S^\circ A$ and $B$ is essential in $S^\circ B$, we see that $R$ is essential in the module $P_R = (S^\circ X, S^\circ T)$. Now $P$ is also a unital subring of $Q$, and we check that $P_P$ is essential in $Q_P$, from which it is easy to infer that $R_R$ is essential in $Q_R$. According to [4, Proposition 1.16], it follows that $S^\circ R = Q$.

4. Triangular splitting rings. This section is devoted to developing necessary and sufficient conditions for a formal triangular matrix ring (with zero socle) to be a splitting ring. In light of §3, we assume throughout this section that:

(a) $A$ is a right nonsingular ring.
(b) $B$ is a nonsingular right $A$-module.
(c) $C$ is a unital subring of $T = \text{End}_A(S^\circ B)$ such that $CB \leq B$.
(d) $R = (\frac{A}{B})$.

For convenience, we label the following three two-sided ideals of $R$:

$$R_{12} = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \quad \text{and} \quad R_{23} = \begin{pmatrix} 0 & 0 \\ B & C \end{pmatrix}.$$

Note that $R(R/R_{12})$ and $(R/R_{23})_R$ are projective, that $R_{12} \in \mathcal{S}(R)$, and that $R_{23} \in L^*(R)$.

**Theorem 5.** Assume that $\text{soc}(R) = 0$. If $R$ is a splitting ring, then

(a) $A$ is a splitting ring.
(b) $B_A$ is injective.
(c) $C_C$ is essential in $T_C$.
(d) All nonsingular right $C$-modules are projective.
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(For characterizations of rings satisfying (d), see [4, Theorems 2.11, 2.12, and 2.15].)

Proof. (a) Note that \( A \cong R/R_{23} \). Since \( R_{23} \) is a two-sided ideal in \( L^*(R) \), [4, Proposition 1.11] says that the singular submodule of any right \( (R/R_{23}) \)-module is the same whether considered as an \( (R/R_{23}) \)-module or as an \( R \)-module. Thus \( R/R_{23} \) must be a splitting ring.

(c)(d) The element \( e = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \) is an idempotent in \( R \) such that \( eR \) is a two-sided ideal and such that \( R(1 - e) \subseteq S(R) \). In view of Proposition 4, we see that \( eRe = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \) and \( e(S^0R)e = (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \). According to Lemmas D and E, it follows that \( C_C \) is essential in \( T_C \) and that all nonsingular right \( C \)-modules are projective. For use in the proof of (b), we note that Lemma D also says that \( Z_c(C) = 0 \) and that \( S^0C = T \).

(b) We proceed via several lemmas. With the exception of Lemma N, we stipulate that Lemmas K through O all include the hypothesis that \( R \) is a splitting ring with \( soc(R_A) = 0 \).

Lemma K. \( S^0B \) is a finitely generated semisimple right \( S^0A \)-module.

Proof. Since \( R_2 \) is nilpotent and therefore contained in the prime radical of \( R \), Theorem 1 says that \( R_2 \) is a finite-dimensional right \( R \)-module. Thus \( B_A \) must be finite dimensional. According to [4, Theorem 1.24], \( S^0B \) is finitely generated and semisimple.

Lemma L. If \( M \) is any simple \( S^0A \)-submodule of \( S^0B \), then there exists an idempotent \( e \in C \) such that \( e(S^0B) = M \) and \( eTe = eCe \).

Proof. In view of Lemma K, \( M \) must be a direct summand of \( S^0B \); hence \( M = f(S^0B) \) for some idempotent \( f \in T \). Inasmuch as the \( A \)-endomorphisms of \( M \) coincide with the \( S^0A \)-endomorphisms, we infer that \( fTf \) is isomorphic to the ring of \( S^0A \)-endomorphisms of \( M \), from which it follows that \( fTf \) is a division ring. Noting that \( T \) is regular and therefore semiprime, we obtain from [6, Proposition 2, p. 63] that \( fTf \) is a minimal right ideal of \( T \).

