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CONVEX HULLS AND EXTREME POINTS OF FAMILIES OF STARLIKE AND CONVEX MAPPINGS

BY
L. BRICKMAN, D. J. HALLENBECK, T. H. MACGREGOR AND D. R. WILKEN(1)

ABSTRACT. The closed convex hull and extreme points are obtained for the starlike functions of order α and for the convex functions of order α. More generally, this is determined for functions which are also k-fold symmetric. Integral representations are given for the hulls of these and other families in terms of probability measures on suitable sets. These results are used to solve extremal problems. For example, the upper bounds are determined for the coefficients of a function subordinate to or majorized by some function which is starlike of order α. Also, the lower bound on $\operatorname{Re}\{f(z)/z\}$ is found for each z ($|z| < 1$) where f varies over the convex functions of order α and $\alpha \geq 0$.

Introduction. In this paper we determine the closed convex hulls and extreme points of families of functions which are generalizations of the starlike and convex mappings. These results allow us to solve a number of extremal problems over related families of analytic functions.

Let Δ denote the unit disk $\{z \in C: |z| < 1\}$ and let A denote the set of functions analytic in Δ. Then A is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of Δ. Let S be the subset of A consisting of the functions f that are univalent in Δ and satisfy $f(0) = 0$ and $f'(0) = 1$. Let K and St denote the subfamilies of S of convex and starlike mappings; that is, $f \in K$ if $f(\Delta)$ is convex, and $f \in St$ if $f(\Delta)$ is starlike with respect to 0.

The problem of studying the convex hulls and the extreme points of various families of univalent functions was initiated by three of the authors in [2]. We shall take advantage of some of the basic results obtained there and generally use the same notation with the exception that $\mathfrak{H}F$ shall now denote the closed convex hull of a family of functions F. $\mathfrak{E}[\mathfrak{H}F]$ shall denote the set of extreme points of $\mathfrak{H}F$. Theorems 2 and 3 in [2] completely determined the sets $\mathfrak{H}K$, $\mathfrak{E}[\mathfrak{H}K]$, $\mathfrak{H}St$ and $\mathfrak{E}[\mathfrak{H}St]$. The present paper contains generalizations of these results.

We consider the family, denoted $St(\alpha)$, of starlike functions of order α introduced in [11] by M. S. Robertson. A function f analytic in Δ belongs to $St(\alpha)$

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if $f(0) = 0$, $f'(0) = 1$ and $\operatorname{Re}\{zf'(z)/f(z)\} > \alpha$ for $|z| < 1$ $(\alpha < 1)$. We note that $St(0) = St$ and that if $0 \leq \alpha < 1$ then $St(\alpha) \subset S$. Our interest in $St(\alpha)$ is for all values of α $(\alpha < 1)$.

We find that $\mathfrak{H}[St(\alpha)]$ consists of the functions represented as

$$f(z) = \int_{|x|=1} \frac{z}{(1 - xz)^{2-2\alpha}} d\mu(x)$$

where μ varies over the probability measures on the unit circle. Also $\mathfrak{E}\mathfrak{H}[St(\alpha)]$ is exactly the set of functions $f(z) = z/(1 - xz)^{2-2\alpha}$, $|x| = 1$. A similar result is given for the class $K(\alpha)$ of convex functions of order α, also introduced by Robertson in [11]. We recall that $f \in K(\alpha)$ if f is analytic in Δ, $f(0) = 0$, $f'(0) = 1$ and $\operatorname{Re}\{zf''(z)/f'(z) + 1\} > \alpha$ for $|z| < 1$ $(\alpha < 1)$. The results about $St(\alpha)$ are contained in a more general theorem we deduce concerning k-fold symmetric, starlike functions of order α, $St_k(\alpha)$. A function f analytic in Δ is called k-fold symmetric $(k = 1, 2, 3, \ldots)$ if its power series has the form $f(z) = \sum_{m=0}^{\infty} a_{mk+1} z^{mk+1}$.

A critical step in our presentation is the proof of Theorem 1. This generalizes Theorem 5 in [2] from positive integers to positive real numbers. This result was recently obtained independently by D. A. Brannan, J. G. Clunie and W. E. Kirwan in [1] by a different method. Consequently, they also obtained $\mathfrak{H}[St(\alpha)]$ and $\mathfrak{E}\mathfrak{H}[St(\alpha)]$.

A further consideration in this paper concerns the class $K(\alpha, \beta)$ of close-to-convex functions of order α and type β introduced by R. J. Libera in [5]. A function f analytic and normalized in Δ belongs to $K(\alpha, \beta)$ if there exists a function g in $St(\beta)$ so that $\operatorname{Re}\{zf'(z)/g(z)\} > \alpha$ $(z \in \Delta)$ $(\alpha < 1, \beta < 1)$. We determine $\mathfrak{H}[K(\alpha, \beta)]$ in the form of an integral over the product of two unit circles of a suitable kernel function. In the special case $\alpha = \frac{1}{2}$ this further simplifies and we then can precisely find the set $\mathfrak{E}\mathfrak{H}[K(\frac{1}{2}, \beta)]$.

We apply these initial results to the solution of a number of extremal problems. For example, we find coefficient estimates for the power series of a function that is either majorized by a subordinate to a function from $St(\alpha)$.

If f and g are analytic in Δ then f is majorized by g if $|f(z)| \leq |g(z)|$ for $|z| < 1$. This relation implies the existence of an analytic function ϕ so that $|\phi(z)| \leq 1$ and $f(z) = \phi(z)g(z)$ for $|z| < 1$. This concept has been studied by several authors. In particular, we point out the following result: If $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is majorized by some function in St, then $|a_n| \leq n$ $(n = 1, 2, \ldots)$ [7]. We obtain the analogous result for $St(\alpha)$ for each $\alpha < 1$.

We recall the definition of subordination between two functions f and g analytic in Δ. Namely, f is subordinate to g if $f(0) = g(0)$ and if there exists an analytic function $\phi(z)$ so that $\phi(0) = 0$, $|\phi(z)| < 1$ and $f(z) = g(\phi(z))$ for $|z| < 1$. In the case g is univalent in Δ then f is subordinate to g is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. This relation is denoted $f \prec g$. A result of W. Rogosinski [13, p. 72] asserts that if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is subordinate to some function in St then $|a_n| \leq n$ $(n = 1, 2, \ldots)$. We generalize this result to $St(\alpha)$ for $\alpha \geq \frac{1}{2}$ and $\alpha \leq 0$. Other coefficient estimates are obtained for functions majorized or subordinate to a k-fold symmetric function in St.

