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APPELL POLYNOMIALS AND DIFFERENTIAL EQUATIONS OF INFINITE ORDER

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ABSTRACT. Let $\Phi(z) = \sum_0^\infty \beta_j z^j$ have radius of convergence r $(0 < r < \infty)$ and no singularities other than poles on the circle $|z| = r$. A complete solution is obtained for the infinite order differential equation $(*)$ $\sum_0^\infty \beta_j u^{(j)}(z) = g(z)$. It is shown that $(*)$ possesses a solution if and only if the function g has a polynomial expansion in terms of the Appell polynomials generated by Φ. The solutions of $(*)$ are expressed in terms of the coefficients which appear in the Appell polynomial expansions of g. An alternate method of solution is obtained, in which the problem of solving $(*)$ is reduced to the problem of finding a solution, within a certain space of entire functions, of a finite order linear differential equation with constant coefficients. Additionally, differential operator techniques are used to study Appell polynomial expansions.

1. **Introduction.** Let $\Phi(z) = \sum_0^\infty \beta_j z^j$ have radius of convergence r $(0 < r < \infty)$ and no singularities other than poles on the circle $|z| = r$. The differential equation we study is

$$\Phi(D)u = g, \tag{1.1}$$

where D denotes the derivative operator and $\Phi(D)$ is the operator which transforms the analytic function u into

$$\sum_{j=0}^{\infty} \beta_j u^{(j)}(z). \tag{1.2}$$

We take for the domain of $\Phi(D)$ the family of entire functions u such that (1.2) is uniformly convergent on every compact set. That very little is lost by this requirement is apparent from the following lemma.

Lemma 1.1. *If v is analytic on a region Ω and $\sum_0^\infty \beta_j v^{(j)}(z)$ is uniformly convergent on some open subset of Ω, then v is the restriction to Ω of an entire function u, and $\sum_0^\infty \beta_j u^{(j)}(z)$ is uniformly convergent on every compact set.*

In the present paper, we are able to determine both the domain and range of $\Phi(D)$ and to obtain a complete solution of (1.1). There is a deep connection between this problem and the problem of expanding an entire function in an

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infinite series of Appell polynomials. The Appell polynomials generated by the function Φ are given by

$$\pi_k(z) = \sum_{j=0}^{k} \beta_{k-j} \frac{z^j}{j!}, \qquad k = 0, 1, 2, \ldots .$$

An entire function g is said to have a $\{\pi_k\}$ expansion if there is a complex sequence $h = \{h_k\}_0^\infty$ such that $\sum_{k=0}^{\infty} h_k \pi_k(z)$ is uniformly convergent on compact sets to $g(z)$. We denote by $\mathcal{E}$ the family of entire functions g which possess $\{\pi_k\}$ expansions.

Let $1/\alpha_q$, $1 \leq q \leq \lambda$, denote the (distinct) poles of Φ on $|z| = r$, and let $m(q)$ denote the order of the pole $1/\alpha_q$. We denote by H the collection of complex sequences h such that each of $\sum_{k=0}^{\infty} \binom{k+m(q)-1}{m(q)-1} \alpha_q^k h_k$ converges, $1 \leq q \leq \lambda$, and we let $\mathcal{U}$ denote the family of entire functions u such that $\{u^{(j)}(0)\}_0^\infty$ belongs to H. Note that every member of $\mathcal{U}$ has exponential type r or less, and that every function of exponential type less than r belongs to $\mathcal{U}$.

Theorem 1.1. *The operator $\Phi(D)$ has domain $\mathcal{U}$ and range $\mathcal{E}$. If the function $g \in \mathcal{E}$ has the Appell expansion $g(z) = \sum_{k=0}^{\infty} h_k \pi_k(z)$, then the function*

$$u(z) = \sum_{k=0}^{\infty} h_k \frac{z^k}{k!}$$

satisfies $\Phi(D)u = g$. Conversely, if $g \in \mathcal{E}$ and u satisfies $\Phi(D)u = g$, then g has the Appell expansion

$$g(z) = \sum_{k=0}^{\infty} u^{(k)}(0)\pi_k(z).$$

Theorem 1.1 reduces the problem of solving $\Phi(D)u = g$ to the problem of expanding g in an Appell series. This problem was solved in [2] and [3]. Before discussing its solution, we consider the homogeneous equation $\Phi(D)u = 0$. Let $\{w_i\}_{i=1}^s$ denote the zeros (according to multiplicity) of Φ in the open disk $|z| < r$, and set $T_0(z) = \prod_{i=1}^s (z - w_i)$, with the convention that $T_0(z) \equiv 1$ if Φ is zero free in $|z| < r$.

Theorem 1.2. $\Phi(D)u = 0$ *if and only if* $T_0(D)u = 0$.

In view of Theorem 1.1, u satisfies $\Phi(D)u = 0$ if and only if $\sum_{k=0}^{\infty} u^{(k)}(0)\pi_k(z)$ is uniformly convergent to 0 on every compact set. The Appell series with this property were characterized in [3]. Theorem 1.2 is, therefore, an immediate consequence of Theorem C of [3].