Observing that \( C/(C \cap fT) \) is a nonsingular right \( C \)-module, we see from (d) that \( C \cap fT = eC \) for some idempotent \( e \in C \). Since \( C_C \) is essential in \( T_C \) by (c), we obtain \( C \cap fT \neq 0 \), whence the minimality of \( fTf \) implies that \( eT = fTf \). Therefore \( e(S^0B) = f(S^0B) = M \). Since \( eT \) is a minimal right ideal of \( T \), it follows as in the proof of [4, Theorem 2.14] that \( eT \) must be a uniserial right \( C \)-module. In particular, \( eC \) must be a characteristic submodule of \( eT \), from which we infer that \( (eTe)(eC) \leq eC \); hence \( eTe = eCe \).

Lemma M. Let \( M \) be any simple \( S^0A \)-submodule of \( S^0B \). If \( I = \{a \in A \mid (M \cap B)a = 0\} \), then \( MI = 0 \).

Proof. Set \( Q = S^0R \) and \( H = (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}) \), and note that \( H \) is a two-sided ideal of \( R \). Since \( S^0I \) is injective, we must have \( S^0I = f(S^0A) \) for some idempotent \( f \in S^0A \). Setting \( g = (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \), we infer that \( H_g \) is essential in \( gQ \), from which it
follows that \( S^\circ H = gQ \). Thus \( R \cap gQ \) is the \( S \)-closure of \( H_R \) in \( R_R \), which is a two-sided ideal of \( R \).

According to Lemma D, \( Rg \) is a unital subring of \( gQg \) and \( Rg \) is an essential right \((Rg)\)-submodule of \( gQg \). Observing that

\[
Rg = \begin{pmatrix} Af & 0 \\ Bf & C \end{pmatrix} \quad \text{and} \quad gQg = \begin{pmatrix} f(S^\circ A)f & fX \\ (S^\circ B)f & T \end{pmatrix},
\]

we infer that \( Af \) is a unital subring of \( f(S^\circ A)f \) and that \( Bf \) is an essential right \((Af)\)-submodule of \( f(S^\circ B)f \).

According to Lemma L, there exists an idempotent \( e \in C \) such that \( e(S^\circ B) = M \). Since \( e \in C \), we have \( eB \leq B \), whence \( eB = M \cap B \). Thus \( eBI = 0 \); hence \( eB(S^\circ I) \) is a sum of epimorphic images of the singular module \( S^\circ I/I \). It follows that \( eB(S^\circ I) = 0 \), whence \( eBf = 0 \). Observing that \( eBf \) is an essential right \((Af)\)-submodule of \( e(S^\circ B)f \), we obtain \( e(S^\circ B)f = 0 \), from which we conclude that \( M I = 0 \).

**Lemma N.** [For this lemma, we only need the hypotheses that soc\((R_R) = 0 \) and that \( A \) is a splitting ring.] Let \( M \) be any simple \( S^\circ A \)-submodule of \( S^\circ B \). If \( I = \{ a \in A \mid Ma = 0 \} \), then \( Z_r(A/I) = M \) is a simple right \( S^\circ(A/I) \)-module, and \( S^\circ(A/I) \) is a simple artinian ring.

**Proof.** Inasmuch as \( M \) is nonsingular, we see that \( I \in L^*(A) \). According to [4, Proposition 1.11], it follows that \( Z_r(A/I) = M \) and that the singular submodule of any right \((A/I)\)-module is the same whether considered as an \((A/I)\)-module or as an \( A \)-module. Since \( A \) is a splitting ring, it follows that \( A/I \) is a splitting ring also. By Proposition 3, we have \( \text{soc}(A_A) = 0 \). Recalling that all simple nonsingular modules are projective [4, pp. 55, 56], we infer that the ring \( A/I \) must have zero right socle. Setting \( P = S^\circ(A/I) \), we thus obtain from Lemma I that \( \text{soc}(P_P) \) is finitely generated. Inasmuch as \( P \) is a regular ring, \( \text{soc}(P_P) \) must thus be a direct summand of \( P_P \).

