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ON STRICTLY CYCLIC ALGEBRAS, $\mathcal{P}$-ALGEBRAS AND REFLEXIVE OPERATORS

BY

DOMINGO A. HERRERO[1] AND ALAN LAMBERT

ABSTRACT. An operator algebra $\mathfrak{A} \subset \mathcal{L}(\mathcal{X})$ (the algebra of all operators in a Banach space $\mathcal{X}$ over the complex field C) is called a "strictly cyclic algebra" (s.c.a.) if there exists a vector $x_0 \in \mathcal{X}$ such that $\mathfrak{A}(x_0) = \{Ax_0 : A \in \mathfrak{A}\} = \mathcal{X}$; x_0 is called a "strictly cyclic vector" for $\mathfrak{A}$. If, moreover, x_0 separates elements of $\mathfrak{A}$ (i.e., if $A \in \mathfrak{A}$ and $Ax_0 = 0$, then $A = 0$), then $\mathfrak{A}$ is called a "separated s.c.a."

$\mathfrak{A}$ is a $\mathcal{P}$-algebra if, given $x_1, \ldots, x_n \in \mathcal{X}$, there exists $x_0 \in \mathcal{X}$ such that $\|Ax_j\| \leq \|Ax_0\|$, for all $A \in \mathfrak{A}$ and for $j = 1, \ldots, n$. Among other results, it is shown that if the commutant $\mathfrak{A}'$ of the algebra $\mathfrak{A}$ is an s.c.a., then $\mathfrak{A}$ is a $\mathcal{P}$-algebra and the strong and the uniform operator topology coincide on $\mathfrak{A}$; these results are specialized for the case when $\mathfrak{A}$ and $\mathfrak{A}'$ are separated s.c.a.'s. (Here, and in what follows, *algebra* means *strongly closed subalgebra on $\mathcal{L}(\mathcal{X})$ containing the identity I on $\mathcal{X}$.*)

In the second part of the paper, it is shown that a large class of bilateral weighted shifts (which includes all the invertible ones) in a Hilbert space are reflexive. The result is used to show that "reflexivity" is neither a "restriction property" nor a "quotient property."

Recall that an algebra $\mathfrak{A}$ is called *reflexive* if, whenever $T \in \mathcal{L}(\mathcal{X})$ and the lattice of invariant subspaces of T contains the corresponding lattice of $\mathfrak{A}$, then $T \in \mathfrak{A}$. [2]

1. A strictly cyclic algebra and its commutant. Let $\mathfrak{A}$ be a strictly cyclic algebra on $\mathcal{X}$, with strictly cyclic vector x_0. We have a commutative diagram:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "S"] \arrow[d, "\pi"'] & \mathcal{X} \\ \mathfrak{A}/\ker S \arrow[ur, "\bar{S}"'] & \end{tikzcd}$$

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(2) After this paper was written, the authors received the preprint *Notes on strictly cyclic operator algebras* by Mary R. Embry. In that paper, M.R. Embry found independent proofs of Theorem 1(i) and Theorem 3 below.

where $S(A) = Ax_0$, ker S is a closed left ideal in $\mathfrak{A}$, π is the canonical projection onto the quotient space and $\bar{S}$ ($S = \bar{S} \circ \pi$) is the quotient map. Clearly, S is continuous, linear and onto, π has the same properties and $\bar{S}$ is an isomorphism of Banach spaces. Moreover, π (and therefore S) is an open map.

Theorem 1. *Let $\mathfrak{A}$ be a subalgebra of $\mathcal{L}(\mathcal{X})$ and let $\mathfrak{A}' = \{T \in \mathcal{L}(\mathcal{X}): TA = AT$, for all $A \in \mathfrak{A}\}$ be the commutant of $\mathfrak{A}$. Assume that $\mathfrak{A}'$ has an s.c. vector x_0; then*

(i) *$\mathfrak{A}$ has the property $(\mathcal{P})$ (i.e., $\mathfrak{A}$ is a $\mathcal{P}$-algebra);*
(ii) *the strong and the uniform operator topologies coincide on $\mathfrak{A}$.*

Proof. (i) By hypothesis, $\mathcal{X} = \mathfrak{A}'x_0$. Thus, given $x_0, \ldots, x_n \in \mathcal{X}$, there exist $A_1, \ldots, A_n \in \mathfrak{A}'$ such that $x_j = A_j x_0$ for $j = 1, \ldots, n$. Moreover, since S is an open map, the A_j's can be chosen in such a way that $\|A_j\| \leq C\|x_j\|$, where C is a positive constant, independent of j (see [5, Lemma 1]).