Let $\Phi(z) = T(z)\varphi(z)$, where T is a polynomial with no zero outside the closed disk $|z| \leq r$, and φ has no zero in $|z| \leq r$. Set

$$Q(z) = \prod_{q=1}^{\lambda} (1 - \alpha_q z)^{\min\{m(q), m-1\}},$$

where $m = \max_{1 \le q \le \lambda} m(q)$ denotes the largest order of a pole of Φ on $|z| = r$. We denote by $\mathcal{G}$ the linear space of entire functions f which satisfy

$$\lim_{n\to\infty} r^{-n}(D^n f)(0) = \lim_{n\to\infty} r^{-n} n^{m-1}(D^n Q(D) f)(0) = 0.$$

In the special case that Φ has only simple poles on $|z| = r$, we have $m = 1$, $Q(z) \equiv 1$, and $\mathcal{G}$ is the space of entire functions f which satisfy

$$f^{(n)}(0) = o(r^n), \qquad n \to \infty. \tag{1.3}$$

In general, (1.3) is a necessary condition that $f \in \mathcal{G}$ and the condition

$$f^{(n)}(0) = o(r^n/n^{m-1}), \qquad n \to \infty,$$

is sufficient.

It was shown in [3] that $\mathcal{E}$ is the image of $\mathcal{G}$ under the differential operator $T(D)$. In particular, if Φ is zero free in the closed disk $|z| \le r$, then $\mathcal{E}$ *is* the space $\mathcal{G}$. We shall prove here that $\mathcal{E} = \mathcal{G}$ under the much weaker condition that Φ has no zero on the circle $|z| = r$. This is an immediate consequence of the following result, together with the observation that $T_0 = T$ if Φ has no zero on the circle $|z| = r$.

Theorem 1.3. *The operator $T_0(D)$ maps $\mathcal{G}$ onto $\mathcal{G}$.*

Corollary 1.1. *Let $T_1(z) = T(z)/T_0(z)$. The operator $T_1(D)$ is a* 1-1 *linear map of $\mathcal{G}$ onto $\mathcal{E}$.*

As a consequence of Corollary 1.1, we note that the space $\mathcal{E}$ is completely determined by the zeros and poles of Φ on its circle of convergence.

It is not hard to show that $\mathcal{E}$ is contained in $\mathcal{G}$, and, therefore, that every member of $\mathcal{E}$ has exponential type r or less. It has been known for a long time [4] that every function of exponential type less than r possesses a $\{\pi_k\}$ expansion. Our differential operator approach yields a slightly stronger version of this important result. Let d denote the largest integer which is the multiplicity of a zero of T_1 (if T_1 is constant, we take $d = 0$).

Theorem 1.4. *If $d > 0$ and $m = 1$, then $\mathcal{E}$ contains every entire function g such that*

$$\sum_{n=0}^{\infty} |g^{(n)}(0)| \frac{n^{d-1}}{r^n} < \infty.$$

If $m > 1$ or $d = 0$, then $\mathcal{E}$ contains every entire function g such that

$$g^{(n)}(0) = o(r^n/n^{m+d-1}), \qquad n \to \infty.$$

In either case, $\mathcal{E}$ contains every entire function g such that

$$g^{(n)}(0) = o(r^n/n^{m+d}), \qquad n \to \infty.$$

The results in [2] pertaining to mapping properties of infinite matrices (regarded as sequence-to-sequence operators) possess differential operator analogues. Let $\theta(z) = 1/\varphi(z) = \sum_{j=0}^{\infty} a_j z^j$, and note that θ is analytic in the closed disk $|z| \leq r$. This insures that the domain of the differential operator $\theta(D)$ includes all entire functions of exponential type r or less.

Theorem 1.5. *The operator* $\varphi(D)$ *has domain* $\mathcal{U}$ *and is a* 1-1 *linear map of* $\mathcal{U}$ *onto* $\mathcal{J}$. *Its inverse is the restriction of* $\theta(D)$ *to* $\mathcal{J}$. *The operator* $\Phi(D)$ *has domain* $\mathcal{U}$ *and satisfies* $\Phi(D)u = T(D)\{\varphi(D)u\}$ *for every* $u \in \mathcal{U}$.

As a consequence of Theorem 1.5, we see that the differential equation $\varphi(D)u = f$ has a solution if and only if $f \in \mathcal{J}$. If $f \in \mathcal{J}$, then one obtains the unique solution of

$$\varphi(D)u = f \tag{1.4}$$

by "dividing" both sides of (1.4) by $\varphi(D)$, i.e., $u = \theta(D)f$.

Theorem 1.5 also allows us to obtain a solution of $\Phi(D)u = g$ by reducing it to a finite order linear differential equation with constant coefficients. The following is an easy consequence of Theorem 1.5.

Theorem 1.6. *Suppose that* g *is an entire function. The infinite order differential equation*

$$\Phi(D)u = g \tag{1.5}$$

has a solution if and only if the differential equation

$$T(D)f = g$$

has a solution f *which belongs to* $\mathcal{J}$. *If* $f \in \mathcal{J}$ *and* $T(D)f = g$, *then* $u = \theta(D)f$ *is a solution of* (1.5). *Conversely, if* u *is a solution of* (1.5), *then the function* $f = \varphi(D)u$ *belongs to* $\mathcal{J}$, *satisfies* $T(D)f = g$, *and has the property that* $u = \theta(D)f$.

Theorem 1.6, while very nearly a restatement of Theorem 1.1, does not depend on Appell polynomials, and is, to that extent, a more "natural" solution of (1.5).

