THE STRONG LAW OF LARGE NUMBERS WHEN THE MEAN IS UNDEFINED

BY

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ABSTRACT. Let $S_n = X_1 + \cdots + X_n$ where $\{X_n\}$ are i.i.d. random variables with $EX_i^+$ = $\infty$. An integral test is given for each of the three possible alternatives $\lim(S_n/n) = +\infty$ a.s.; $\lim(S_n/n) = -\infty$ a.s.; $\lim sup(S_n/n) = +\infty$ and $\lim inf(S_n/n) = -\infty$ a.s. Some applications are noted.

1. Introduction. Let $\{X_n\}$ be a sequence of independent identically distributed random variables and put $S_n = X_1 + \cdots + X_n$, $n \geq 1$. It is well known that if $EX_i$ is defined in the sense that one or both of $EX_i^+$, $EX_i^-$ ($x^+ = \max(x,0)$, $x^- = \max(-x,0)$) is finite then

$$P\left\{\lim_{n \to \infty} (S_n/n) = EX_i\right\} = 1.$$ 

If however $EX_i^+ = EX_i^- = \infty$ then $EX_i$ is undefined and (1.1) is meaningless. In this case Kesten [5, Corollary 3, p. 1195] has proved the following.

Theorem 1. If $EX_i^+ = EX_i^- = \infty$ then one of the following alternatives must prevail:

(i) $P\{\lim(S_n/n) = +\infty\} = 1$;
(ii) $P\{\lim(S_n/n) = -\infty\} = 1$;
(iii) $P\{\lim sup(S_n/n) = +\infty$ and $\lim inf(S_n/n) = -\infty\} = 1$.

In this paper we shall give a simple necessary and sufficient criterion, in the form of an integral test, for each of (i)–(iii).

2. Notation and statement of results. Let $X$ stand for any of the random variables $\{X_n\}$ and assume $P\{X = 0\} \neq 1$. Put $F(t) = P\{X \leq t\}$ and define the following quantities:

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\[ m_-(x) = \int_{-\infty}^{0} F(y) \, dy = xF(-x) + \int_{-\infty}^{0} |y| \, dF(y), \]
\[ m_+ (x) = \int_{0}^{\infty} [1 - F(y)] \, dy = x[1 - F(x)] + \int_{0}^{\infty} y \, dF(y), \]
\[ J_+ = J_+(X) = \int_{0}^{\infty} \frac{x}{m_-(x)} \, dF(x), \]
\[ J_- = J_-(X) = \int_{-\infty}^{0} \frac{|x|}{m_+(|x|)} \, dF(x) = J_+(\neg X). \]

The integrand in \( J_+ \), \( J_- \) is bounded near \( x = 0 \) whenever \( F(0-) \neq 0 \) or \( 1 - F(0) \neq 0 \) respectively. If \( P\{X < 0\} = F(0-) = 0 \) define \( J_+ = EX^+ \) and if \( P\{X > 0\} = 1 - F(0) = 0 \) define \( J_- = E|X| = EX^- \).

Note the following properties: as \( t \to \infty \), \( m_+(t) \to EX^+ \), \( m_-(t) \to EX^- \) and, since \( m_+ \) and \( m_- \) are nondecreasing,

\[ (2.1) J_+ < cEX^+, \quad J_- < cEX^- \]

for some \( c < \infty \) whether or not \( EX^+ \), \( EX^- \) are finite.

**Theorem 2.** (No assumptions on \( EX^+_t \).)

(a) \( J_+ = \infty \) if and only if \( P\{\sup (S_n/n) = +\infty\} = 1 \);

(b) \( J_- = \infty \) if and only if \( P\{\inf (S_n/n) = -\infty\} = 1 \);

(c) \( J_+ < \infty \) if and only if \( P\{\lim (S_n/n) = +\infty\} = 1 \);

(d) \( J_- < \infty \) if and only if \( P\{\lim (S_n/n) = -\infty\} = 1 \).