Thus, if $B \in \mathfrak{A}$, then

$$\|Bx_j\| = \|BA_jx_0\| = \|A_jBx_0\| \leq \|A_j\| \cdot \|Bx_0\| \leq \|B(Mx_0)\|,$$

where $M = C \max\{\|x_j\|: j = 1, \ldots, n\}$.

Hence, $\mathfrak{A}$ is a $\mathcal{P}$-algebra.

(ii) Let $\{A_\lambda : \lambda \in \Lambda$, a directed set$\}$ be a net in $\mathfrak{A}$ converging strongly to A ($\in \mathfrak{A}$). Let $x = Bx_0$, $B \in \mathfrak{A}'$ be chosen s.t. $\|x\| \leq 1$ and $\|B\| \leq C$. Then

$$\|(A_\lambda - A)x\| = \|(A_\lambda - A)Bx_0\| = \|B(A_\lambda - A)x_0\| \leq C\|(A_\lambda - A)x_0\|.$$

Therefore

$$\lim_{\lambda \in \Lambda} \|A_\lambda - A\| \leq C \lim_{\lambda \in \Lambda} \|(A_\lambda - A)x_0\| = 0;$$

i.e., strongly convergent nets are actually norm-convergent. It follows immediately that the strong and the uniform topologies coincide on $\mathfrak{A}$.

Now, most of the results of [6, §3] can be "translated" to these algebras (or sums of them). Consider, in particular, the four algebras associated with an operator $T \in \mathcal{L}(\mathcal{X})$ defined in [6]; i.e., the algebras $\mathfrak{A}_T$ and $\mathfrak{A}_T^a$ generated by the polynomials and by the rational functions (with poles outside of the spectrum) in T, resp., and the commutant $\mathfrak{A}'_T$ and the double commutant $\mathfrak{A}''_T$ of T. Then $\mathfrak{A}_T \subset \mathfrak{A}_T^a \subset \mathfrak{A}''_T \subset \mathfrak{A}'_T$, and the corresponding lattices of invariant subspaces satisfy the reverse inclusions. The above mentioned results yield the following

Corollary 1. *If $T \in \mathcal{L}(\mathcal{X})$ and $\mathfrak{A}'_T$ is an s.c.a., then any two of the algebras $\mathfrak{A}_T$, $\mathfrak{A}_T^a$, $\mathfrak{A}''_T$ and $\mathfrak{A}'_T$ are equal if and only if the corresponding lattices of invariant subspaces are equal.*

Let $\{x_1, x_2\}$ be linearly independent in $\mathcal{X}$; if x_0 is any nonzero vector in $\mathcal{X}$, then (by the Hahn-Banach theorem) there exists a continuous linear functional x^* on

$\mathcal{X}$ such that $x^*(x_0) = 1$. If $\mathcal{M} = \ker(x^*)$, then $\mathcal{X} = \mathcal{M} \oplus \{\lambda x_0 \colon \lambda \in \mathbf{C}\}$ and the linear map T_0 defined by $T_0|_{\mathcal{M}} = I|_{\mathcal{M}}$, $T_0 x_0 = 0$, belongs to $\mathcal{L}(\mathcal{X})$. Moreover, it is clear that either $T_0 x_1 \neq 0$ or $T_0 x_2 \neq 0$; thus we have the following

Lemma 2. *If* $\dim \mathcal{X} > 1$, *then* $\mathcal{L}(\mathcal{X})$ *does not have the property* ($\mathcal{P}$).

Hence, $\mathfrak{A} = \{\lambda I \colon \lambda \in \mathbf{C}\}$ is a $\mathcal{P}$-algebra whose commutant $\mathfrak{A}' = \mathcal{L}(\mathcal{X})$ is an s.c.a., but not a $\mathcal{P}$-algebra ($\dim \mathcal{X} > 1$). Here is an open problem: Assume that $\mathfrak{A}$ is a $\mathcal{P}$-algebra. Does it imply that $\mathfrak{A}''$ is a $\mathcal{P}$-algebra too?