2. **Proofs of Theorems 1.1 and 1.3.** It was established in [2] that a sequence h belongs to H if and only if each of the series $\sum_{j=k}^{\infty} \beta_{j-k} h_j$, $0 \leq k < \infty$, converges (the additional hypothesis in [2] that Φ have no zeros in the disk $|z| \leq r$ was not used in the proof of this).

Lemma 2.1. *Suppose* v *is analytic on some region containing the complex number* w, *and*

$$\sum_{j=0}^{\infty} \beta_j v^{(j)}(z) \tag{2.1}$$

is uniformly convergent on a neighborhood of w. *Then the sequence* $\{v^{(j)}(w)\}_0^{\infty}$ *belongs to* H, *and* v *is the restriction of an entire function.*

Proof. Let k denote a nonnegative integer, differentiate (2.1) k times, and set $z = w$. Therefore

$$\sum_{j=0}^{\infty} \beta_j v^{(j+k)}(w) = \sum_{j=k}^{\infty} \beta_{j-k} v^{(j)}(w)$$

converges, so that $\{v^{(j)}(w)\}_0^\infty$ belongs to H. From the definition of H, we obtain

$$\limsup_{j\to\infty} |v^{(j)}(w)|^{1/j} \leq r. \tag{2.2}$$

Therefore the series

$$u(z) = \sum_{j=0}^{\infty} v^{(j)}(w)\frac{(z-w)^j}{j!}$$

defines an entire function u which agrees with v on a neighborhood of w. Note that (2.2) implies that u is of exponential type r or less.

Theorem 2.1. *If u is entire, w is a complex number, and $\{u^{(j)}(w)\}_0^\infty$ belongs to H, then*

$$\sum_{j=0}^{\infty} \beta_j u^{(j)}(z) = \sum_{k=0}^{\infty} u^{(k)}(w)\pi_k(z-w)$$

for all z, and both series are uniformly convergent on every compact set.

Proof. It is no loss of generality to take $w = 0$. To see this, replace z by $z + w$ and let $u_1(z) = u(z + w)$.

Taking $w = 0$, we have that $\{u^{(j)}(w)\}_0^\infty = \{u^{(j)}(0)\}_0^\infty$ belongs to H, and it follows from this [2] that $\sum_{k=0}^{\infty} u^{(k)}(0)\pi_k(z)$ is uniformly convergent on compact sets to an entire function g. To complete the proof, we show that

$$\lim_{n\to\infty}\left\{g(z) - \sum_{j=0}^{n} \beta_j u^{(j)}(z)\right\} = 0$$

uniformly on compact sets.

For notational simplicity, set $h_k = u^{(k)}(0)$. We have

$$\begin{aligned}
\sum_{j=0}^{n} \beta_j u^{(j)}(z) &= \sum_{j=0}^{n} \beta_j \sum_{k=j}^{\infty} h_k \frac{z^{k-j}}{(k-j)!} \\
&= \sum_{k=0}^{\infty} h_k \sum_{j=0}^{\min\{n,k\}} \beta_j \frac{z^{k-j}}{(k-j)!} \\
&= \sum_{k=0}^{n} h_k \sum_{j=0}^{k} \beta_j \frac{z^{k-j}}{(k-j)!} + \sum_{k=n+1}^{\infty} h_k \sum_{j=0}^{n} \beta_j \frac{z^{k-j}}{(k-j)!} \\
&= \sum_{k=0}^{n} h_k \pi_k(z) + \sum_{k=n+1}^{\infty} h_k \sum_{j=0}^{n} \beta_j \frac{z^{k-j}}{(k-j)!}.
\end{aligned}$$

Therefore

$$g(z) - \sum_{j=0}^{n} \beta_j u^{(j)}(z) = \sum_{k=n+1}^{\infty} h_k \left\{ \pi_k(z) - \sum_{j=0}^{n} \beta_j \frac{z^{k-j}}{(k-j)!} \right\}$$
$$= \sum_{k=n+1}^{\infty} h_k \sum_{j=n+1}^{k} \beta_j \frac{z^{k-j}}{(k-j)!}.$$

To estimate this quantity, we use the poles of Φ on $|z| = r$ to obtain an asymptotic estimate for β_j. For $1 \leq q \leq \lambda$, let

$$\sum_{t=1}^{m(q)} \frac{c_t(q)}{(1 - \alpha_q z)^{m(q)+1-t}}$$

denote the sum of the negative powers in the Laurent expansion of Φ at α_q^{-1}. Then

$$\Phi(z) = \sum_{q=1}^{\lambda} \sum_{t=1}^{m(q)} \frac{c_t(q)}{(1 - \alpha_q z)^{m(q)+1-t}} + \Phi_2(z),$$

where $\Phi_2(z) = \sum_{j=0}^{\infty} \beta_j' z^j$ is analytic in a disk $|z| \leq r_1$, $r_1 > r$. We then have

$$\beta_j = \sum_{q=1}^{\lambda} \sum_{t=1}^{m(q)} c_t(q) \binom{j + m(q) - t}{m(q) - t} \alpha_q^j + \beta_j'.$$

Therefore

$$\sum_{k=n+1}^{\infty} h_k \sum_{j=n+1}^{k} \beta_j \frac{z^{k-j}}{(k-j)!} = R_n(z) + \sum_{q=1}^{\lambda} \sum_{t=1}^{m(q)} U_{qtn}(z),$$

where $R_n(z) = \sum_{k=n+1}^{\infty} h_k \sum_{j=n+1}^{k} \beta_j' z^{k-j}/(k-j)!$ and

$$U_{qtn}(z) = \sum_{k=n+1}^{\infty} h_k \sum_{j=n+1}^{k} \binom{j + m(q) - t}{m(q) - t} \alpha_q^j \frac{z^{k-j}}{(k-j)!}.$$

The proof that $\lim_{n\to\infty} R_n(z) = 0$ uniformly on compact sets is straightforward and is omitted. The following lemma completes the proof of Theorem 2.1.