Our last consideration is the question of determining the set $\{f(z)/z\}$ where z is fixed in Δ and f varies in $K(\alpha)$. We completely find this variability region if $\alpha \geq \frac{1}{2}$. Also we determine $\min_{f \in K(\alpha)} \min_{|z|=r} \operatorname{Re}\{f(z)/z\}$ for each r $(0 < r < 1)$ and $\alpha \geq 0$. As a consequence it follows that $\operatorname{Re}\{f(z)/z\} > (1 - 2^{2\alpha-1})/(1 - 2\alpha)$ $(z \in \Delta)$, if $f \in K(\alpha)$ and $\alpha \geq 0$. This generalizes the result corresponding to the case $\alpha = 0$ proved by A. Marx [9] and E. Strohhäker [14]; namely, that if $f \in K$ then $\operatorname{Re}\{f(z)/z\} > \frac{1}{2}$ $(z \in \Delta)$. Some of the problems discussed in this paper were considered earlier by the second author in his doctoral thesis. His thesis contains special cases of some of the results presented here.

1. A product theorem and a geometric mean theorem for the family $\mathfrak{F}_p$.

Lemma 1. *Let* $|u| < 1$, $|v| < 1$, $p > 0$, $q > 0$. *Then*

$$(1) \qquad (1-u)^{-p}(1-v)^{-q} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 t^{p-1}(1-t)^{q-1}[1 - \{tu + (1-t)v\}]^{-p-q}\, dt.$$

Proof. The identity

$$(2) \qquad (1-z)^{-b} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-c}\, dt,$$

where $c > b > 0$ and $z \notin [1, +\infty)$, is well known in the theory of the hypergeometric function [10, page 206]. From $|u| < 1$, $|v| < 1$, it follows that $(u-v)/(1-v) \notin [1, +\infty)$. Indeed $1 - (u-v)t/(1-v) = 0 \Rightarrow tu + (1-t)v = 1 \Rightarrow t \notin [0,1]$. Hence letting $z = (u-v)/(1-v)$, $b = p$, $c = p + q$, we obtain

$$\left(1 - \frac{u-v}{1-v}\right)^{-p} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 t^{p-1}(1-t)^{q-1}\left(1 - \frac{u-v}{1-v}t\right)^{-p-q} dt,$$

$$\left(\frac{1-u}{1-v}\right)^{-p} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 t^{p-1}(1-t)^{q-1}\left[\frac{1 - \{tu + (1-t)v\}}{1-v}\right]^{-p-q} dt.$$

The required result now follows upon multiplication by $(1-v)^{-p-q}$. We remark that a self-contained proof can be obtained by expanding both sides of (1) in a power series in u and v and comparing coefficients.

Theorem 1. *Let X be the unit circle, $\mathcal{P}$ the set of probability measures on X, and $\mathfrak{F}_p$ $(p > 0)$ the class of functions f on Δ given by*

$$(3) \qquad f(z) = \int_X \frac{d\mu(x)}{(1-xz)^p} \quad (\mu \in \mathcal{P}).$$

Then

$$\mathcal{P}_p \circ \mathcal{P}_q \subset \mathcal{P}_{p+q} \quad (p > 0, q > 0). \tag{4}$$

Proof. As explained in [2], it is sufficient to show that if $|x| = 1$, $|y| = 1$, then the function $(1 - xz)^{-p}(1 - yz)^{-q}$ belongs to $\mathcal{P}_{p+q}$. By Lemma 1 we have the equation

$$(1 - xz)^{-p}(1 - yz)^{-q} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 t^{p-1}(1-t)^{q-1}[1 - c(t)z]^{-p-q}\, dt,$$

where $z \in \Delta$ and $c(t) = tx + (1-t)y$. Since $(\Gamma(p+q)/\Gamma(p)\Gamma(q))t^{p-1}(1-t)^{q-1}\, dt$ is a probability measure on $[0,1]$, we now need only prove that $[1 - c(t)z]^{-p-q}$ belongs to $\mathcal{P}_{p+q}$ for each t in $[0,1]$. For $t = 0$ or $t = 1$ this is obvious while for $0 < t < 1$ Poisson's integral formula gives

$$[1 - c(t)z]^{-p-q} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left[\frac{e^{i\theta} + c(t)}{e^{i\theta} - c(t)}\right](1 - e^{i\theta}z)^{-p-q}\, d\theta.$$

Corollary 1. *If* $0 < p < q$, *then* $\mathcal{P}_p \subset \mathcal{P}_q$.

Proof. The above theorem together with the identity

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1 - e^{i\theta}z)^{q-p}} = 1 \quad (z \in \Delta)$$

implies the result.

Remarks. The identity (2) used to prove Lemma 1 leads one to define an operator $T_{cb}: A \to A$ by

$$T_{cb}f(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} f(tz)\, dt \quad (c > b > 0).$$

In terms of power series

$$T_{cb} \sum_0^\infty a_n z^n = a_0 + \sum_1^\infty \frac{b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)} a_n z^n.$$

It is not difficult to show that T_{cb} is a linear homeomorphism of A onto itself and that $T_{cb}(\mathcal{P}_c) = \mathcal{P}_b$. Specializing $b = 1$, $c = p > 1$, we obtain a neat characterization of the class $\mathcal{P}_p$:

$$T_{p1}f(z) = (p-1) \int_0^1 (1-t)^{p-2} f(tz)\, dt \quad (p > 1).$$

Hence $f \in \mathcal{P}_p$ $(p > 1) \Leftrightarrow f(0) = 1$ and $\mathrm{Re}\{(p-1)\int_0^1 (1-t)^{p-2} f(tz)\, dt\} > \frac{1}{2}$.