The results of [5] can be "specialized" here. In fact, we have

Theorem 2. *Let* $\mathfrak{A}$ *be an s.c.a. with strictly cyclic vector* x_0. *Then every cyclic vector of* $\mathfrak{A}$ *is actually a strictly cyclic vector. Moreover, the set of all these vectors is open in* $\mathcal{X}$ *and coincides with* $C_{\mathfrak{A}} = \{Ax_0 \colon A \in \mathfrak{A}$ *and* A *has a left inverse* (mod $\ker S$) *in* $\mathfrak{A}\}$, *where* $\ker S$ *is the kernel of the map* S *defined above.*

Proof. If x is a cyclic vector of $\mathfrak{A}$, then $\mathfrak{A}x$ is a *dense linear manifold* of $\mathcal{X}$ invariant under $\mathfrak{A}$. Since, according to [5, Lemma 2] the only linear manifold of $\mathcal{X}$ satisfying these two conditions is $\mathcal{X}$ itself, we conclude that $\mathfrak{A}x = \mathcal{X}$. In other words, x is a strictly cyclic vector.

Clearly, if $x = Ax_0 \in C_{\mathfrak{A}}$, then there exists $B \in \mathfrak{A}$ such that $BA = I + L$, for some $L \in \ker S$. Therefore, $x_0 = BAx_0 \in \mathfrak{A}x$; hence, $\mathcal{X} = \mathfrak{A}x_0 \subset \mathfrak{A}x$, which implies that x is a strictly cyclic vector of $\mathfrak{A}$.

Conversely, if $x = Ax_0$ is cyclic (and therefore, it is strictly cyclic), then

$$Ix_0 = x_0 \in \mathcal{X} = \mathfrak{A}x = \{BAx_0 \colon B \in \mathfrak{A}\}.$$

Therefore, there exists $B \in \mathfrak{A}$ s.t. $(I - BA)x_0 = 0$; i.e., $I - BA = L \in \ker S$. Hence $BA = I - L$, i.e., B is a left inverse of A, modulo $\ker S$. Therefore $\chi \in C_{\mathfrak{A}}$.

Finally, to see that $C_{\mathfrak{A}}$ is an open subset of $\mathcal{X}$, observe that the mapping S is *open* and (see, for example, [10, Chapter I]) $S^{-1}(C_{\mathfrak{A}})$ is open in $\mathfrak{A}$.

2. Separated strictly cyclic algebras.

Lemma 3 [10, p. 3]. *Let* $\mathfrak{B}$ *be a Banach algebra with identity* e; *let* $\mathfrak{B}_L = \{L_b =$ *left multiplication by* b; $b \in \mathfrak{B}\}$ *and let* $\mathfrak{B}_R$ (*right multiplications*) *be similarly defined* (*i.e.*, $b \to L_b$ *and* $b \to R_b$ *are the regular left and right, resp., representations of* $\mathfrak{B}$). *Then*:

(i) $\mathfrak{B}_L$ *and* $\mathfrak{B}_R$ *are separated s.c.a.'s of* $\mathcal{L}(\mathfrak{B})$ (*i.e.*, $I \in \mathfrak{B}_L$, $I \in \mathfrak{B}_A$ *and* $\mathfrak{B}_L$, $\mathfrak{B}_R$ *are strongly closed in* $\mathcal{L}(\mathfrak{B})$).

(ii) $\mathfrak{B}_L' = \mathfrak{B}_R$; $\mathfrak{B}_R' = \mathfrak{B}_L$.

Theorem 3. *Let* $\mathfrak{A}$ *be a separated s.c.a.*; *then*

(i) $\mathfrak{A}'$ *is also a separated s.c.a.*;

(ii) $\mathfrak{A}'' = \mathfrak{A}$;

(iii) $\mathfrak{A}$ *and* $\mathfrak{A}'$ *have the property* ($\mathcal{P}$).