**Remark.** It follows from the four alternatives presented in Theorem 2 and the Hewitt-Savage 0-1 law that if both \( J_+ \) and \( J_- \) are finite the sequence \( \{S_n/n\} \) must be bounded with probability 1. But this is the case if and only if \( E|X| < \infty \) (and then \( \lim (S_n/n) = EX \) a.s.). From this and (2.1) we conclude

\[ J_+ + J_- < \infty \quad \text{if and only if} \quad E|X| < \infty. \]

This is a purely analytic fact. For a direct analytic proof that \( J_+ + J_- < \infty \) implies \( E|X| < \infty \), see note 7 below.

**Corollary 1.** Assume \( E|X| = \infty \). Then at most one of \( J_+ \), \( J_- \) is finite and

(a) \( P\{\lim (S_n/n) = +\infty\} = 1 \iff J_- < \infty \);

(b) \( P\{\lim (S_n/n) = -\infty\} = 1 \iff J_+ < \infty \);

(c) \( P\{\lim (S_n/n) = -\infty \text{ and } \lim (S_n/n) = +\infty\} = 1 \iff J_+ = J_- = \infty \).

**Proof.** This corollary follows immediately from Theorem 2 and the preceding remark.

**Corollary 2.** If \( E|X| = \infty \) and \( P\{X < 0\} \neq 0 \) then \( P\{S_n > 0 \text{ i.o.}\} = 0 \) or 1 according as \( \sum_1^{\infty} (1/n)P\{S_n > 0\} \) converges or diverges, according as \( \int_{0}^{\infty} (x/\int_0^{\infty} F(-y) \, dy) \, dF(x) \) is finite or infinite.
Proof. Corollary 1 and Spitzer's test [4, p. 415, Theorem 2].

Corollary 3. Let \( \{S_t\} \), \( t \geq 0 \), be a process on \( R^1 \) with stationary independent increments and

\[
\frac{1}{t} \log E e^{\theta S_t} = ib\theta - \frac{\sigma^2}{2} \theta^2 + \int \left( e^{\theta x} - 1 - \frac{i\theta x}{1 + x^2} \right) d\lambda(x).
\]

Put \( \lambda_-(y) = \lambda([-(\infty, y]), y < 0, \) and assume \( \lambda_-(2a) \neq 0 \) for some \( a > 0 \). Then

\[
\limsup_{t \to \infty} \frac{S_t}{t} = +\infty \quad \text{a.s. iff } \int_0^{\infty} \left( x/ \int_a^x \lambda_-(y) dy \right) d\lambda(x) = \infty.
\]

Proof. Write \( S_t = S'_t + S''_t \) (in distribution) where

\[
\frac{1}{t} \log E e^{\theta S'_t} = ib\theta - \frac{\sigma^2}{2} \theta^2 + \int_{|x| \leq a} \left( e^{\theta x} - 1 - i\theta x \right) d\lambda(x),
\]

\[
\frac{1}{t} \log E e^{\theta S''_t} = \int_{|x| > a} \left( e^{\theta x} - 1 \right) d\lambda(x).
\]

Then \( \lim_{t \to \infty} (S'_t/t) = ES'_t \), finite, \( (E|S'_t|^r < \infty \) for all \( r > 0 \) \) and hence

\[
\limsup_{t \to \infty} \frac{S_t}{t} = +\infty \quad \text{a.s. iff } \limsup_{t \to \infty} \frac{S''_t}{t} = +\infty \quad \text{a.s.}
\]

Now \( S''_t \) is a compound Poisson process: \( S''_t = X_1 + \cdots + X_N \), see [3, p. 504, p. 555 and p. 571] where the i.i.d. random variables \( \{X_i\} \) have distribution \( P\{X_n \in I\} = \beta^{-1} \lambda(I \cap [-a,a]^c) \), \( \beta = \lambda([-a,a]^c) \) \((0 < \beta < \infty \) by \( \lambda_-(2a) \neq 0 \) and properties of Levy measures) and the Poisson process \( N_t \) has rate \( \beta \).

Therefore \( \lim_{t \to \infty} (N_t/t) = \beta \) a.s., so

\[
\beta^{-1} \limsup_{t \to \infty} \frac{S''_t}{t} = \limsup_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} \quad \text{a.s.}
\]

and the conclusion of the corollary follows from Theorem 2(a).