Lemma 2.2. *If* $1 \leq q \leq \lambda$ *and* $1 \leq t \leq m(q)$, *then* $\lim_{n\to\infty} U_{qtn}(z) = 0$ *uniformly on compact sets.*

Proof of Lemma 2.2. From [2, Lemma 2.7] we have

$$|U_{qtn}(z)| \leq V(z) \sup_{0 \leq i < \infty} \left| \sum_{k=n+1+i}^{\infty} \binom{k + m(q) - 1}{m(q) - 1} \alpha_q^k h_k \right|,$$

where $V(z)$ denotes the total variation of the sequence $\{y_k\}_{k=n+1}^{\infty}$ given by

$$y_k = \sum_{j=n+1}^{k} \binom{j + m(q) - t}{m(q) - t} \binom{k + m(q) - 1}{m(q) - 1}^{-1} \frac{(z/\alpha_q)^{k-j}}{(k-j)!}.$$

Replace the summation index j by $k-j$ and set $y'_k = y_{k+n+1}$. After some simplification, one obtains

$$y'_k = \sum_{j=0}^{k} \binom{k+n+1-j+m(q)-t}{m(q)-t} \binom{k+n+m(q)}{m(q)-1}^{-1} \frac{(z/\alpha_q)^j}{j!}.$$

The sequence $\{y'_k\}_{k=0}^{\infty}$ has total variation $V(z)$; also, $|y'_0| = \binom{n+1+m(q)-t}{m(q)-t}\binom{n+m(q)}{m(q)-1}^{-1} \leq 1$. Set

$$x_{kj} = \binom{k+n+1-j+m(q)-t}{m(q)-t} \binom{k+n+m(q)}{m(q)-1}^{-1}, \quad 0 \leq j \leq k < \infty.$$

Then

$$y'_k - y'_{k+1} = \sum_{j=0}^{k} (x_{kj} - x_{k+1,j}) \frac{(z/\alpha_q)^j}{j!} - x_{k+1,k+1} \frac{(z/\alpha_q)^{k+1}}{(k+1)!},$$

and

$$\left| x_{k+1,k+1} \frac{(z/\alpha_q)^{k+1}}{(k+1)!} \right| \leq \frac{|z/\alpha_q|^{k+1}}{(k+1)!}.$$

Therefore

$$\begin{aligned} V(z) &= |y'_0| + \sum_{k=0}^{\infty} |y'_k - y'_{k+1}| \\ &\leq 1 + \sum_{k=0}^{\infty} \frac{|z/\alpha_q|^{k+1}}{(k+1)!} + \sum_{k=0}^{\infty} \sum_{j=0}^{k} |x_{kj} - x_{k+1,j}| \frac{|z/\alpha_q|^j}{j!} \\ &= \exp|z/\alpha_q| + \sum_{j=0}^{\infty} \frac{|z/\alpha_q|^j}{j!} \sum_{k=j}^{\infty} |x_{kj} - x_{k+1,j}|. \end{aligned}$$

The method used to prove Lemma 3.2 of [2] allows one to establish that $\sum_{k=j}^{\infty} |x_{kj} - x_{k+1,j}| \leq 2$. Therefore $V(z) \leq 3\exp|z/\alpha_q| = 3e^{|rz|}$. This completes the proof of Lemma 2.2 and also the proof of Theorem 2.1. Note that Lemma 1.1 is a consequence of Lemma 2.1 and Theorem 2.1.

Proof of Theorem 1.1. It follows from Theorem 2.1 (with $w = 0$) that $\mathcal{U}$ is contained in the domain of $\Phi(D)$, and from Lemma 2.1 (with $w = 0$) that $\mathcal{U}$ contains the domain of $\Phi(D)$. Therefore the domain of $\Phi(D)$ is $\mathcal{U}$; using Theorem 2.1 again, we see that the range of $\Phi(D)$ is contained in $\mathcal{E}$. Suppose now that $g \in \mathcal{E}$. Then there is a complex sequence h such that $g(z) = \sum_{k=0}^{\infty} h_k \pi_k(z)$ uniformly on compact sets. From the Convergence Theorem in [2] it follows that $h \in H$. Consequently the function $u(z) = \sum_{k=0}^{\infty} h_k z^k/k!$ belongs to $\mathcal{U}$, and from Theorem 2.1 we have

$$(\Phi(D)u)(z) = \sum_{k=0}^{\infty} u^{(k)}(0)\pi_k(z) = \sum_{k=0}^{\infty} h_k \pi_k(z) = g(z).$$

Therefore the range of $\Phi(D)$ is $\mathcal{E}$. Suppose now that $g \in \mathcal{E}$ and that u satisfies $\Phi(D)u = g$. From Theorem 2.1 we obtain

$$g(z) = (\Phi(D)u)(z) = \sum_{k=0}^{\infty} u^{(k)}(0)\pi_k(z),$$

and the proof of Theorem 1.1 is complete.