Theorem 2. *Let X be the unit circle, $\mathcal{P}$ the set of probability measures on X, and $p > 0$. Then given $\mu \in \mathcal{P} \;\exists\; \nu \in \mathcal{P}$ such that*

$$\exp\left\{\int_X -p\log(1 - xz)\,d\mu(x)\right\} = \int_X (1 - xz)^{-p}\,d\nu(x). \tag{5}$$

Proof. The integral $\int_X -p\log(1 - xz)\,d\mu(x)$ can be approximated locally uniformly in Δ by sums $\sum_n -p\mu_n\log(1 - x_n z)$ where $\mu_n > 0$, $\sum_n \mu_n = 1$, and $|x_n| = 1$. (See Theorem 1 of [2].) Consequently $\exp\{\int_X -p\log(1 - xz)\,d\mu(x)\}$ is approximated by the products $\prod_n (1 - x_n z)^{-p\mu_n}$. By Theorem 1, such a product belongs to $\mathcal{F}_p$. Since $\mathcal{F}_p$ is closed in A our theorem follows.

2. The convex hulls and extreme points of $St_k(\alpha)$ and $K(\alpha)$.

Theorem 3. *Let X be the unit circle $\{z: |z| = 1\}$, $\mathcal{P}$ the set of probability measures on X, $\alpha < 1$, k any positive integer, and $\mathcal{F}$ the set of functions f on Δ defined by*

$$f_\mu(z) = \int_X \frac{z}{(1 - xz^k)^{(2-2\alpha)/k}}\,d\mu(x) \quad (\mu \in \mathcal{P}). \tag{6}$$

Then $\mathcal{F} = \mathfrak{H}St_k(\alpha)$, the map $\mu \to f_\mu$ is one-to-one, and the extreme points of $\mathfrak{H}St_k(\alpha)$ are precisely the functions $z \to z/(1 - xz^k)^{(2-2\alpha)/k}$, $|x| = 1$.

Proof. Suppose that $f \in St_k(\alpha)$ and $f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \cdots$. We let

$$g(z) = z(1 + a_{k+1}z + a_{2k+1}z^2 + \cdots)^k.$$

Then $g(z^k) = f^k(z)$ and $zf'(z)/f(z) = z^k g'(z^k)/g(z^k)$. Hence $\operatorname{Re}\{zg'(z)/g(z)\} > \alpha$, and this, by a familiar calculation involving Herglotz's formula, is equivalent to

$$g(z) = z\exp\left\{\int_X -(2 - 2\alpha)\log(1 - xz)\,d\mu(x)\right\}$$

for some $\mu \in \mathcal{P}$. Therefore

$$f^k(z) = z^k\exp\left\{\int_X -(2 - 2\alpha)\log(1 - xz^k)\,d\mu(x)\right\},$$

$$f(z) = z\exp\left\{\int_X -(2 - 2\alpha)/k\,\log(1 - xz^k)\,d\mu(x)\right\}.$$

By means of Theorem 2 we conclude that $f \in \mathcal{F}$. Thus $\mathfrak{H}St_k(\alpha) \subset \mathcal{F}$. Conversely, since each kernel function $z \to z/(1 - xz^k)^{(2-2\alpha)/k}$ belongs to $St_k(\alpha)$, $\mathcal{F} \subset \mathfrak{H}St_k(\alpha)$.

If $f_{\mu_1} = f_{\mu_2}$ for $\mu_1 \in \mathcal{P}$, $\mu_2 \in \mathcal{P}$, then

$$\int_X \frac{d\mu_1(x)}{(1 - xz)^{(2-2\alpha)/k}} = \int_X \frac{d\mu_2(x)}{(1 - xz)^{(2-2\alpha)/k}} \quad (z \in \Delta).$$

It follows that $\int_X x^n\, d\mu_1(x) = \int_X x^n\, d\mu_2(x)$ $(n = 0, 1, 2, \ldots)$, so $\mu_1 = \mu_2$. Hence the map $\mu \to f_\mu$ is one-to-one, and the assertion about extreme points follows from Theorem 1 of [2].

Remarks. (1) We shall emphasize the special case of Theorem 3 corresponding to $k = 1$. In particular, we mention that $\mathfrak{E}\mathfrak{H}[St(\alpha)]$ consists of the functions $f(x) = z/(1 - xz)^{2-2\alpha}$, $|x| = 1$. The case of Theorem 3 corresponding to $\alpha = 0$ shows that $\mathfrak{E}\mathfrak{H}[St_k]$ consists of the functions $f(z) = z/(1 - xz^k)^{2/k}$, $|x| = 1$.

(2) It is a known result that each function in K is starlike of order $\frac{1}{2}$ (see [9] and [14]). Also, a slight modification of an argument of R. M. Robinson (see [12, p. 32]) shows that if f is starlike of order $\frac{1}{2}$ then $\operatorname{Re}\{f(z)/z\} > \frac{1}{2}$ $(z \in \Delta)$. If we let R denote the set of functions f analytic in Δ so that $f(0) = 0$, $f'(0) = 1$ and $\operatorname{Re}\{f(z)/z\} > \frac{1}{2}$, then these relations may be expressed $K \underset{\neq}{\subset} St(\frac{1}{2}) \underset{\neq}{\subset} R$, the inequalities being easy to show and generally known. From this point of view it is interesting to note that, with $k = 1$ and $\alpha = \frac{1}{2}$, Theorem 3 asserts that $\mathfrak{H}K = \mathfrak{H}[St(1/2)] = R$.

Theorem 4. *Let X be the unit circle, $\mathcal{P}$ the set of probability measures on X, $\alpha < 1$, and $\mathcal{F}$ the set of functions f_μ on Δ defined by*

$$f_\mu(z) = \int_X \frac{1}{(1 - 2\alpha)x}\left[\frac{1}{(1 - xz)^{1-2\alpha}} - 1\right] d\mu(x) \quad (\mu \in \mathcal{P}) \tag{7}$$

if $\alpha \neq \frac{1}{2}$, and

$$f_\mu(z) = \int_X -\frac{1}{x}\log(1 - xz)\, d\mu(x) \quad (\mu \in \mathcal{P}) \tag{7'}$$

if $\alpha = \frac{1}{2}$. Then $\mathcal{F} = \mathfrak{H}K(\alpha)$, the map $\mu \to f_\mu$ is one-to-one, and the extreme points of $\mathfrak{H}K(\alpha)$ are precisely the kernel functions in (7) *if $\alpha \neq \frac{1}{2}$, and in* (7′) *if $\alpha = \frac{1}{2}$.*

Proof. Let $A_0 = \{f \in A : f(0) = 0\}$. Then the operator T defined by $Tf(z) = \int_0^z f(w)/w\, dw$ is a linear homeomorphism of A_0 onto itself with $TSt(\alpha) = K(\alpha)$. All the assertions of Theorem 4 follow from this and Theorem 3 with $k = 1$.