3. Notes. (1) Suppose \( F(x) \leq 1 - c/x^\alpha \) for \( x \geq b > 0 \) and \( \int_0^\infty |x|^{\beta} dF(x) < \infty \) for some \( 0 < \alpha < \beta < 1 \). Then \( EX_1^+ = \infty \) and \( J \leq c_1 \int_0^\infty |x|^{\alpha} dF(x) \) \( < \infty \). Hence \( S_n/n \to +\infty \) with probability 1. This example is due to C. Derman and H. Robbins [2].

(2) Suppose \( F(x) = L(|x|)/|x|^\alpha \), \( x \leq -a \leq 0 \) where \( L \) is slowly varying at \( \infty \) and \( 0 < \alpha < 1 \). Then by Karamata's theorem on regularly varying functions, see [4, p. 281], we have

\[
EX_1^- \geq c \int_a^\infty \frac{L(x)}{x^\alpha} dx = \infty
\]
and
\[
x/m_-(x) \sim x^\alpha \int_a^\infty y^{-\alpha} L(y) \, dy \sim \frac{(1 - \alpha)x^\alpha}{L(x)} = \frac{1 - \alpha}{F(-x)}
\]
as \(x \to \infty\). Hence by Corollary 1
\begin{equation}
(3.1) \quad P(\lim S_n = -\infty) = P(\lim S_n/n = -\infty) = 1
\end{equation}
if and only if
\begin{equation}
(3.2) \quad E(1/F(-X_1^+)) < \infty.
\end{equation}

This example is due to Williamson [7, part (i) of Theorem on p. 866].

In that same paper Williamson conjectured that for arbitrary \(F\) (3.2) is necessary and sufficient for (3.1). Here is a counterexample: Let \(F\) have a density \(F'(x) = f(x)\) such that
\[
f(x) \sim \frac{1}{x^2 \log x}, \quad f(-x) \sim \frac{1}{x^2 (\log x)^{1/2}}, \quad x \to \infty.
\]
Then \(1 - F(x) \sim (x \log x)^{-1}\), \(m_+(x) \sim \log \log x\), \(F(-x) \sim x^{-1/(\log x)^{1/2}}\) and \(m_-(x) \sim 2(\log x)^{1/2}\) as \(x \to \infty\). Hence \(J_+ < \infty\) and \(J_- = \infty\) and (3.1) holds. But (3.2) fails since
\[
E(1/F(-X_1^+)) \sim \int_0^\infty x^{-1/(\log x)^{1/2}} \, dx = \infty.
\]

(3) If the tails of \(F\) satisfy
\begin{equation}
(3.3) \quad 0 < c_1 \leq (1 - F(t))/F(-t) \leq c_2 < \infty, \quad t \geq 0,
\end{equation}
then an integration by parts shows that \(J_+\) and \(J_-\) both diverge or converge together. Hence the random walk \(\{S_n\}\) generated by an \(F\) satisfying (3.3) and \(|X_1| = \infty\) is always of the oscillating type; case (iii) of Theorem 1, whether or not it is transient.

(4) Suppose \(F(-x) \sim x^{-2 \log \log x} \quad F'(x) \sim x^{-2}, \quad x \to \infty\). Here the left tail predominates: \(1 - F(x) = o(F(-x))\) as \(x \to \infty\); nevertheless, \(\lim \sup (S_n/n) = +\infty\) and \(\lim \inf (S_n/n) = -\infty\) with probability 1, since \(m_+(x) \sim \log x\), and \(m_-(x) \sim \log x \log \log x\) as \(x \to \infty\), so for some \(a > 0\),
\[
J_+ \geq \lim_{t \to \infty} \int_a^t \frac{dx}{x \log x \log \log x} = \lim_{t \to \infty} \log \log \log x \left|_{x = t} \right| = \infty,
\]
\[
J_- \geq \lim_{t \to \infty} \int_a^t \frac{\log \log x}{x \log x} \, dx = \infty.
\]
One should note that the random walk \(\{S_n\}\) of this example is transient, i.e. \(\lim |S_n| = \infty\) a.s. This follows from the asymptotic estimates \(|1 - \varphi(\theta)| \sim |\theta|m_+(1/|\theta|)\), \(\Re(1 - \varphi(\theta)) = O(|1 - \varphi(\theta)|/\log(1/|\theta|))\) as \(\theta \to 0\) where \(\varphi(\theta) = Ee^{i \theta x}\). See [3, Lemma 1].
(5) Theorem 1 guarantees that \( \lim \sup |S_n/n| = \infty \) with probability 1 whenever \( E X_i^+ = EX_i^- = \infty \). However, it need not happen that

\[
(3.4) \quad P(\lim \inf |S_n/n| = \infty) = 1.
\]