Proof of Theorem 1.3. If $T_0(z) \equiv 1$, there is nothing to prove, since in this case $T_0(D) = I$, the identity operator. If $T_0(z) = \prod_{i=1}^{s} (z - w_i)$, $s \geq 1$, we have the operator factorization $T_0(D) = \prod_{i=1}^{s} (D - w_i I)$, so that it is sufficient to prove that, if $|w_0| < r$, then $D - w_0 I$ maps $\mathcal{G}$ onto $\mathcal{G}$. Clearly $D - w_0 I$ maps $\mathcal{G}$ into $\mathcal{G}$, since $\mathcal{G}$ is a linear space which is closed under differentiation. If $f \in \mathcal{G}$, then so does $\sum_{j=1}^{\infty} f^{(j-1)}(0)z^j/j!$; therefore we can exclude the case $w_0 = 0$ and reduce the problem to that of showing that every $F \in \mathcal{G}$ is the image of some $f \in \mathcal{G}$ under the transformation $wD - I$, where w is a complex constant such that $|w| > 1/r$.

Suppose that $F \in \mathcal{G}$; the differential equation with boundary condition

$$wf' - f = F, \qquad f(0) = 0, \tag{2.3}$$

has a unique solution f. We shall show that this function belongs to $\mathcal{G}$. Set $Q(z) = \sum_{t=0}^{\tau} C_t z^t$; it is easy to verify that a function g belongs to $\mathcal{G}$ if and only if

$$\lim_{n\to\infty} r^{-n} g^{(n)}(0) = \lim_{n\to\infty} n^{m-1} r^{-n} \sum_{t=0}^{\tau} C_t g^{(n+t)}(0) = 0. \tag{2.4}$$

Let k denote a positive integer, differentiate (2.3) k times, multiply by w^k, and set $z = 0$. This yields

$$w^{k+1} f^{(k+1)}(0) - w^k f^{(k)}(0) = w^k F^{(k)}(0). \tag{2.5}$$

Since $f(0) = 0$, we have $wf'(0) = F(0)$, which corresponds to $k = 0$ in (2.5). If we sum equation (2.5) over $0 \leq k < n$, we obtain

$$w^n f^{(n)}(0) = \sum_{k=0}^{n-1} w^k F^{(k)}(0).$$

Therefore the solution of (2.3) satisfies

$$f^{(n)}(0) = \sum_{k=0}^{n-1} w^{k-n} F^{(k)}(0), \qquad 1 \leq n < \infty.$$

To complete the proof, we use the fact that (2.4) holds if $g = F$ to prove that (2.4) holds if $g = f$.

We have

$$r^{-n} f^{(n)}(0) = \sum_{k=0}^{n-1} (rw)^{k-n} r^{-k} F^{(k)}(0).$$

Since $|rw| > 1$, the infinite matrix

$$\begin{aligned} M_{nk} &= (rw)^{k-n}, \qquad 0 \le k < n, \\ &= 0, \qquad\qquad k \ge n, \end{aligned}$$

transforms sequences with limit 0 into sequences with limit 0. Therefore $\lim_{k\to\infty} r^{-k}F^{(k)}(0) = 0$ implies $\lim_{n\to\infty} r^{-n}f^{(n)}(0) = 0$.

The remainder of the proof is similar in nature but more difficult. We again use the fact that sufficient (and also necessary) conditions for an infinite matrix N to transform sequences with limit 0 into sequences with limit 0 are that

(i) $$\lim_{n\to\infty} N_{nk} = 0, \qquad k = 0, 1, 2, \ldots,$$

and

(ii) $$\sup_{0\le n<\infty} \sum_{k=0}^{\infty} |N_{nk}| < \infty.$$

We have

$$n^{m-1}r^{-n}\sum_{t=0}^{\tau} C_t f^{(n+t)}(0) = n^{m-1}r^{-n}\sum_{t=0}^{\tau} C_t \sum_{k=0}^{n+t-1} w^{k-n-t}F^{(k)}(0).$$

Now

$$\begin{aligned} \sum_{t=0}^{\tau} C_t \sum_{k=0}^{n+t-1} w^{k-n-t}F^{(k)}(0) &= \sum_{t=0}^{\tau} C_t \sum_{k=-t}^{n-1} w^{k-n}F^{(k+t)}(0) \\ &= \sum_{t=0}^{\tau} C_t \sum_{k=-t}^{0} w^{k-n}F^{(k+t)}(0) + \sum_{t=0}^{\tau} C_t \sum_{k=1}^{n-1} w^{k-n}F^{(k+t)}(0). \end{aligned}$$

It is easy to show that

$$\lim_{n\to\infty} n^{m-1}r^{-n}\sum_{t=0}^{\tau} C_t \sum_{k=-t}^{0} w^{k-n}F^{(k+t)}(0) = 0.$$