Theorem 5. *Let X^2 be the torus $\{(x,y) : |x| = 1, |y| = 1\}$, $\mathcal{P}$ the set of probability measures on X^2, $\alpha < 1$, $\beta < 1$, and $\mathcal{F}$ the class of functions f_μ on Δ defined by*

$$f_\mu(z) = \int_{X^2} \frac{1 + (1 - 2\alpha)xz}{(1 - xz)(1 - yz)^{2-2\beta}}\, d\mu(x,y) \quad (\mu \in \mathcal{P}). \tag{8}$$

Then $\mathcal{F} = \mathfrak{H}K'(\alpha,\beta)$, where $K'(\alpha,\beta)$ is the set of derivatives of functions belonging to $K(\alpha,\beta)$.

Proof. Let $f \in K(\alpha,\beta)$. Then, by definition, $\exists\ g \in St(\beta)$ such that $\operatorname{Re} zf'(z)/g(z) > \alpha$ for $z \in \Delta$. It follows that

$$zf'(z)/g(z) = \int_X \frac{1 + (1 - 2\alpha)xz}{1 - xz} d\mu_1(x) \quad (z \in \Delta)$$

for some probability measure μ_1 on the unit circle X. By Theorem 3, there is a measure μ_2 such that

$$g(z) = \int_X \frac{z}{(1 - yz)^{2-2\beta}} d\mu_2(y) \quad (z \in \Delta).$$

Hence

$$f'(z) = \int_{X^2} \frac{1 + (1 - 2\alpha)xz}{(1 - xz)(1 - yz)^{2-2\beta}} d\mu(x,y) \quad (z \in \Delta)$$

where $\mu = \mu_1 \times \mu_2$. Thus $f' \in \mathfrak{I}$ and therefore $\mathfrak{H}K'(\alpha,\beta) \subset \mathfrak{I}$. Conversely it is clear that each kernel function in (8) belongs to $K'(\alpha,\beta)$. Hence $\mathfrak{I} \subset \mathfrak{H}K'(\alpha,\beta)$.

Remarks. (1) We know (see [2, Theorem 1]) that the extreme points of $\mathfrak{H}K'(\alpha,\beta)$ are contained among the above kernel functions but we have not determined which kernel functions are extreme points. Of course Theorem 5 can nevertheless be quite useful in solving certain linear extremal problems over the class $K(\alpha,\beta)$.

(2) By means of our Theorem 1 we observe that each kernel function in (8) belongs to the class $\mathcal{G}$ defined by

$$g_\lambda(z) = \int_{X^2} \frac{1 + (1 - 2\alpha)xz}{(1 - yz)^{3-2\beta}} d\lambda(x,y) \quad (\lambda \in \mathcal{P}).$$

Hence $\mathfrak{I} \subset \mathcal{G}$. It appears, however, that $\mathfrak{I} \neq \mathcal{G}$ in general. But if $\alpha = \frac{1}{2}$, reasoning based on Theorem 1 yields $\mathfrak{I} = \mathcal{G}$ and leads to a precise theorem concerning $\mathfrak{H}K(\frac{1}{2},\beta)$ and $\mathfrak{E}\mathfrak{H}K(\frac{1}{2},\beta)$.

3. Coefficient estimates for functions majorized by or subordinate to functions in $St(\alpha)$ or St_k. In this section we take advantage of Theorem 3 to solve extremal problems. We present our results for the case $k = 1$, α arbitrary and the case $\alpha = 0$, k arbitrary, that is, for starlike functions of order α and for k-fold symmetric starlike functions $St_k = St_k(0)$. We specifically use the precise determination of the sets $\mathfrak{E}\mathfrak{H}[St(\alpha)]$ and $\mathfrak{E}\mathfrak{H}[St_k]$.

The extremal problems considered are all concerned with maximizing a continuous convex functional over a compact subfamily $\mathfrak{I}$ of A. Recall that a real valued functional J is convex if

$$J(tf + (1 - t)g) \leq tJ(f) + (1 - t)J(g) \quad (0 \leq t \leq 1).$$

For such functionals we have $\max_{f\in\mathfrak{I}} J(f) = \max_{f\in\mathfrak{E}\mathfrak{H}[\mathfrak{I}]} J(f)$. In many cases J is either the real part of or the modulus of a complex valued continuous linear functional.

As an illustration of this approach we deduce the result of Robertson [11]: if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $St(\alpha)$, then

$$|a_n| \leq \frac{(2-2\alpha)(3-2\alpha)\cdots(n-2\alpha)}{(n-1)!} \quad (n = 2,3,\ldots).$$

Namely, we need only consider the set $\mathfrak{E}\mathfrak{H}[St(\alpha)]$ which consists of the functions $f(z) = z/(1-xz)^{2-2\alpha}$, $|x| = 1$, according to Theorem 3. For such functions clearly

$$|a_n| = \frac{(2-2\alpha)(3-2\alpha)\cdots(n-2\alpha)}{(n-1)!}.$$

Another illustration depends upon the fact that if f is analytic in Δ, $p \geq 1$, and $0 < r < 1$, then the functional

$$\|f\| = \left\{\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p}$$

is convex. If $f^{(n)}$ denotes the nth derivative of f $(n = 0,1,2,\ldots)$ and if $\mathfrak{I}$ is any compact subset of A and $p \geq 1$, then $\max_{f\in\mathfrak{I}}\|f^{(n)}\| = \max_{f\in\mathfrak{E}\mathfrak{H}[\mathfrak{I}]}\|f^{(n)}\|$. This was pointed out in [8]. If we apply this result and use the fact, given by Theorem 3, that $\mathfrak{E}\mathfrak{H}[St(\alpha)]$ is the set of functions $f(z) = z/(1-xz)^{2-2\alpha}$, $|x| = 1$, we find that

$$\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} |f_0^{(n)}(re^{i\theta})|^p \, d\theta$$

for each f in $St(\alpha)$ $(n = 0,1,2,\ldots)$, $0 \leq r < 1$, $p \geq 1$. Here we can take $f_0(z) = z/(1-z)^{2-2\alpha}$, because the functional $\|f^{(n)}\|$ is constant on $\mathfrak{E}\mathfrak{H}[St(\alpha)]$.