In fact, given any nonnegative number \( c \) there is a random walk \( \{S_n\} \) with \( EX_i^\pm = \infty \) such that

\[
P(\lim \sup |S_n/n| = \infty \text{ and } \lim \inf |S_n/n| = c) = 1.
\]

For the proof see [5, Theorem 7, p. 1196].

**Problem.** Find a simple integral test equivalent to (3.4). In this connection note Remark 2, p. 1182 in [5].

(6) Put \( \varphi(\theta) = E e^{i\theta x} \). The following assertions are equivalent (see Binmore-Katz [1], also [5, Theorem 6 and Remark 5, p. 1195]):

\[
(3.5) \quad \lim (S_n/n) = +\infty \text{ a.s.}
\]

\[
(3.6) \quad \lim_{b \to \infty} \left( \int_{-\infty}^{\infty} \frac{e^{ib\theta} - 1}{i\theta} \log \left( \frac{1 - e^{-ib\theta} \varphi(\theta)}{1 + \theta^2} \right)^{-1} \, d\theta \right) < \infty,
\]

for every \( a > 0 \);

\[
(3.7) \quad \sum_{n=1}^{\infty} \frac{1}{n} P(S_n \leq an) < \infty, \quad \text{for every } a > 0.
\]

(The convergence of this series, for one \( a \), is of course, Spitzer's criterion for \( P(S_n - an \leq 0 \text{ i.o.}) = 0 \).) Thus (3.5)–(3.7) are each equivalent to \( J < \infty \), \( J_+ = \infty \).

**Problem.** Find a “nonprobabilistic” proof that (3.6) is equivalent to \( J < \infty \), \( J_+ = \infty \).

(7) As noted previously the assertions

\[
(3.8) \quad J_+ + J_- < \infty
\]

and

\[
(3.9) \quad E|X_0| < \infty
\]

are equivalent due to Theorem 2 and (2.1). Here is another proof that (3.8) \( \Rightarrow \) (3.9).

**Proposition.** Let \( H \) be a distribution on \([0, \infty)\) with \( H(0) < 1 \) and put \( m(x) = \int_0^x [1 - H(y)] \, dy \); then

\[
I(H) = \int_0^\infty \frac{x}{m(x)} \, dH(x) < \infty \leftrightarrow \int_0^\infty x \, dH(x) < \infty.
\]
Proof that (3.8) $\Rightarrow$ (3.9) from the Proposition. Let $H(x) = P(|X_1| \leq x) = F(x) - F(-x -), x > 0$, then

$$m(x) = \int_0^x [1 - F(y) + F(-y)] \, dy = m_+(x) + m_-(x)$$

and

$$J_+ + J_- = \int_0^\infty \frac{x}{m_-(x)} \, dF(x) + \int_0^\infty \frac{|x|}{m_+(|x|)} \, dF(x)$$

$$\geq \int_0^\infty \frac{x}{m(x)} \, dH(x) = I(H).$$

Consequently, $J_+ + J_- < \infty \Rightarrow I(H) < \infty \Rightarrow \int_0^\infty x \, dH(x) = \int_0^\infty |x| \, dF(x)$ is infinite.

Proof of the Proposition. The implication $\int_0^\infty x \, dH(x) < \infty \Rightarrow I(H) < \infty$ is clear so let us assume $I(H) < \infty$. Note first that $m$ is absolutely continuous on bounded intervals and

$$m'(x) = 1 - H(x) \leq \frac{m(x)}{x} \text{ a.e. } x > 0$$

(the exceptional set where $m'$ does not exist is at most countable); consequently the function $x \to x/m(x), x > 0$, is absolutely continuous on intervals $[a, b], 0 < a < b < \infty$ and is nondecreasing because

$$(3.10) \quad \frac{|x/m(x)|}{x} = \frac{m(x) - x[1 - H(x)]}{m^2(x)} \geq 0 \text{ a.e.}$$