Now

$$\begin{aligned} n^{m-1}r^{-n}\sum_{t=0}^{\tau} C_t \sum_{k=1}^{n-1} w^{k-n}F^{(k+t)}(0) &= n^{m-1}(rw)^{-n}\sum_{k=1}^{n-1} w^k \sum_{t=0}^{\tau} C_t F^{(k+t)}(0) \\ &= \frac{n^{m-1}}{(rw)^n}\sum_{k=1}^{n-1}\frac{(rw)^k}{k^{m-1}}k^{m-1}r^{-k}\sum_{t=0}^{\tau} C_t F^{(k+t)}(0) \\ &= \sum_{k=1}^{n-1}(rw)^{k-n}\left(\frac{n}{k}\right)^{m-1}\left\{k^{m-1}r^{-k}\sum_{t=0}^{\tau} C_t F^{(k+t)}(0)\right\}. \end{aligned}$$

To complete the proof, it suffices to show that the infinite matrix

$$N_{nk} = 0 \qquad \text{if } k = 0 \text{ or } k \geq n,$$
$$= (rw)^{k-n}(n/k)^{m-1} \qquad \text{if } 1 < k < n,$$

has properties (i) and (ii). Property (i) is obvious. To establish (ii), set $\rho = |rw|^{-1} < 1$. Then

$$\sum_{k=0}^{\infty} |N_{nk}| = \sum_{k=1}^{n-1} \rho^{n-k}\left(\frac{n}{k}\right)^{m-1}$$
$$= \sum_{k=1}^{n-1} \rho^{k}\left(\frac{n}{n-k}\right)^{m-1} = \sum_{k=1}^{n-1} k^{m-1}\rho^{k}\left\{\frac{n}{k(n-k)}\right\}^{m-1}.$$

Since $k(n-k)$ is least at $k = 1$ and $k = n-1$, we have

$$\sum_{k=0}^{\infty} |N_{nk}| \leq \left(\frac{n}{n-1}\right)^{m-1} \sum_{k=1}^{n-1} k^{m-1}\rho^{k} \leq 2^{m-1} \sum_{k=1}^{\infty} k^{m-1}\rho^{k},$$

and this completes the proof.

Proof of Corollary 1.1. The operator $T(D)$ maps $\mathcal{G}$ onto $\mathcal{E}$ and has the factorization $T(D) = T_1(D)\{T_0(D)\}$. In view of Theorem 1.3, we need only show that the restriction of $T_1(D)$ to $\mathcal{G}$ is 1-1. If $T_1(z)$ is a constant, there is nothing to prove. If not, $T_1(D)$ has a factorization into differential operators of order 1. It is easily verified that these differential operators are 1-1 on $\mathcal{G}$, and this completes the proof.

3. **Proofs of Theorems 1.5 and 1.6.** Let $v(z) = v_1(z)v_2(z)$, where v_1 and v_2 are functions analytic in a neighborhood of 0. For convenience, we shall sometimes use $v_1(D)v_2(D)$ to denote the operator $v(D)$. Since we shall always denote operator composition by $v_1(D)\{v_2(D)\}$, no confusion will result.

Lemma 3.1. *If v_1 and v_2 are functions analytic in the closed disk $|z| \leq r$, then $v_1(D)\{v_2(D)f\} = \{v_1(D)v_2(D)\}f$ for every entire function f of exponential type r or less.*

Proof. Suppose that f is of exponential type r or less. For $\delta > 0$ we have

$$\limsup_{j\to\infty} |f^{(j)}(0)|^{1/j} \leq r < r(1+\delta).$$

Therefore the quantity $K_\delta = \sup_{0\leq j<\infty} |f^{(j)}(0)|/\{r(1+\delta)\}^j$ is finite. Consequently,

$$|f^{(k)}(z)| \leq \sum_{j=k}^{\infty} |f^{(j)}(0)| \frac{|z|^{j-k}}{(j-k)!}$$
$$\leq K_\delta\{r(1+\delta)\}^k \sum_{j=k}^{\infty} \{r(1+\delta)\}^{j-k}\frac{|z|^{j-k}}{(j-k)!}$$
$$= K_\delta\{r(1+\delta)\}^k \exp\{r(1+\delta)|z|\}.$$

Suppose that v is analytic in the closed disk $|z| \leq r$. Choose $\delta > 0$ so that v is analytic in the closed disk $|z| \leq r(1 + \delta)$. The previous estimate insures that

$$\{v(D)f\}(z) = \sum_{k=0}^{\infty} \frac{v^{(k)}(0)}{k!} f^{(k)}(z)$$

is uniformly convergent on every compact set. Note also that

$$|\{v(D)f\}(z)| \leq K_\delta \exp\{r(1 + \delta)|z|\} \sum_{k=0}^{\infty} \frac{|v^{(k)}(0)|}{k!} \{r(1 + \delta)\}^k,$$

so that $v(D)f$ is of exponential type $r(1 + \delta)$ or less; consequently $v(D)f$ is of exponential type r or less since δ is arbitrary.