We generalize Robertson's coefficient inequality to functions subordinate to a fixed arbitrary function in $St(\alpha)$. The initial step in the argument depends upon a fact proved in [8]. Suppose that $\mathcal{G}$ is a compact subset of A and that $\mathfrak{I}$ is the class of functions subordinate to some function in $\mathcal{G}$. Let $\mathfrak{I}_0$ be the subset of $\mathfrak{I}$ of functions subordinate to some function in $\mathfrak{E}\mathfrak{H}[\mathcal{G}]$. If J is a complex valued, continuous, linear functional on A, then $\max_{f\in\mathfrak{I}}|J(f)| = \max_{f\in\mathfrak{I}_0}|J(f)|$.

A similar result was obtained in [8] when the relation between the families $\mathfrak{I}$ and $\mathcal{G}$ is given by majorization. This information and the result of Theorem 3 will be used to find coefficient bounds for functions majorized by functions in $St(\alpha)$ or in St_k. The further details of these arguments are similar to those presented in [7] for majorization by functions in St or in K. The central idea is a method introduced by E. Landau (see [4, p. 21]), which has been effectively used by several authors to solve extremal problems for the family of bounded, analytic functions in Δ. The sharpness of our results about majorization depends on a result of S. Kakeya ([3]; also see [4, p. 20]) as used by Landau. This is the

Theorem. *If* $d_0 > d_1 > d_2 > \cdots > d_n > 0$, *then the polynomial* $q(z) = d_0 + d_1 z + d_2 z^2 + \cdots + d_n z^n$ *does not vanish for* $|z| \leq 1$.

Our proof of Theorem 7 depends on a result of W. Rogosinski (see [13, p. 64]), which is convenient to state now. Suppose that $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is subordinate to $F(z) = \sum_{n=1}^{\infty} A_n z^n$ in Δ. If, for $1 \leq k \leq n$, the numbers A_k are nonnegative, nonincreasing, and convex,then

$$(9) \qquad |a_n| \leq A_1 \quad (k = 1,2,\ldots,n).$$

If, for $1 \leq k \leq n$, the numbers A_k are nonnegative, nondecreasing, and convex, then

$$(10) \qquad |a_n| \leq A_n.$$

Theorem 6. *Let* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *be analytic in* Δ *and be majorized by some function in* $St(\alpha)$. *If* $\alpha \geq 0$, *then*

$$(11) \qquad \begin{aligned} |a_n| \leq 1 + [1-\alpha]^2 + \left[\frac{(1-\alpha)(2-\alpha)}{2}\right]^2 \\ + \cdots + \left[\frac{(1-\alpha)(2-\alpha)\cdots(n-1-\alpha)}{(n-1)!}\right]^2 \quad (n = 1,2,\ldots). \end{aligned}$$

If $\alpha < 0$, *then* $|a_n| \leq (2-2\alpha)(3-2\alpha)\cdots(n-2\alpha)/(n-1)!$ $(n = 1,2,\ldots)$. *All inequalities are sharp.*

Proof. Because of the results mentioned from [8] it suffices to assume that f is majorized by some function in $\mathfrak{E}\mathfrak{H}[St(\alpha)]$. According to Theorem 3 we may thus assume that $f(z) = \phi(z)z/(1-xz)^{2-2\alpha}$, $|x| = 1$ and ϕ is analytic in Δ with $|\phi(z)| \leq 1$. Without loss of generality we may assume $x = 1$.

Using Cauchy's formula for the coefficients of $f(z)$ we have

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{\phi(z)}{z^{n+1}} \frac{z}{(1-z)^{2-2\alpha}} dz, \qquad \text{where } 0 < r < 1.$$

If we expand $z/(1-xz)^{2-2\alpha}$ in a power series and integrate term by term, the "tail" of the series integrates to zero leaving

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{\phi(z)}{z^n}\left\{1 + \frac{2-2\alpha}{1!}z + \frac{(2-2\alpha)(3-2\alpha)}{2!}z^2 + \cdots + \frac{(2-2\alpha)(3-2\alpha)\cdots(n-2\alpha)}{(n-1)!}z^{n-1}\right\} dz.$$

Let $b_k = (2-2\alpha)(3-2\alpha)\cdots(k+1-2\alpha)/k!$ $(k = 1,2,\ldots)$. It follows from Cauchy's theorem that $a_n = (1/2\pi i)\int_{|z|=r}(\phi(z)/z^n)p(z)\,dz$, where p is any function analytic in Δ with a power series which begins

(12) $p(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_{n-1} z^{n-1} + b_n^* z^n + b_{n+1}^* z^{n+1} + \cdots.$

The function $g(z) = 1/(1-z)^{2-2\alpha}$ has the power series $g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$. If we let $h(z) = \sqrt{g(z)} = 1/(1-z)^{1-\alpha} = 1 + \sum_{k=1}^{\infty} d_k z^k$ and $q(z) = 1 + d_1 z + d_2 z^2 + \cdots + d_{n-1} z^{n-1}$ then as $h^2(z) = g(z)$ it follows that $q^2(z)$ is a polynomial of degree $2(n-1)$ with the form of $p(z)$ described by (12). Therefore,

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{\phi(z)}{z^n} q^2(z)\, dz, \tag{13}$$

$$|a_n| \leq \frac{1}{r^{n-1}} \frac{1}{2\pi} \int_0^{2\pi} |q^2(re^{i\theta})|\, d\theta$$
$$= \frac{1}{r^{n-1}} \{1 + |d_1|^2 r^2 + |d_2|^2 r^4 + \cdots + |d_{n-1}|^2 r^{2(n-1)}\}.$$

Since this inequality holds for each r $(0 < r < 1)$, we conclude that

$$|a_n| \leq 1 + |d_1|^2 + |d_2|^2 + \cdots + |d_{n-1}|^2$$
$$= 1 + [1-\alpha]^2 + \left[\frac{(1-\alpha)(2-\alpha)}{2!}\right]^2 + \cdots$$
$$+ \left[\frac{(1-\alpha)(2-\alpha)\cdots(n-1-\alpha)}{(n-1)!}\right]^2.$$

This proves the inequality (11).

