ALMOST EVERYWHERE CONVERGENCE OF VILENKIN-FOURIER SERIES

BY

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ABSTRACT. It is shown that the partial sums of Vilenkin-Fourier series of functions in $L^q(G)$, $q > 1$, converge almost everywhere, where $G$ is a zero-dimensional, compact abelian group which satisfies the second axiom of countability and for which the dual group $X$ has a certain bounded subgroup structure. This result includes, as special cases, the Walsh-Paley group $2^n$, local rings of integers, and countable products of cyclic groups for which the orders are uniformly bounded.

Introduction. Let $X$ denote the dual group of a compact, abelian, zero-dimensional group $G$, which satisfies the second axiom of countability. Then $X$ is a discrete, countable, abelian, torsion group. N. Ja. Vilenkin [14] showed $X$ is the union of subgroups $\{X_s\}_{s=0}^{\infty}$, $X_s \subset X_{s+1}$, such that $X_{s+1}/X_s$ is of prime order $p_{s+1}$. Vilenkin also placed an ordering on $X$. Such a pair $(G,X)$ is called a Vilenkin system. A Vilenkin system is said to be bounded if $\sup_s p_s < \infty$.

For $f \in L^1(G)$, let $S_nf$ denote the $n$th partial sum of the Fourier series with respect to $X$. In this work we prove that $S_nf$ converges to $f$ almost everywhere for each $f$ in $L^q(G)$, $1 < q \leq \infty$. Special cases of this result include the Walsh-Paley series [11], Fourier series on the ring of integers of a local field [8], and countable products of cyclic groups with uniformly bounded orders [10].

In 1966, L. Carleson [3] established the a.e. convergence of the trigonometric Fourier series for $L^2(T)$ where $T$ denotes the circle. This result was extended to $L^q(T)$, $q > 1$, by R. Hunt [6]. The $L^2$ result for the Walsh-Paley system was first established by P. Billard [1] and later improved by R. Hunt [7]. P. Sjölin [12] then proved the $L^q$ result for the Walsh-Paley system. R. Hunt and M. Taibleson [8] established the result on local rings of integers for $L^q$, $q > 1$, and certain Orlicz spaces. Recently, R. Moore [10] established the result for $L^q(G)$, $q > 1$, where $G$ is a countable product of discrete cyclic groups $\mathbb{Z}/p_i$ which satisfies $\sup_i p_i < \infty$. All of these results are based on Carleson’s original proof [3] with various modifications and simplifications. A different unpublished proof was recently discovered by C. Fefferman.

The proof given here is also based on Carleson’s proof [3]. The simplifications used in the $L^2$ proof are closely related to those used in [7] while the $L^q$ result is
based on the proof in [8]. In this proof great use is made of the subgroup structures of $X$ and $G$.

This work has been divided into four main chapters. In Chapter I the essentials of Vilenkin systems are reviewed. In Chapter II preliminary results are collected. A new proof of Paley's theorem [11], [15] based on the Calderón-Zygmund decomposition [2] is given. In Chapter III the result is proved for $L^2(G)$. Finally, in Chapter IV the main result is extended to $L^q(G)$, $1 < q < 2$.

I. VILENKIN SYSTEMS

The groups $G$ and $X$. Let $G$ be a zero-dimensional, compact, abelian group which satisfies the second axiom of countability. The dual group of $G$, $X$, is a discrete, countable, abelian, torsion group [4, Theorems 24.15 and 24.26]. Vilenkin [14] proved the existence of a sequence of finite subgroups of $X$, $\{X_s\}_{s=0}^{\infty}$, which satisfy

1. $X_0 = \{\chi_0\}$, the identity character;
2. $X_s \subseteq X_{s+1}$;
3. $X = \bigcup_{s=0}^{\infty} X_s$;
4. $X_s / X_{s+1}$ is of prime order $p_s$;
5. there exists a sequence $\{q_s\}_{s=0}^{\infty}$ in $X$ such that $q_s \in X_{s+1} \setminus X_s$ and $q_s^{p_{s+1}} \in X_s$.

Such a pair of groups $(G, X)$ as described above is called a Vilenkin system. A Vilenkin system is said to be bounded if $\sup_s p_s = p < \infty$. Throughout this work, we deal solely with a bounded Vilenkin system.