Noting that \( M \) is finitely generated as an \( S^\circ A \)-module, we see from [4, Proposition 1.15] that \( M \) is a direct summand of \( S^\circ B \), from which it follows that \( M_A \) is injective. Since \( MI = 0 \), \( M \) is also an injective right \((A/I)\)-module. Now \( M \) is nonsingular as an \( A \)-module and thus as an \((A/I)\)-module; hence we obtain \( S^\circ_A/I M = M \). Therefore \( M \) is a right \( P \)-module.

Inasmuch as \( S^\circ_A M = M \), the simplicity of \( M \) implies that \( M \) is indecomposable as an \( A \)-module, from which we infer that \( M \) must also be indecomposable as a \( P \)-module. Noting from [4, Proposition 1.15] that all finitely generated \( P \)-submodules of \( M \) are direct summands of \( M \), we conclude that \( M \) is a simple \( P \)-module.

Observing that \( \{ x \in P \mid Mx = 0 \} \cap (A/I) = 0 \), we obtain \( \{ x \in P \mid Mx = 0 \} = 0 \). Therefore \( M \) is a faithful simple \( P \)-module; hence \( P \) is a primitive
ring. In particular, $P$ is a prime ring. The module $M_P$ is also nonsingular and simple, hence projective, from which we see that $P$ must contain a minimal right ideal. Thus $\text{soc}(P_P)$ is a nonzero two-sided ideal in the prime ring $P$; hence the left annihilator of $\text{soc}(P_P)$ must be zero. It follows that $\text{soc}(P_P)$ is an essential right ideal of $P$. Since $\text{soc}(P_P)$ is also a direct summand of $P$, we conclude that $\text{soc}(P_P) = P$, i.e., $P$ is a semisimple ring. Inasmuch as $P$ is prime, it must therefore be simple artinian.

**Lemma O.** $B_A$ is injective.

**Proof.** Suppose not. Since $S^0 B$ is an injective $A$-module, we obtain $B < S^0 B$. In view of Lemma K, $S^0 B$ must contain a simple $S^0 A$-submodule $M$ such that $M \leq B$, i.e., $M \cap B < M$. Setting $I = \{a \in A \mid (M \cap B)a = 0\}$ and $P = S^0(A/I)$, we see from Lemmas M and N that $MI = 0$, $Z_r(A/I) = 0$, $M$ is a simple right $P$-module, and $P$ is a simple artinian ring. Note that since $\{x \in P \mid (M \cap B)x = 0\} \cap (A/I) = 0$, we obtain $\{x \in P \mid (M \cap B)x = 0\} = 0$.

According to Lemma L, there exists an idempotent $e \in C$ such that $e(S^0 B) = M$ and $eTe = eCe$. Inasmuch as $CB \leq B$, we infer that $M \cap B$ is a left $(eTe)$-submodule of $M$.

The $P$-endomorphisms of $M$ coincide with the $(A/I)$-endomorphisms, and the $A$-endomorphisms of $M$ are just the left multiplications by the elements of $eTe$; hence we may identify $eTe$ with the endomorphism ring of $M_P$. Since $M_P$ is simple and $P$ is a simple artinian ring, we infer that $eTe$ is a division ring, that $M$ is a finite-dimensional left vector space over $eTe$, and that $P$ is the ring of all linear transformations on $M$. However, $M \cap B$ is a proper subspace of $M$, and no nonzero element of $P$ annihilates $M \cap B$, which is impossible.

**Theorem 6.** Assume that $\text{soc}(R_R) = 0$. If the following conditions are satisfied, then $R$ is a splitting ring:

(a) $A$ is a splitting ring.

(b) $B_A$ is injective.

(c) $C_C$ is essential in $T_C$.

(d) All nonsingular right $C$-modules are projective.

**Proof.** Once again we organize the proof as a series of lemmas. We stipulate that each of Lemmas P through U contains conditions (a)–(d) in its hypotheses.