To show that it is sharp we need to take advantage of the assumption $\alpha \geq 0$. If $\alpha > 0$ then the sequence $\{d_n\}$ is nonnegative and strictly decreasing since $d_n - d_{n+1} = ((1-\alpha)(2-\alpha)\cdots(n-\alpha)/(n+1)!)\alpha$. Due to the result of Kakeya the polynomial $q(z)$ does not vanish for $|z| \leq 1$. Therefore, $\Phi(z) = z^{n-1}q(1/z)/q(z)$ is analytic for $|z| \leq 1$. If $|z| = 1$, then $|\Phi(z)| = 1$ and thus $|\Phi(z)| \leq 1$ for $|z| \leq 1$. With this function in the place of $\phi(z)$, equation (13) implies that

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{\Phi(z)}{z^n} q^2(z)\, dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{z} q\left(\frac{1}{z}\right) q(z)\, dz$$
$$= \frac{1}{2\pi} \int_0^{2\pi} |q(e^{i\theta})|^2\, d\theta = 1 + |d_1|^2 + |d_2|^2 + \cdots + |d_{n-1}|^2.$$

As this is the nth coefficient of the function $f(z) = \Phi(z)z/(1-z)^{2-2\alpha}$ our sharpness assertion is proved for $\alpha > 0$. For $\alpha = 0$ inequality (10) is the same as $|a_n| \leq n$ and this is sharp as illustrated by the function $f(z) = F(z) = z/(1-z)^2$.

Next we consider the case $\alpha < 0$ and again note that we may assume that $|f(z)| \leq |F(z)|$ where $F(z) = z/(1-z)^{2-2\alpha}$. Set $g(z) = (1-z)^{-2\alpha} f(z) = b_1 z + b_2 z^2 + \cdots$. Then $|g(z)| \leq |z/(1-z)^2|$, that is, g is majorized by the Koebe function $k(z) = z/(1-z)^2$. As $k \in St$ the case $\alpha = 0$ of this theorem implies that $|b_n| \leq n$ $(n = 1, 2, \ldots)$. The coefficients of the power series for $(1-z)^{2\alpha}$ are all positive since $\alpha < 0$ and thus the nth coefficient of f is maximal when $b_k = k$ for $k = 1, 2, \ldots, n$. That is $|a_n| \leq A_n$ where $F(z) = z/(1-z)^{2-2\alpha} = z + \sum_{n=2}^{\infty} A_n z^n$. This proves the inequality

$$|a_n| \leq (2-2\alpha)(3-2\alpha)\cdots(n-2\alpha)/(n-1)!,$$

and it is sharp if $f(z) = F(z) = z/(1-z)^{2-2\alpha}$.

Theorem 7. *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in Δ and be subordinate to some function in $St(\alpha)$. If $\alpha \leq 0$ then*

$$(14) \qquad |a_n| \leqq \frac{(2-2\alpha)(3-2\alpha)\cdots(n-2\alpha)}{(n-1)!} \qquad (n = 1, 2, \ldots).$$

If $\frac{1}{2} \leq \alpha < 1$ then

$$(15) \qquad |a_n| \leq 1 \qquad (n = 1, 2, \ldots).$$

Proof. Since the family $St(\alpha)$ is compact the arguments given in [8] show that in order to maximize $|a_n|$ we need only consider the functions f which are subordinate to a function in $\mathfrak{E}\mathfrak{H}[St(\alpha)]$. Therefore, by Theorem 3, f has the form $f(z) = \phi(z)/[1 - x\phi(z)]^{2-2\alpha}$, where $|x| = 1$, ϕ is analytic for $|z| < 1$, $|\phi(z)| < 1$, and $\phi(0) = 0$. The function $\phi(z)/x$ has the same properties as ϕ and a function f and xf have nth coefficients with the same modulus. This implies that we may assume that $x = 1$, that is, that f is subordinate to $F(z) = z/(1-z)^{2-2\alpha}$.

If we set $F(z) = \sum_{n=1}^{\infty} A_n z^n$ then

$$A_1 = 1 \quad \text{and} \quad A_n = (2-\alpha)(3-2\alpha)\cdots\frac{(n-2\alpha)}{(n-1)!}$$

for $n = 2, 3, \ldots$. The sequence $\{A_n\}$ consists of nonnegative real numbers. Since

$$A_2 - A_1 = 1 - 2\alpha \quad \text{and}$$

$$A_{n+1} - A_n = \left((2-2\alpha)(3-2\alpha)\cdots\frac{(n-2\alpha)}{n!}\right)[1-2\alpha]$$

for $n \geq 2$, we see that $\{A_n\}$ is nondecreasing for $\alpha \leq \frac{1}{2}$ and nonincreasing for $\alpha \geq \frac{1}{2}$. Also, $A_1 - 2A_2 + A_3 = \alpha(2\alpha - 1)$ and

$$A_n - 2A_{n+1} + A_{n+2} = 2\frac{(2-2\alpha)(3-2\alpha)\cdots(n-2\alpha)}{(n+1)!}[\alpha(2\alpha-1)] \quad \text{for } n \geq 2.$$

Thus, $\{A_n\}$ is convex if $\alpha \leq 0$ or if $\alpha \geq \frac{1}{2}$.

A direct application of the result of Rogosinski given by inequalities (9) and (10) concludes the proof. The sharpness of inequality (14) is exhibited by $f(z) = F(z) = z/(1 - z)^{2-2\alpha}$. The sharpness of inequality (15) is shown by $f(z) = F(z^n) = z^n/(1 - z^n)^{2-2\alpha}$. We cannot prove what result holds when $0 < \alpha < \frac{1}{2}$ although it appears likely that inequality (14) also holds in this range of α.

Theorem 8. *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in Δ and be majorized by some function in $St_k(0) = St_k$. Then*

$$|a_{mk+j}| \leq 1 + \left[\frac{1}{k}\right]^2 + \left[\frac{(1/k)(1/k+1)}{2!}\right]^2 + \cdots + \left[\frac{(1/k)(1/k+1)\cdots(1/k+m-1)}{m!}\right]^2$$
$$(j = 1,2,\ldots,k; m = 0,1,2,\ldots).$$

This inequality is sharp.