Proof. Let $\mathfrak{A}$ be a separated s.c.a. with separating strictly cyclic vector x_0; then (see [8, Lemma 2.2]) there exists a constant $C \geq 1$ s.t., if $A_x \in \mathfrak{A}$ is the (unique!) operator satisfying $A_x x_0 = x$, then $\|x\| \leq \|A_x\| \leq C\|x\|$.

These two inequalities show that $\mathfrak{A}$ is isomorphic (as a Banach space) with $\mathcal{X}$. Moreover, if for $x, y \in \mathcal{X}$ the product $x \cdot y$ is defined by

$$x \cdot y = A_x \cdot A_y x_0,$$

and $\mathcal{X}$ is re-normed with $\|x\|' = \|A_x\|$, then $\mathfrak{B} = (\mathcal{X}, +, \cdot, \|\cdot\|')$ is a Banach algebra with identity x_0, $\|\cdot\|'$ is equivalent to $\|\cdot\|$, $\|x_0\|' = 1$ and $\mathfrak{A}$ can be identified with $\mathfrak{B}_L$. In other words: a separated s.c.a. is essentially the algebra of all "left multiplications" in a Banach algebra with identity. Using this characterization and Lemma 3, we obtain (i) and (ii).

Finally, using (i), (ii) and Theorem 1(i), we obtain (iii).

Thus, if $\mathfrak{A}$ is a separated s.c.a., Theorem 2 applies both to $\mathfrak{A}$ and $\mathfrak{A}'$. Moreover, in this case we also have

Corollary 4. *Let $\mathfrak{A}$ be a separated s.c.a.; then*

(i) *$\mathfrak{A}$ and $\mathfrak{A}'$ have the same separating cyclic vectors (which are actually strictly cyclic vectors for both algebras); this set of vectors is open in $\mathcal{X}$ and can be identified with*

$$SC = \{Ax_0\colon\ A \in \mathfrak{A} \text{ and } A \text{ is invertible}\},$$

where x_0 is an arbitrary separating s.c. vector of $\mathfrak{A}$.

(ii) *Let $A_x \in \mathfrak{A}$, $A'_x \in \mathfrak{A}'$ be the operators defined by*

$$A_x x_0 = A'_x x_0 = x;$$

then $\sigma(A_x) = \sigma(A'_x)$ ($\sigma(T)$ = spectrum of T).

Proof. (i) Identify $\mathcal{X}$ with a Banach algebra $\mathfrak{B}$ with identity $e = x_0$ as in Theorem 3. Then SC is the set of all invertible elements of $\mathfrak{B}$ and, clearly, every element of SC is a separating strictly cyclic vector of $\mathfrak{B}_L$.

On the other hand, if $f \in \mathfrak{B}$ is a separating s.c. vector of $\mathfrak{B}_L$, then Theorem 2 says that f must have a left inverse in $\mathfrak{B}$. If f is actually invertible, then $f \in SC$, and we are done. Otherwise, there exist two *different* elements, $b, c \in \mathfrak{B}$, s.t. $bf = cf = e$, and therefore

$$0 = (L_b - L_c)f = (L_b - L_c)L_f e = 0 \cdot e,$$

even when $(L_b - L_c) \neq 0$. Hence, f does not separate points of $\mathfrak{B}_L$, contradicting our assumption.

Similarly, we can prove that the set of all separating s.c. vectors of $\mathfrak{A}'$ also coincides with SC.

That SC is open in $\mathfrak{B}$ is immediate (see [10, Chapter I]).

(ii) Using the above identification, consider the following statements:

(1) $\lambda \in \sigma(L_b)$;

(2) $L_b - \lambda I$ has no inverse in $\mathfrak{B}_L$;

(3) $b - \lambda e$ has no inverse in $\mathfrak{B}$;

(4) $R_b - \lambda I$ has no inverse in $\mathfrak{B}_R$;

(5) $\lambda \in \sigma(R_b)$.

It is not hard to check that each of these statements is equivalent to the next one. Therefore, $\sigma(L_b) = \sigma(R_b)$.

Remarks. (a) The definition of "algebra with the property ($\mathcal{P}$)" was introduced in [6]. For "strictly cyclic algebras," see [7], [9], [10].