Now

$$\begin{aligned}
\{v_1(D)\{v_2(D)f\}\}(z) &= \sum_{j=0}^{\infty} \frac{v_1^{(j)}(0)}{j!} \sum_{k=0}^{\infty} \frac{v_2^{(k)}(0)}{k!} f^{(k+j)}(z) \\
&= \sum_{j=0}^{\infty} \frac{v_1^{(j)}(0)}{j!} \sum_{k=j}^{\infty} \frac{v_2^{(k-j)}(0)}{(k-j)!} f^{(k)}(z) \\
&= \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} \frac{v_1^{(j)}(0)}{j!} \frac{v_2^{(k-j)}(0)}{(k-j)!} \right\} f^{(k)}(z) \\
&= \{\{v_1(D)v_2(D)\}f\}(z),
\end{aligned}$$

provided that the interchange in order of summation is valid. To see that this is the case, note that

$$\sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} \frac{|v_1^{(j)}(0)|}{j!} \frac{|v_2^{(k-j)}(0)|}{(k-j)!} \right\} |f^{(k)}(z)|$$

does not exceed

$$K_\delta \exp\{r(1 + \delta)|z|\} \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} \frac{|v_1^{(j)}(0)|}{j!} \frac{|v_2^{(k-j)}(0)|}{(k-j)!} \right\} \{r(1 + \delta)\}^k,$$

which is finite, provided that δ is chosen so that both v_1 and v_2 are analytic in the closed disk $|z| \leq r(1 + \delta)$.

Lemma 3.2. *If v_1 is polynomial and v_2 is analytic in a neighborhood of 0, then $v_1(D)\{v_2(D)f\} = \{v_1(D)v_2(D)\}f$ for every f in the domain of $v_2(D)$.*

Proof. The interchange in order of summation is trivial to verify in this case, since the "outer" sum is finite.

Theorem 3.1. *If $u \in \mathcal{U}$, then $\theta(D)\{\varphi(D)u\} = u$.*

Proof. Set $P(z) = \prod_{q=1}^{\lambda} (1 - \alpha_q z)^{m(q)}$ and note that the functions $v_1 = \theta/P$, $v_2 = P$, and $v_3 = P\varphi$ are analytic in the closed disk $|z| \leq r$. In the proof of

Theorem 1.1 the possibility that $T(z) \equiv 1$ is not excluded. Therefore $\varphi(D)$ has domain $\mathcal{U}$ and range $\mathcal{G}$. From Lemma 3.2 we have $P(D)\{\varphi(D)u\} = v_3(D)u$. The function $f = \varphi(D)u$ is of exponential type r or less; from Lemma 3.1 we obtain

$$v_1(D)\{P(D)f\} = \theta(D)f = \theta(D)\{\varphi(D)u\}.$$

Since u is of exponential type r or less, we have

$$\begin{aligned}\theta(D)\{\varphi(D)u\} &= v_1(D)\{P(D)f\} = v_1(D)\{P(D)\{\varphi(D)u\}\} \\ &= v_1(D)\{v_3(D)u\} = \{v_1(D)v_3(D)\}u = u.\end{aligned}$$

Proof of Theorem 1.5. Most of Theorem 1.5 follows from Theorem 1.1, since in Theorem 1.1 the case $T(z) \equiv 1$ is not excluded, and in this case $\Phi = \varphi$. The factorization $\Phi(D)u = T(D)\{\varphi(D)u\}$ is a consequence of Lemma 3.1. That $\varphi(D)$ is 1-1 follows from Theorem 3.1, as does the assertion that the inverse of $\varphi(D)$ is the restriction to $\mathcal{G}$ of $\theta(D)$.

Proof of Theorem 1.6. The second sentence of Theorem 1.6 is equivalent to the assertion that $\mathcal{E}$ is the range of $\Phi(D)$, and this has already been established. Suppose that $f \in \mathcal{G}$ and $T(D)f = g$. Set $u = \theta(D)f$. From Theorem 1.5 we have $u \in \mathcal{U}$ and $\Phi(D)u = T(D)\{\varphi(D)u\} = T(D)f = g$.

Suppose now that u is such that $\Phi(D)u = g$. From Theorem 1.1 we have $u \in \mathcal{U}$. Therefore the function $f = \varphi(D)u$ belongs to $\mathcal{G}$ by Theorem 1.5. Also from Theorem 1.5 we obtain $u = \theta(D)f$ and

$$g = \Phi(D)u = T(D)\{\varphi(D)u\} = T(D)f,$$

which completes the proof.

4. **Proof of Theorem 1.4.** Suppose that $m = 1$ and $d > 0$. Set

$$S(z) = \frac{1}{T_1(z)} = \sum_{j=0}^{\infty} s_j z^j.$$

Since S is zero free and has only poles on its circle of convergence, the operators $S(D)$ and $T_1(D)$ are controlled by Theorem 1.5. Let $\mathcal{U}(S)$ and $\mathcal{G}(S)$ denote the spaces obtained from $\mathcal{U}$ and $\mathcal{G}$ by replacing Φ by S. Since $m = 1$, $\mathcal{G}$ is the collection of all entire f such that

$$f^{(n)}(0) = o(r^n), \qquad n \to \infty. \tag{4.1}$$

Every member of $\mathcal{G}(S)$ must satisfy (4.1) and one other growth condition; therefore $\mathcal{G}(S)$ is contained in $\mathcal{G}$. If one writes out explicitly the condition that a function g belongs to $\mathcal{U}(S)$, it follows easily that $\mathcal{U}(S)$ contains every entire g such that

$$\sum_{n=0}^{\infty} |g^{(n)}(0)| \frac{n^{d-1}}{r^n} < \infty. \tag{4.2}$$

From Theorem 1.5 we see that $T_1(D)$ maps $\mathcal{G}(S)$ onto $\mathcal{U}(S)$. Therefore every g which satisfies (4.2) is the image under $T_1(D)$ of an f which belongs to $\mathcal{G}(S)$, and therefore to $\mathcal{G}$. Since $\mathcal{E}$ is the image of $\mathcal{G}$ under $T_1(D)$, the proof of the first half is complete.