Proof. The proof is similar to that given for Theorem 6. We may assume that the majorant is in $\mathfrak{E}\mathfrak{H}[St_k(0)]$ and hence has the form $F(z) = z/(1 - xz^k)^{2/k}$ $(|x| = 1)$, according to Theorem 3. As before we let $x = 1$.

If we set $g(z) = 1/(1 - z^k)^{2/k}$, $h(z) = \sqrt{g(z)} = 1/(1 - z^k)^{1/k} = 1 + \sum_{m=1}^{\infty} d_{mk} z^{mk}$ and $q(z) = 1 + d_k z^k + d_{2k} z^{2k} + \cdots + d_{mk} z^{mk}$, then the argument given in the proof of Theorem 6 shows that

$$a_{mk+j} = \frac{1}{2\pi i}\int_{|z|=r} \frac{\phi(z)}{z^{mk+j}} q^2(z)\,dz \qquad (0 < r < 1).$$

Therefore,

$$|a_{mk+j}| \leq \frac{1}{r^{mk+j-1}}\frac{1}{2\pi}\int_0^{2\pi} |q(re^{i\theta})|^2\,d\theta = \frac{1}{r^{mk+j-1}}\{1 + |d_k|^2 r^{2k} + |d_{2k}|^2 r^{4k} + \cdots + |d_{mk}|^2 r^{2mk}\}.$$

Since this inequality holds for each r $(0 < r < 1)$ we conclude that

$$|a_{mk+j}| \leq 1 + |d_k|^2 + |d_{2k}|^2 + \cdots + |d_{mk}|^2 = 1 + \left[\frac{1}{k}\right]^2 + \left[\frac{(1/k)(1/k+1)}{2!}\right]^2 + \cdots + \left[\frac{(1/k)(1/k+1)\cdots(1/k+m-1)}{m!}\right]^2.$$

This proves the required inequality.

To prove that the result is sharp, recall that the case $k = 1$ was already done in Theorem 6. If $k \geq 2$ then $1 > d_k > d_{2k} > \cdots > d_{mk} > 0$ and so by Kakeya's theorem $q(z)$ does not vanish for $|z| \leq 1$. The function

$$\Phi(z) = z^{mk+j-1}q(1/z)/q(z)$$

is analytic for $|z| \leq 1$, and since $|\Phi(z)| = 1$ for $|z| = 1$, we conclude that $|\Phi(z)| \leq 1$ for $|z| \leq 1$. If we let $f(z) = \Phi(z)[z/(1 - z^k)^{2/k}]$ then

$$\begin{aligned} a_{mk+j} &= \frac{1}{2\pi i}\int_{|z|=r} \frac{\Phi(z)}{z^{mk+j}} q^2(z)\,dz = \frac{1}{2\pi i}\int_{|z|=1} \frac{1}{z} q\left(\frac{1}{z}\right) q(z)\,dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} |q(e^{i\theta})|^2\,d\theta = 1 + |d_k|^2 + |d_{2k}|^2 + \cdots + |d_{mk}|^2. \end{aligned}$$

Theorem 9. *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in Δ and be subordinate to some function in $St_k(0)$. Then, $|a_n| \leq 1$ for $n = 1, 2, \ldots$ and for $k \geq 2$.*

Proof. We may assume that f is subordinate to a function in $\mathfrak{E}\mathfrak{H}[St_k(0)]$, that is, f is subordinate to a function $F(z) = z/(1 - xz^k)^{2/k}$, where $|x| = 1$. Again, it is easy to see that we may assume that $x = 1$. But, for the functions f subordinate to $F(z) = z/(1 - z^k)^{2/k}$ it has already been shown by W. Rogosinski [13, p. 67] that $|a_n| \leq 1$ $(n = 1, 2, \ldots)$. This depends on the fact that the nonzero coefficients of $F(z)$ are nonnegative, nonincreasing and convex. The sharpness of this inequality is exhibited by the function $f(z) = F(z^n) = z^n/(1 - z^{nk})^{2/k}$.

4. **Majorants for $f(z)/z$ where f is in $K(\alpha)$.** We shall take advantage of Theorem 4 to solve some extremal problems over $K(\alpha)$. As in the previous section we use the exact knowledge of $\mathfrak{E}\mathfrak{H}[K(\alpha)]$. Here we shall be interested only in the case $\alpha \geq 0$

Theorem 10. *If $f \in K(\alpha)$ and $0 \leq \alpha < 1$, $\alpha \neq \frac{1}{2}$, then*

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} \geq \frac{1}{1 - 2\alpha}\frac{1}{|z|}(1 - (1 + |z|)^{2\alpha-1}).$$

If $\alpha = \frac{1}{2}$, then $\operatorname{Re}\{f(z)/z\} \geq (1/|z|)\log(1 + |z|)$. *This inequality is sharp for each α and each z.*

Proof. The case $\alpha = 0$ is found in [9] and [14] and follows from $\operatorname{Re}\{f(z)/z\} > \frac{1}{2}$ $(|z| < 1)$ and Lindelöf's principle for subordination. In the range $0 < \alpha < 1$ we reduce the problem to one involving only the extreme points as follows: On the disc $\{|z| \leq r\}$ the minimum value of $\operatorname{Re}\{f(z)/z\}$ over the family $K(\alpha)$ is attained at some point z_0. The continuous convex functional $\operatorname{Re}\{f(z_0)/z_0\}$ is now minimized over the extreme points. Restricting our attention to extreme points, according to Theorem 4 we may assume that

$$f(z) = \int_0^z (1 - w)^{-(2-2\alpha)}\, dw.$$

Since $0 < 2 - 2\alpha < 2$ we may put $b = 2 - 2\alpha$ and $c = 2$ in the identity (2) discussed in the proof of Lemma 1. This gives

$$(1 - z)^{-(2-2\alpha)} = \frac{\Gamma(2)}{\Gamma(2 - 2\alpha)\Gamma(2\alpha)} \int_0^1 t^{1-2\alpha}(1 - t)^{2\alpha-1}(1 - tz)^{-2}\, dt.$$

Therefore

$$f(z) = \frac{\Gamma(2)}{\Gamma(2 - 2\alpha)\Gamma(2\alpha)} \int_0^1 t^{1-2\alpha}(1 - t)^{2\alpha-1} \frac{z}{1 - tz}\, dt$$

and

$$\frac{f(z)}{z} = \frac{\Gamma(2)}{\Gamma(2 - 2\alpha)\Gamma(2\alpha)} \int_0^1 t^{1-2\alpha}(1 - t)^{2\alpha-1} \frac{1}{1 - tz}\, dt.$$

From the last equation it is clear that min $\{\operatorname{Re} f(z)/z : |z| \leq r < 1\}$ occurs when $z = -r$. (Note that Re $1/(1 - tz) \geq 1/(1 + tr)$ for every $t \in [0, 1]$ and every z with $|z| \leq r$.)