(b) If $\mathfrak{A}$ is a separated s.c.a., then $\mathfrak{A}'' = \mathfrak{A}$ (by Theorem 3). Therefore, if $A \in \mathfrak{A}$ is invertible in $\mathcal{L}(\mathcal{X})$, then $A^{-1} \in \mathfrak{A}$. However, if A has only a left (or right) inverse $B \in \mathcal{L}(\mathcal{X})$, but A is not invertible, then $B \notin \mathfrak{A}$, in general.

For example (see [8]), if $\mathcal{X}$ is a Hilbert space with orthonormal basis $\{e_n\}$ $(n \geq 0)$ and T is the unilateral weighted shift defined by $Te_n = (n+2)/(n+1)e_{n+1}$, then $\mathfrak{A} = \mathfrak{A}_T = \mathfrak{A}'_T$ is a maximal abelian separated s.c.a. and T has a right inverse $L \in \mathcal{L}(\mathcal{X})$; but it is clear (since L does not commute with T!) that $L \notin \mathfrak{A}$.

3. **Reflexive bilateral weighted shifts.** Examples of separated s.c.a.'s generated by a unilateral weighted shift, or by an invertible bilateral weighted shift and its inverse, can be found in [2], [8], [9]. In particular, the "Donoghue weighted shift" D in the Hilbert space $\mathcal{H}$ (D is defined by $De_n = 2^{-n}e_{n+1}$, $n = 0, 1, 2, \ldots$) generates the maximal abelian separated s.c.a. $\mathfrak{A}_D$. This example shows that a $\mathcal{P}$-algebra need not be a reflexive one. (Furthermore, $\{2^{n(n-1)/2}D^n\}$ $n = 1, 2, \ldots,$ converges *weakly* to 0; but this sequence does not converge in the strong topology of $\mathcal{L}(\mathcal{X})$. Hence, "strong" cannot be replaced by "weak" in Theorem 1.)

We want to show here that this is no longer the case if D is replaced by a bilateral weighted shift T such that T and T^{-1} generate a separated s.c.a. In fact, our result includes a more general class of shift operators.

In [1], J.A. Deddens remarked that, using the same techniques as in D. Sarason's paper [11], it is possible to prove that a unilateral weighted shift on a Hilbert space with nontrivial (i.e., other than 0) compression spectrum is reflexive. This result can be extended to a class of bilateral weighted shifts, as follows:

Theorem 4. *Let B be a bilateral weighted shift in a Hilbert space $\mathcal{H}$ with ONB $\{e_n\}$ $(-\infty < n < +\infty)$, defined by $Be_n = a_n e_{n+1}$, where $\{a_n\}$ $(-\infty < n < +\infty)$ is a bounded sequence of positive reals. Assume that either*

$$I^+ = \liminf_{n\to+\infty}\left(\prod_{j=0}^{n-1} a_j\right)^{1/n} > 0,$$

or

$$Q^- = \liminf_{n\to+\infty}\left(\prod_{j=1}^{n} a_{-j}\right)^{1/n} > 0.$$

Then B is reflexive.

Proof. Let L be a bounded operator in $\mathcal{H}$ and assume that $\mathcal{J}_L \supset \mathcal{J}_B$ (where $\mathcal{J}$ denotes the lattice of invariant subspaces). Assume that $I^+ > 0$ and for each integer m, let $\mathcal{H}_m$ be the closed linear span of $\{e_n\colon\ n \geq m\}$. The restriction $B\,|_{\mathcal{H}_m}$ of B to $\mathcal{H}_m$ is a unilateral weighted shift in $\mathcal{H}_m$ with nontrivial compression spectrum. In fact, the compression spectrum of $B\,|_{\mathcal{H}_m}$ contains the open disc of radius I^+ (see [2], [4]). It follows from [1] and [12] that

$$L\,|_{\mathcal{H}_m} = (\text{strong})\lim_{N\to+\infty}\sum_{n=0}^{N}\left(1-\frac{n}{N+1}\right)c_n(B\,|_{\mathcal{H}_m})^n \tag{1}$$