The subgroups $G_s$. Let $G_s$ denote the annihilator of $X_s$. That is

$$G_s = \{x \in G : \chi(x) = 1 \text{ for all } \chi \in X_s\}.$$  

Then each $G_s$ is a compact, open subgroup of $G$. In addition, the sequence $\{G_s\}_{s=0}^{\infty}$ satisfies $G_0 = G$, $G_s \supset G_{s+1}$, and $\bigcap_{s=0}^{\infty} G_s = \{e\}$, the identity of $G$. Vilenkin [14] proved that for each $s$, there exists $x_s \in G_s \setminus G_{s+1}$ such that $q_s(x_s) = \exp\{2\pi i / p_{s+1}\}$. He also proved that each $x \in G$ has a unique representation of the form $x = \sum_{i=0}^{\infty} b_i x_i$ where $0 \leq b_i < p_{s+1}$. Consequently,

$$G_s = \left\{x \in G : x = \sum_{i=0}^{\infty} b_i x_i \text{ with } b_0 = b_1 = \cdots = b_{s-1} = 0\right\},$$

and each coset of $G_s$ in $G$ has a representation of the form $x + G_s$ with $x = \sum_{i=0}^{s-1} b_i x_i$, $0 \leq b_i < p_{s+1}$.

Each subgroup, $G_s$, is itself a zero-dimensional, compact, abelian group which satisfies the second axiom of countability. Its dual group can be identified with $X/X_s$ [4, Theorem 24.5]. Thus if $(G, X)$ is a bounded Vilenkin system with bound $p$, then so is $(G_s, X/X_s)$ for any $s \geq 0$. 

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The orderings of $X$ and $X/X_s$. As the choice of the sequence $(\varphi_j)_{j=0}^\infty$ is not unique, we assume a particular choice has been made. Having done so, the following ordering, introduced by Vilenkin [14], can be placed on $X$: Let $m_0 = 1$ and let $m_r = \prod_{j=0}^r p_j$ for $r \geq 1$. Then each natural number $n$ can be uniquely expressed as $n = \sum_{r=0}^\infty \alpha_r m_r$ where $0 \leq \alpha_r < p_{r+1}$, and only finitely many of the $\alpha_r$'s are nonzero. Then we define $X_n$ by the formula

$$X_n = \prod_{r=0}^\infty \varphi_r^{\alpha_r}.$$  

With this ordering we have

1. $X_s = \{X_n : 0 \leq n < m_s\}, s = 0, 1, 2, \ldots$;
2. $X_s = \{X_n : n \text{ is of the form } \sum_{r=0}^\infty \alpha_r m_r\}$;
3. if $n = \alpha_r m_r + k, 0 \leq k < m_r$, then $X_n = (X_{m_r})^{\alpha_r} \cdot X_k$.

For the sake of brevity, we shall write the dual group of $G_s$ simply as $\{X_n : n = \sum_{r=0}^\infty \alpha_r m_r\}$. The set $(X_{m_r} : n = \sum_{r=0}^\infty \alpha_r m_r)$ has an ordering induced by $X$. This ordering in turn induces an ordering on $X/X_s$, which is the one we use.

Notation. Throughout this work $\mu$ will denote the normalized Haar measure on $G$. By an interval $\omega$, we shall mean any coset of $G_s$ in $G$ for some $s \geq 0$. If $\omega = \sum_{i=0}^{s-1} b_i x_i + G_s$, then $\mu(\omega) = \mu(G_s) = m_s^{-1}$. If $\omega \in G/G_1$, we define $\omega^* = G_s$. If $\omega = \sum_{i=0}^{s-1} b_i x_i + G_s, s > 1$, we define $\omega^*$ as

$$\omega^* = \sum_{i=0}^{s-1} b_i x_i + G_{s-1}.$$  

Since there are $p_{s-1}$ intervals $\omega$ with the same $\omega^*$, we have

$$\mu(\omega^*) = p_{s-1} \mu(\omega) \leq p \mu(\omega).$$  

Let $n = \sum_{r=0}^\infty \alpha_r m_r$ and let $\omega \in G/G_s$. Then we define $\mu(\omega)$ as the integer $\sum_{r=0}^\infty \alpha_r m_r$. Then if $x \in \omega \in G/G_s$ is of the form $x = \sum_{i=0}^{s-1} b_i x_i + g_s, g_s \in G_s$, we have

$$X_n(x) = \left\{ \prod_{r=0}^{s-1} \left( X_{m_r} \left( \sum_{i=0}^{r-1} b_i x_i \right) \right)^{\alpha_r} \right\} X_{m_s}(x).$$