**Lemma P.** Any direct sum of copies of $B_A$ is injective.

**Proof.** Since $T$ is the endomorphism ring of the nonsingular injective module $S^0 B$, [4, Proposition 1.17] says that $T$ is a regular, right self-injective ring. In light of condition (c), we see from [4, Proposition 1.16] that $Z_r(C) = 0$ and $S^0 C = T$. According to [4, Theorem 2.11], condition (d) implies that $C_C$ is finite dimensional, whence [4, Theorem 1.26] says that $T$ is a semisimple ring.
Now there exist orthogonal idempotents $e_1, \ldots, e_n \in T$ such that $e_1 + \cdots + e_n = 1$ and each $e_i Te_i$ is a division ring. Observing from condition (b) that $S^o B = B$, we see that $B = e_1 B \otimes \cdots \otimes e_n B$, and that each $e_i B$ is an indecomposable $A$-module, hence an indecomposable $S^o A$-module. According to [4, Proposition 1.15], every finitely generated $S^o A$-submodule of $B$ is a direct summand of $B$, from which we infer that the modules $e_i B$ are simple $S^o A$-modules.

To show that any direct sum of copies of $B_A$ is injective, it suffices to show that any direct sum of copies of $M_A$ is injective, where $M$ is any one of the modules $e_i B$. Setting $I = \{a \in A \mid Ma = 0\}$, we obtain from Lemma N that $Z_e(A/I) = 0$ and that $S^o (A/I)$ is a simple artinian ring. According to [4, Theorem 1.26], all direct sums of nonsingular injective $(A/I)$-modules are injective. Inasmuch as $M$ is a nonsingular injective $A$-module, it must also be a nonsingular injective $(A/I)$-module; hence any direct sum $\bigoplus M_i$ of copies of $M$ must be injective as an $(A/I)$-module. Noting that $\prod M_i$ is an $(A/I)$-module, we infer that $\bigoplus M_i$ is a direct summand of $\prod M_i$, from which it follows that $\bigoplus M_i$ is injective as an $A$-module.

**Lemma Q.** If $N$ is any nonsingular right $R$-module, then $NR_{23}$ is a direct summand of $N$.

**Proof.** The module $NR_2$ must be isomorphic to $F/K$ for some direct sum $F$ of copies of $R_2$ and some $K \in L^*(F)$. In view of Lemma P, we infer that $F$ is injective as a right $(R/R_{23})$-module. Inasmuch as $K \in L^*(F)$, it follows that $K$ must be a direct summand of $F$, and thus that $NR_2$ is injective as an $(R/R_{23})$-module. Noting that $NR_{12}$ is an $(R/R_{23})$-module which contains $NR_2$, we conclude that $NR_{23} = NR_2 \otimes W$ for some $W$.

Since $R_2$ is essential in $R_{23}$, it follows that $R_{23}/R_2$ is a singular right $R$-module. Noting that $NR_{23}/NR_2$ is a sum of epimorphic images of $R_{23}/R_2$, we see that $NR_{23}/NR_2$ is singular. Inasmuch as $NR_2$ is nonsingular, it follows that $NR_2$ is essential in $NR_{23}$. We now take the equation $NR_2 \cap W = 0$ and infer from this that $NR_{23} \cap W = 0$. Checking that $N = NR_{23} + NR_{12} = NR_{23} + W$, we conclude that $N = NR_{23} \otimes W$.

**Lemma R.** If $N$ is any nonsingular right $R$-module, then $N/NR_{12}$ is a projective right $(R/R_{12})$-module.

**Proof.** In view of condition (d), it suffices to show that $N/NR_{12}$ is nonsingular as an $(R/R_{12})$-module. Since $N_R$ is nonsingular, there exists a monomorphism $N \to \prod Q_i$, where each $Q_i$ is a copy of $Q = S^o R$. Inasmuch as $R(R/R_{12})$ is finitely generated and projective, we obtain another monomorphism

$N \otimes_R (R/R_{12}) \to (\prod Q_i) \otimes_R (R/R_{12}) \to \prod [Q \otimes_R (R/R_{12})]$. 
Thus \( N/NR_{12} \) is embedded in a direct product of copies of \( Q/QR_{12} \); hence it suffices to show that \( Q/QR_{12} \) is nonsingular as an \((R/R_{12})\)-module. Using Proposition 4 to check that

\[
\begin{pmatrix}
S^oA & X \\
S^oB & T
\end{pmatrix}
\]

we infer that it suffices to prove that \( X_C \) and \( T_C \) are nonsingular.