Since $I(H) < \infty$ we see that

$$\epsilon(t) = \frac{t}{m(t)} [1 - H(t)] \leq \int_t^\infty \frac{x}{m(x)} \, dH(x) \to 0 \text{ as } t \to \infty$$

and it follows on integrating by parts in $\int_t^\infty (x/m(x)) \, dH(x)$ that

$$\int_b^\infty [1 - H(x)] \, d(x/m(x))$$

is finite for any $b > 0$. Choosing $b > 0$ so large that $1 - \epsilon(x) \geq \frac{1}{2}$ for $x \geq b$ and noting (3.10) and the absolute continuity of $\log m(x)$ on bounded intervals $[b, B], b > 0$, gives

$$\lim_{t \to \infty} \log \frac{m(t)}{m(b)} = \int_b^\infty \frac{m'(x)}{m(x)} \, dx = 2 \int_b^\infty \frac{m'(x)}{m(x)} [1 - \epsilon(x)] \, dx$$

$$= 2 \int_b^\infty [1 - H(x)] \frac{x}{m(x)} \, dx < \infty.$$

But this implies $\lim_{t \to \infty} m(t) = \int_0^\infty [1 - H(x)] \, dx < \infty$ which in turn implies $\int_0^\infty x \, dH(x) < \infty$. 

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Note. One can also prove the above proposition by observing $I(H) < \infty \Rightarrow \int_0^\infty y dH(y) \sim m(x)$ as $x \to \infty$, hence

$$
\int_0^\infty x \left( \int_0^x y dH(y) \right) dH(x) < \infty
$$

and then $\int_0^\infty x dH(x) < \infty$ by the Abel-Dini theorem.

4. Proof of Theorem 2. We prove Theorem 2 in a series of lemmas, each having independent interest.

**Lemma 1.** Let $G$ be any probability distribution concentrated on $[0, \infty)$ (but not all the mass at the origin). Put

$$
U(t) = \sum_{n=0}^\infty G^{*n}(t), \quad m(t) = \int_0^t [1 - G(x)] dx
$$

where $G^{*n}$ is the $n$-fold convolution. Then

(4.1) $1 \leq m(t)U(t)/t \leq 2$ for all $t > 0$

and

(4.2) $\min(1, a/2) \leq U(at)/U(t) \leq \max(1, 2a)$

for all $t > 0$, $a > 0$.

**Proof.** $U$ satisfies the renewal equation $U = 1 + G * U$, see [4, p. 186] or, equivalently,

$$
1 = \int_0^x [1 - G(x - y)] dU(y), \quad x \geq 0.
$$

Integrating this over $0 \leq x \leq t$ gives

$$
t = \int_0^t dU(y) \int_y^t [1 - G(x - y)] dx = \int_0^t m(t - y) dU(y).
$$

Since $m$ is nondecreasing

$$
m\left( \frac{t}{2} \right)U\left( \frac{t}{2} \right) \leq \int_0^{t/2} m(t - y) dU(y) \leq t \leq m(t)U(t)
$$

and (4.1) follows. To get (4.2) note that $m$ and $U$ are nondecreasing so

$$
1 \leq \frac{U(at)}{U(t)} \leq \frac{2at}{m(at)U(t)} \leq \frac{2at}{m(t)U(t)} \leq 2a
$$

for $a \geq 1$, $t \geq 0$. Similarly, $U(at)/U(t) \geq a/2$ for $a \leq 1$.

(f) Intervals of integration are closed unless otherwise indicated.
Corollary. An integral of the form \( \int_0^\infty \sum_{n=0}^\infty G^*(ax) \, dF(x) \) either converges for all \( a > 0 \) or diverges for all \( a > 0 \), according as \( \int_0^\infty (x/m(x)) \, dF(x) \) converges or diverges, \( m(x) = \int_0^x [1 - G(y)] \, dy \).

For Lemmas 2–5 let \( \{X_n\} \) be a sequence of i.i.d. random variables with distribution \( F \) such that \( F(0-) = P(X_1 < 0) \neq 0 \).

**Lemma 2.** Let \( a > 0 \) be fixed and put \( A_0 = \Omega = \text{certain event}, A_1 = \{X_1 > 0\} \) and \( A_n = \{X_n^- + \cdots + X_{n-1}^- < aX_n^+\}, n > 1 \).