For $\alpha \neq \frac{1}{2}$, $f(z)/z = (1/(1 - 2\alpha))(1/z)[(1 - z)^{2\alpha-1} - 1]$. Hence

$$\min \operatorname{Re}\left\{\frac{f(z)}{z} : |z| \leq r\right\} = \frac{1}{1 - 2\alpha} \frac{1}{|z|}\{1 - (1 + |z|)^{2\alpha-1}\}.$$

For $\alpha = \frac{1}{2}$, $f(z) = -\log(1 - z)$ and min $\operatorname{Re}\{f(z)/z : |z| \leq r\} = (1/|z|)\log(1 + |z|)$. Since the sharpness of the inequalities is evident, this completes the proof of the theorem.

Corollary. *If $f \in K(\alpha)$ and $0 \leq \alpha < 1$, $\alpha \neq \frac{1}{2}$, then*

$$\operatorname{Re}\{f(z)/z\} > (1 - 2^{2\alpha-1})/(1 - 2\alpha) \qquad \text{for } |z| < 1.$$

If $\alpha = \frac{1}{2}$, then $\operatorname{Re}\{f(z)/z\} > \log 2$ *for* $|z| < 1$.

Remark. The result $\operatorname{Re}\{f(z)/z\} > 1/(1 + |z|)$ for f in $K = K(0)$ is also a consequence of what is directly developed in this paper. Specifically, this case of Theorem 4 implies that $f(z)/z = \int_X (1/(1 - xz))\, du(x)$ and hence

$$\operatorname{Re}\frac{f(z)}{z} = \int_X \operatorname{Re}\frac{1}{1 - xz}\, d\mu(x) \geq \int_X \frac{1}{1 + |z|}\, d\mu(x) = \frac{1}{1 + |z|} > \frac{1}{2}.$$

The original proof given in [2] for determining $\mathfrak{H}K$ depended on knowing the result $\operatorname{Re}\{f(z)/z\} > \frac{1}{2}$ for f in K, but that is not needed and the arguments here are independent of that fact.

The following result was proved by R. J. Libera in [6].

Lemma 2. *Let f be analytic and univalent for $|z| < 1, f(0) = 0$, and let $f(\Delta)$ be convex. If $g(z) = (1/z)\int_0^z f(w)\,dw$ then g is also univalent and convex for $|z| < 1$.*

Lemma 3. *Let $K(z,\alpha) = (1-2\alpha)^{-1}[(1-z)^{2\alpha-1} - 1]$, $\alpha \neq \frac{1}{2}$. Let $K(z,\frac{1}{2}) = \log(1/(1-z))$. If $1 > \alpha \geq \frac{1}{2}$, then $K(z,\alpha)/z$ is univalent and convex for $|z| < 1$.*

Proof. We have $K(z,\alpha)/z - 1 = (1/z)\int_0^z (K'(w,\alpha) - 1)\,dw$. Since $K'(z,\alpha) = 1/(1-z)^{2-2\alpha}$, a simple computation shows that $K'(w,\alpha) - 1$ is a convex map for $1 > \alpha \geq \frac{1}{2}$. (Note that $(1-z)^{-p}$ is a convex map for $0 < p \leq 1$.) Hence, by Lemma 2, $K(z,\alpha)/z - 1$ is univalent and convex, and consequently so is $K(z,\alpha)/z$.

Theorem 11. *If $f(z) \in K(a)$ and $\frac{1}{2} \leq \alpha < 1$, then $f(z)/z \prec K(z,\alpha)/z$ in Δ.*

Proof. We know by Theorem 4 that

$$f(z) = \int_X \frac{1}{1-2\alpha}\frac{1}{x}[(1-xz)^{2\alpha-1} - 1]\,d\mu(x) \qquad \text{if } \alpha \neq \tfrac{1}{2}$$

and

$$f(z) = \int_X -\frac{1}{x}\log(1-xz)\,d\mu(x) \qquad \text{for } \alpha = \tfrac{1}{2}.$$

Hence $f(z)/z = \int_X (K(xz,\alpha)/xz)\,d\mu(x)$. By Lemma 3, for $\frac{1}{2} \leq \alpha < 1$, it follows that

$$\frac{f(z)}{z} \prec \frac{K(z,\alpha)}{z} \qquad \text{for } |z| < 1.$$

Remarks. We remark that Theorem 10 (for $\frac{1}{2} \leq \alpha < 1$) follows from this result. We also note that Theorem 10 can be obtained when $\frac{1}{2} \leq \alpha < 1$ by appropriately integrating the lower bound on $\operatorname{Re} f'(z)$ for any $f(z)$ in $K(\alpha)$. One uses the known fact that $f'(z) \prec 1/(1-z)^{2-2\alpha}$ to obtain the lower bound on $\operatorname{Re} f'(z)$. This approach is independent of knowledge of the extreme points of $K(\alpha)$.

We conjecture that Theorem 11 holds for all α, $0 \leq \alpha < 1$. This would be a consequence of the expected result that $K(z,\alpha)/z$ is univalent and convex for $|z| < 1$ for $0 < \alpha < \frac{1}{2}$. Writing $H(z,\alpha) = K(z,\alpha)/z$ this is equivalent to showing that $zH'(z,\alpha)$ is univalent and starlike. We can show by lengthy computation that $zH'(z,\alpha)$ is univalent for $|z| < 1$, which, incidentally, affords another proof of Theorem 10.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT ALBANY, ALBANY, NEW YORK 12222