As in Lemma P, we have \( Z_r(C) = 0 \) and \( S^oC = T \), whence \( T_C \) is nonsingular. Now consider any \( \text{element} \ f \in Z(X_C) \). Since \( f \) maps \( S^oB \) into the nonsingular module \( S^oA \), we have \( \ker f \in L^*(S^oB) \). Inasmuch as \( S^oB \) is injective, it follows that \( \ker f = e(S^oB) \) for some indempotent \( e \in T \). We now infer that \( fT \cong (1 - e)T \), from which it follows that \( (fT)_c \) is nonsingular, and thus \( f = 0 \). Therefore \( Z(X_C) = 0 \).

**Lemma S.** Let \( n \) be any positive integer, and let \( K \in L^*(B^n) \). If \( J = \{ x \in C^n \mid xB \leq K \} \), then \( JB = K \).

**Proof.** As in Lemma P, we have \( Z_r(C) = 0 \) and \( S^oC = T \). In light of condition (d), we see from [4, Theorem 2.5] that \( C \) is right semihereditary and that \( Z[(T \otimes C T)_c] = 0 \). Then [4, Lemma 2.2] says that \( cT \) is flat, while [4, Lemma 1.25] shows that the natural map \( T \otimes C T \rightarrow T \) is an isomorphism.

Inasmuch as \( B_4 \) is injective, we have \( B = S^oB \); hence \( B \) is a left \( T \)-module. Then \( T_B \) is flat because \( T \) is a regular ring, and we infer from the flatness of \( cT \) that \( cB \) must be flat.

Setting \( L = \{ x \in T^n \mid xB \leq K \} \), we note that \( L \) is a right \( T \)-submodule of \( T^n \). We have a monomorphism \( C^n/J \rightarrow T^n/L \), from which we obtain another monomorphism \( (C^n/J) \otimes C B \rightarrow (T^n/L) \otimes C B \). Now \( (C^n/J) \otimes C B \) is naturally isomorphic to \( B^n/JB \), and we also have natural isomorphisms

\[
(T^n/L) \otimes C T \rightarrow (T^n/L) \otimes T \rightarrow (T^n/L) \otimes C B \rightarrow (T^n/L) \otimes T \rightarrow B^n/LB;
\]

hence we conclude that the natural map \( B^n/JB \rightarrow B^n/LB \) is injective. Therefore \( JB = LB \).

Inasmuch as \( B^n \) is injective and \( K \in L^*(B^n) \), \( K \) must be a direct summand of \( B^n \). Thus there exists an idempotent \( n \times n \) matrix \( p \) over \( T \) such that \( pB^n = K \). Given any \( x \in K \), we can obtain \( x = u_1 b_1 + \cdots + u_n b_n \) for appropriate choices of \( u_i \in T^n \) and \( b_i \in B \). Since each \( pu_i B \leq pB^n = K \), we see that each \( pu_i \in L \). Observing that \( x = px \), it follows that \( x \in LB \). Therefore \( K = LB = JB \).

**Lemma T.** If \( N \) is any nonsingular right \( R \)-module, then \( \text{Tor}^R_k(N, R/R_{23}) = 0 \).

**Proof.** We may assume, without loss of generality, that \( N \) is finitely generated, and we shall prove that the map \( N \otimes_R R_{23} \rightarrow N \) is injective. Inasmuch as \( NR_{23} \) is a direct summand of \( N \) by Lemma Q, the map \( f: NR_{23} \otimes_R R_{23} \rightarrow N \otimes_R R_{23} \) is
injective. Noting that $R_{23}$ is idempotent, we see that $f$ is also surjective and hence an isomorphism. Thus it suffices to prove that $NR_{23} \otimes_R R_{23} \rightarrow NR_{23}$ is injective.