(i) If \( \sum_{n=0}^\infty P(A_n) < \infty \) then

\[
\limsup\left(\frac{X_n^+}{(X_1^- + \cdots + X_n^-)}\right) \leq \frac{1}{a} \quad \text{a.s.}
\]

(ii) If \( \sum_{n=0}^\infty P(A_n) = \infty \) then

\[
\limsup\left(\frac{X_n^+}{(X_1^- + \cdots + X_n^-)}\right) \geq \frac{1}{a} \quad \text{a.s.}
\]

(We define \( X_n^+(\omega)/0 = \infty \) if \( X_n(\omega) > 0 \).)

**Proof.** Assertion (i) follows from the first Borel-Cantelli lemma. To prove (ii) assume \( \sum P(A_n) = \infty \). Since \( P(A_n \text{ i.o.}) \) is either 0 or 1 by the Hewitt-Savage 0-1 law, it suffices to show

\[
P(A_n \text{ i.o.}) > 0.
\]

Now for \( m > n, A_n \cap A_m \subset A_n \cap \{X_n^- + \cdots + X_m^- < aX_m^+\} \) so

\[
P(A_n \cap A_m) \leq P(A_n)P(A_{m-n})
\]

by independence and stationarity of \( \{X_n\} \). Put \( Z_n = \sum_{k=0}^n I_{A_k} = \text{number of } A_k \) which occur up to time \( n \). Then (4.4) gives

\[
EZ_n^2 \leq 2 \sum_{i=0}^n P(A_i) \sum_{j=i}^n P(A_{j-i}) \leq 2 \left[ \sum_{i=0}^n P(A_i) \right]^2 = 2(EZ_n)^2
\]

and hence

\[
P(\limsup (Z_n/EZ_n) \geq 1) > 0
\]

by the generalized Borel-Cantelli lemma, cf. [6]. But this clearly implies (4.3) since \( EZ_n = \sum_{i=0}^n P(A_k) \to \infty \).

**Lemma 3.** \( \limsup \left(\frac{X_n^+}{(X_1^- + \cdots + X_n^-)}\right) = 0 \) or \( \infty \) with probability 1, according as \( J_+ = \int_0^\infty x/m_+(x) \, dF(x) \) is finite or infinite where \( m_-(x) = \int_0^x F(-y) \, dy \).

**Proof.** Let \( A_n \) be as in Lemma 2. Then since \( X_n^- = 0 \) on \( A_n \) we have
\[ P\{X_1^- + \cdots + X_n^- < aX_n^+\} = P(A_n) \]
\[ = \int_0^\infty P\{X_1^- + \cdots + X_n^- < ay\} P\{X_n^+ \in dy\} \]
\[ = \int_0^\infty G^{(n-1)^+}(ay) \, dF(y) \leq \int_0^\infty G^{(n-1)^+}(ay) \, dF(y) \]
where \( G(t) = P(X_1^- \leq t), \ t \geq 0. \) If \( 0 < b < a \) then clearly
\[ P(A_n) \geq \int_0^\infty G^{(n-1)^+}(by) \, dF(y). \]
Therefore from the corollary to Lemma 1 \( \sum_{n=1}^\infty P\{X_1^- + \cdots + X_n^- < aX_n^+\} \) converges or diverges for all \( a > 0 \) according as \( J_+ \) is finite or infinite. The desired conclusion now follows immediately from Lemma 2.

Lemma 4. If
\[ \limsup \frac{X_1^+ + \cdots + X_n^-}{X_1^- + \cdots + X_n^-} = \infty \ \text{a.s.,} \]
then \( EX_1^+ = \infty \) and \( \limsup \frac{S_n}{n} = \infty \) a.s., where \( S_n = X_1 + \cdots + X_n. \)