Suppose that $m > 1$. If $d = 0$, there is nothing to prove, since in this case $\mathcal{E} = \mathcal{G}$. Since no pole of S on $|z| = r$ is of order greater than d, we have the estimate

$$S^{(j)}(0)/j! = s_j = O(j^{d-1}/r^j), \qquad j \to \infty.$$

Since $m \geq 2$, the growth condition $g^{(j)}(0) = o(r^j/j^{m+d-1}), j \to \infty$, implies

$$g^{(j)}(0) = o(r^j/j^{d+1}), \qquad j \to \infty. \tag{4.3}$$

We see from Theorem 1.1 that (4.3) is sufficient to guarantee that g belongs to $\mathcal{U}(S)$, the domain of $S(D)$. Set $f = S(D)g$. From Lemma 3.2 we have $g = T_1(D)\{S(D)g\} = T_1(D)f$. To complete the proof, we need only show that $f \in \mathcal{G}$. We establish this by showing that $f^{(n)}(0) = o(r^n/n^{m-1})$, $n \to \infty$.

We have

$$f^{(n)}(0) = (S(D)g)^{(n)}(0) = \sum_{j=n}^{\infty} s_{j-n} g^{(j)}(0).$$

Therefore

$$\begin{aligned} |r^{-n} f^{(n)}(0)| &= \left| \sum_{j=n}^{\infty} \binom{j+m}{m} g^{(j)}(0) s_{j-n} r^{-n} \binom{j+m}{m}^{-1} \right| \\ &\leq \left\{ \sup_{j \geq n} \left| \binom{j+m}{m} g^{(j)}(0) s_{j-n} r^{-n} \right| \right\} \sum_{j=n}^{\infty} \binom{j+m}{m}^{-1} \\ &= \frac{m}{m-1} \binom{n+m-1}{m-1}^{-1} \sup_{j \geq n} \left| \binom{j+m}{m} g^{(j)}(0) s_{j-n} r^{-n} \right|. \end{aligned}$$

There exist constants K_1 and K_2 such that, if $1 \leq n \leq j$, then

$$\binom{j+m}{m} \leq K_1 j^m \quad \text{and} \quad |s_{j-n}| \leq K_2 j^{d-1}/r^{j-n}.$$

Therefore

$$\left| \binom{j+m}{m} g^{(j)}(0) s_{j-n} r^{-n} \right| \leq K_1 K_2 \frac{j^{m+d-1}}{r^j} |g^{(j)}(0)|,$$

and we have

$$\binom{n+m-1}{m-1} r^{-n} |f^{(n)}(0)| \leq \frac{K_1 K_2 m}{m-1} \sup_{j \geq n} \left| \frac{j^{m+d-1}}{r^j} g^{(j)}(0) \right|.$$

Since $\lim_{j\to\infty}(j^{m+d-1}/r^j)g^{(j)}(0) = 0$, the proof is complete.

5. **An example.** Take $\Phi(z) = (1+z)^2/(1-z)$. Then $\mathcal{G}$ is the set of entire f such that $f^{(n)}(0) = o(1)$, $n \to \infty$, and

$$\pi_k(z) = 4\sum_{j=0}^{k-2} z^j/j! + \frac{3z^{k-1}}{(k-1)!} + \frac{z^k}{k!}.$$

It follows from the Convergence Theorem in [2] that convergence of $\sum_{k=0}^{\infty} h_k \pi_k(z)$ for one value of z implies uniform convergence on every compact set. The sequence $\{\pi_k\}_0^\infty$ is biorthogonal to the sequence of linear functionals $\{\mathcal{L}_k\}_0^\infty$ given by

$$\mathcal{L}_k(g) = \sum_{j=k}^{\infty} (-1)^{j-k}(2j-2k+1)g^{(j)}(0), \qquad k = 0, 1, 2, \cdots.$$

Every entire g for which the functionals $\mathcal{L}_k(g)$ are defined has the formal basic series expansion

$$g(z) \sim \sum_{k=0}^{\infty} \mathcal{L}_k(g)\pi_k(z).$$

(We use the term "basic series" in the sense of Whittaker [5].) It follows from [3, Theorem C] that the $\{\pi_k\}$ expansions are unique. In spite of this, not every convergent $\{\pi_k\}$ expansion is a basic series expansion. To see this, take

$$g(z) = \sum_{j=0}^{\infty} (2j^2 + 8j + 7)\frac{z^j}{(j+3)!}$$

and note that $g = f'' + 2f' + f$, where $f(z) = (e^z - 1)/2z$. Since $f \in \mathcal{G}$, it follows from [3, Theorem B] that g has the convergent $\{\pi_k\}$ expansion

$$g(z) = \sum_{k=0}^{\infty} \{f^{(k)}(0) - f^{(k+1)}(0)\}\pi_k(z) = \sum_{k=0}^{\infty} \frac{\pi_k(z)}{2(k+1)(k+2)}.$$

On the other hand, the series defining $\mathcal{L}_k(g)$ is easily seen to be divergent for every k, $0 \le k < \infty$.

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