Since $N$ is finitely generated, we obtain $NR_{23} \cong R_{23}^n/H$ for some positive integer $n$ and some $H \in \text{L}^*(R_{23})$. We check that $H = (0,0)$ for some $K \in \text{L}^*(B^n)$ and some $J \leq C^n$. Since $H$ is a submodule of $R_{23}$, we must have $JB \leq K$. Given any $x \in C^n$ for which $xB \leq K$, we see that $(x,J) \in H$, whence $x \in J$. Therefore $J = \{x \in C^n \mid xB \leq K\}$; hence according to Lemma S we obtain $JB = K$.

Now $HR_{23} = (0,0) = H$; hence the map $H/HR_{23} \rightarrow R_{23}^n/(R_{23}^n)R_{23}$ is injective. Inasmuch as $R_{23}^n$ is projective, it follows that $\text{Tor}_1^R(R_{23}/H, R/R_{23}) = 0$, i.e., $\text{Tor}_1^R(NR_{23}, R/R_{23}) = 0$. Therefore $NR_{23} \otimes_R R_{23} \rightarrow NR_{23}$ is injective.

**Lemma U.** $R$ is a splitting ring.

**Proof.** We must show that $\text{Ext}_1^R(N, W) = 0$ whenever $N$ is a nonsingular right $R$-module and $W$ is a singular right $R$-module. Since it suffices to show that $\text{Ext}_1^R(N, WR_{12}) = 0$ and $\text{Ext}_1^R(N, WR_{12}) = 0$, we may assume that either $WR_{12} = 0$ or $WR_{23} = 0$.

**Case I.** $WR_{12} = 0$. Consider any short exact sequence $E: 0 \rightarrow W \rightarrow V \rightarrow N \rightarrow 0$ of right $R$-modules. Since $R(R/R_{12})$ is projective, we obtain another exact sequence $E^*: 0 \rightarrow W \rightarrow V/WR_{12} \rightarrow N/NR_{12} \rightarrow 0$. According to Lemma R, $N/NR_{12}$ is a projective right $(R/R_{12})$-module; hence $E^*$ splits, from which we infer that $E$ splits.

**Case II.** $WR_{23} = 0$. Consider any short exact sequence $E: 0 \rightarrow W \rightarrow V \rightarrow N \rightarrow 0$ of right $R$-modules. Noting from Lemma T that $\text{Tor}_1^R(N, R/R_{23}) = 0$, we obtain another exact sequence $E^*: 0 \rightarrow W \rightarrow V/WR_{23} \rightarrow N/NR_{23} \rightarrow 0$. Inasmuch as $R_{23}$ is a two-sided ideal in $L^*(R)$, [4, Proposition 1.11] says that the singular submodule of any right $(R/R_{23})$-module is the same whether considered as an $(R/R_{23})$-module or as an $R$-module. In particular, $W$ must be a singular $(R/R_{23})$-module. Considering that $NR_{23}$ is a direct summand of $N$ by Lemma Q, we see that $N/NR_{23}$ is nonsingular as an $R$-module and hence as an $(R/R_{23})$-module. Inasmuch as $R/R_{23}$ is a splitting ring by (a), it follows that $E^*$ splits, from which we conclude that $E$ splits.

5. **Conclusion.** Combining Theorems 2, 5, and 6, we obtain the following structure theorem for splitting rings with zero socle:

**Theorem 7.** Let $R$ be a right nonsingular ring with zero socle. Then $R$ is a splitting ring if and only if $R$ is isomorphic to a ring of the form $(A \oplus C)$, where

(a) $A$ is a semiprime right nonsingular splitting ring.
(b) $B$ is a nonsingular injective right $A$-module.
(c) $C$ is a unital subring of $T = \text{End}_A(B)$.
(d) $C_c$ is essential in $T_c$.
(e) All nonsingular right $C$-modules are projective.
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