Proof. Equation (4.5) implies that the event \( X_1^+ \geq 2(X_1^- + \cdots + X_{n-1}^-) \) takes place with probability 1 for infinitely many \( n \). For such an \( n \) we have
\[ S_n = X_n^+ - (X_1^- + \cdots + X_{n-1}^-) + X_1^+ + \cdots + X_{n-1}^+. \]
Hence, \( S_n/n \geq (|X_1| + \cdots + |X_{n-1}|)/n \) infinitely often with probability 1. However, this implies
\[ \limsup \frac{S_n}{n} \geq \liminf \frac{|X_1| + \cdots + |X_n|}{n} \ \text{a.s.} \]
But
\[ \lim \frac{|X_1| + \cdots + |X_n|}{n} = E|X_1| \ \text{a.s.} \]
(whether or not \( E|X_1| \) is finite), and
\[ EX_1^+ = \lim \frac{X_1^+ + \cdots + X_n^+}{n} \geq \limsup \frac{S_n}{n} \ \text{a.s.} \]
since \( X_1^+ + \cdots + X_n^+ \geq X_1 + \cdots + X_n = S_n. \) It follows from (4.6)–(4.8) that \( EX_1^+ \geq E|X_1| \) which, since we are assuming \( P(X_1 < 0) > 0, \) is impossible unless \( EX_1^+ = E|X_1| = \infty. \) From (4.6) and (4.7) it now follows that \( \limsup \frac{S_n}{n} = \infty \) with probability 1.
Lemma 5. If $EX_1^+ = \infty$ and if $P\{S_n > 0 \ i.o.\} > 0$, then

$$\lim \sup (X_n^+ / (X_1^- + \cdots + X_n^-)) = \infty \text{ with probability 1.}$$

This remarkable fact is due to Kesten [5, Theorem 5, p. 1190]. We omit the proof.

Proof of Theorem 2. Note first that we may assume

$$P\{X_1 < 0\} \cdot P\{X_1 > 0\} \neq 0.$$  

(If, for example, $P\{X_1 \geq 0\} = 1$, $P\{X_1 = 0\} \neq 1$ then $EX_1 = EX_1^+ < \infty$ if and only if $J_+ < \infty$; see note 7, §3, and Theorem 2 follows from (1.1).)

Clearly the theorem is symmetric in + and – (replace $X_n$ by $-X_n$, then $J_+$ becomes $J_-$, etc.). Thus, (b) follows from (a) and (d) follows from (c).

Proof of (a). If $J_+ = \infty$ then, by Lemmas 3 and 4, $P(\lim \sup (S_n/n) = +\infty) = 1$. Suppose that $P(\lim \sup (S_n/n) = +\infty) = 1$. Then $EX_1^+ = \infty$ (for otherwise by (1.1) we would have $\lim (S_n/n) = EX_1^+ - EX_1^- \neq +\infty$), and obviously $P(S_n > 0 \ i.o.) = 1$. Hence, by Lemmas 5 and 3, $J_+ = \infty$.

Proof of (c). Assume $J_+ = \infty$ and $J_- < \infty$. We want to show

$$(4.9) \quad P(\lim (S_n/n) = +\infty) = 1.$$  

By parts (a) and (b) we have

$$(4.10) \quad P(\lim \sup (S_n/n) = +\infty \text{ and } \lim \inf (S_n/n) > -\infty) = 1.$$  

Also, $EX_1^+ = \infty$ by (2.1). If $EX_1^- < \infty$ then (4.9) follows from (1.1). If, however, $EX_1^- = \infty$ then (4.9) follows from Theorem 1; we must be in case (i) by (4.10). The converse that (4.9) implies $J_+ = \infty$ and $J_- < \infty$ follows from parts (a) and (b) since (4.9) implies

$$P(\lim \sup (S_n/n) = \lim \inf (S_n/n) = +\infty \neq -\infty) = 1.$$  

Added in proof. I have recently learned of a paper A note on fluctuations of random walks without the first moment by Tashio Mori, Yokohama Math. J. 20 (1972), 51–55. He has obtained, independently, an integral criterion for $P(S_n > 0 \ i.o.) = 1$ when $E|X_1| = \infty$. His criterion is not expressed in terms of the tails of $F$, however. Mr. Mori’s remark in §1 of his paper that Williamson’s result is not true without regular variation of $F^-$ is somewhat misleading: Williamson’s result is false even if the tails are regularly varying, (with exponent 1), see Note 2 in §3 above.

References

2. C. Derman and H. Robbins, The strong law of large numbers when the first moment does not exist,
THE STRONG LAW OF LARGE NUMBERS


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