where the "Taylor coefficients" c_n $(n \geq 0)$ are uniquely determined by L. Moreover, since L maps $\mathcal{H}_m$ into itself for each m, it also follows that L commutes with B on the *dense* linear manifold $\bigcup\{\mathcal{H}_m\colon\ -\infty < m < +\infty\}$ of $\mathcal{H}$; hence $LB = BL$ and therefore (by a slight modification of the proof given in [3]),

$$L = \lim_{N\to+\infty}\sum_{n=-N}^{+N}\left(1-\frac{n}{N+1}\right)c_n' B^n \quad \text{(in the strong topology)}, \tag{2}$$

where the "Laurent coefficients" $\{c_n'\}$ $(-\infty < n < +\infty)$ are uniquely determined by L. Since $L\,|_{\mathcal{H}_m} = P_m L P_m = L P_m$, where P_m denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_m$, multiplying (2) by P_m on both sides and comparing this expression with (1) it follows that $c_n' = c_n$, for all $n \geq 0$, $c_n' = 0$, for $n < 0$ and $L \in \mathfrak{A}_B$, the strong closure of the polynomials in B. In other words, $\mathfrak{A}_B$ is a reflexive algebra.

If, instead of $I^+ > 0$ we assume that $Q^- > 0$, the result can be proved in the same way, replacing B by B^* and observing (see [1]) that $\mathfrak{A}_B$ is reflexive if and only if $\mathfrak{A}_{B^*}$ is reflexive.

Observe that, in the proof of Theorem 4, we do not use the fact that $\mathcal{J}_L$ contains *all* the invariant subspaces of B; in fact, we merely need to assume that $\mathcal{J}_L$ contains every $\mathcal{M} \in \mathcal{J}_B$ such that $\mathcal{M} \subset \mathcal{H}_m$ for some integer m. If B is an *invertible* bilateral weighted shift, then $I^+(B) > 0$ and $Q^-(B) > 0$ (see [2], [4]) and 0 belongs to the spectrum of $B\,|_{\mathcal{M}}$ for any invariant subspace $\mathcal{M}$ which is contained in $\mathcal{H}_m$, for some m. Since B is invertible, $\mathcal{M} \notin \mathcal{J}_B^a$, the lattice of analytically invariant subspaces of B (see [6, §2]). Thus, we obtain the following

Corollary 5. *An invertible bilateral weighted shift B in a Hilbert space $\mathcal{H}$ is always reflexive. Moreover, if L is any bounded linear operator in $\mathcal{H}$ such that $\mathcal{J}_L \supset \mathcal{J}_B\backslash\mathcal{J}_B^a$, then $L \in \mathfrak{A}_B$.*

Finally, we shall provide a counterexample for the following question:

Let $T \in \mathcal{L}(\mathcal{X})$ be a reflexive operator, $\mathcal{M} \in \mathcal{J}'_T$, and $R = T|_{\mathcal{M}}$; is R a reflexive operator on $\mathcal{M}$? ($\mathcal{J}'_T$ = lattice of $\mathfrak{A}'_T$.)

In the case when $\mathcal{X}$ is finite dimensional, the answer is yes, as it follows from [1, Theorem 2]. This is no longer true if $\mathcal{X}$ is infinite dimensional; in fact, if B is defined as in Theorem 4, with $a_n = 1$, for $n \leq 0$, $a_n = 2^{-n}$, for $n > 0$, then $I^+ = 0$, $Q^- = 1$ and B is reflexive. Since B is not invertible, $\mathfrak{A}'_B = \mathfrak{A}_B$ (see [3]) and therefore $\mathcal{J}'_B = \mathcal{J}_B$. However $B|_{\mathcal{H}_0}$ is a "Donoghue unilateral weighted shift" [2] and therefore, it is not reflexive (see [1]).

Similarly, it is possible to show that if T is reflexive, $\mathcal{M} \in \mathcal{J}'_T$ and $\overline{T}$ is the operator induced by T on $\mathcal{X}/\mathcal{M}$, then $\overline{T}$ is not necessarily reflexive.

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Department of Mathematics, State University of New York at Albany, Albany, New York 12203

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506 (Current address of Alan Lambert)

Current address (Domingo A. Herrero): Universidade Estadual de Campinas, Campinas, São Paulo, Brasil