Consequently, $X_n(x) = A(\omega)X_{m_s}(x)$ as $x$ ranges over $\omega$ where $A(\omega)$ is a constant of modulus 1 depending only on $\omega$. We also define

$$c_n(\omega) = c_n(\omega; f) = \mu(\omega)^{-1} \int_\omega f(t)X_{m_s}(t) \, d\mu(t),$$

and

$$C_n(\omega^*) = C_n(\omega^*; f) = \max_{\omega} |c_m(\omega)|,$$

where the maximum is taken over all $\omega'$ with $\omega'^* = \omega^*$. Throughout this work $A$ will denote a constant, which may vary from line to line, depending only on the bound $p = \sup_s p_s$. 

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Fourier series and Dirichlet kernels. The Fourier series of a function \( f \) in \( L^1(G) \) is the series \( \sum_{i=0}^{\infty} c_i \chi_i(x) \) where \( c_i = \int_G f(t) \chi_i(t) \, dp(t) \). For the \( n \)th partial sums, \( S_n f = \sum_{i=0}^{n-1} c_i \chi_i \), we have

\[
S_n f(x) = (f \ast D_n)(x) = \int_G f(t) D_n(x - t) \, dp(t),
\]

where \( D_n(x) = \sum_{i=0}^{n-1} \chi_i(x) \) is the Dirichlet kernel of order \( n \). Vilenkin [14] derived the following formulas:

\[
D_n(x) = m_n I_n(x),
\]

where \( I_n \) is the characteristic function of \( G_n \). Also if \( n = \sum_{s=0}^{\infty} \alpha_s m_s \),

\[
D_n(x) = \chi_n(x) \sum_{s=0}^{\infty} D_{m_s}(x) \Phi_{m_s}(x),
\]

with the appropriate interpretation if \( \alpha_s = 0 \) or 1. For convenience, we write

\[
D_n(x) = \chi_n(x) \sum_{s=0}^{\infty} D_{m_s}(x) \Phi_{m_s}(x).
\]

We define the modified \( n \)th partial sum, \( S_n^* f \), by the formula

\[
S_n^* f = \chi_n S_n(f \chi_n).
\]

It follows that \( S_n^* f = f \ast D_n^* \) where

\[
D_n^* = \sum_{s=0}^{\infty} D_{m_s} \Phi_{m_s}.
\]

II. PRELIMINARY RESULTS

The modified kernels \( D_n^* \). The modified kernels \( D_n^* \) satisfy the following two properties, which will be used in the proof of the main result. Let \( n = \sum_{s=0}^{\infty} \alpha_s m_s \). Then

\[
D_n^* = \sum_{s=0}^{\infty} \left( \sum_{k=m_{s+1} - \alpha m_s}^{m_s - 1} \chi_k \right),
\]

where the inner sum is 0 if \( \alpha_s = 0 \).

If \( \omega \in G/G_s \), \( s > 0 \), and \( x \notin \omega \),

\[
D_n^*(x - t) \text{ is constant as } t \text{ ranges over } \omega.
\]

To prove (12) it suffices to prove

\[
D_n \Phi_{m_s} = \sum_{k=m_{s+1} - \alpha m_s}^{m_s - 1} \chi_k
\]

since \( D_n^* = \sum_{s=0}^{\infty} D_{m_s} \Phi_{m_s} \). Using (1)(v), (7), and (4)(iii) we have
This completes the proof of (12).

To prove (13) we consider an interval \( \omega = \sum_{i=0}^{r-1} b_i x_i + G_r \). Each \( t \in \omega \) is of the form

\[
t = \sum_{i=0}^{r-1} b_i x_i + g_s(t)
\]

where \( g_s(t) \in G_s \). Let \( x \in \sum_{i=0}^{r-1} c_i x_i + G_r \). Then

\[
x = \sum_{i=0}^{r-1} c_i x_i + g_s(x)
\]

where \( g_s(x) \in G_s \). Since \( x \notin \omega \), it follows that \( b_i \neq c_i \) for some \( 0 \leq i \leq s - 1 \).

Let \( \nu \) denote the smallest such \( i \). Then by (2) it follows that \( x - t \in G_{r+1} \setminus G_r \) for all \( t \in \omega \). By (7) and (8) we have,

\[
D^*_s(x - t) = \sum_{r=0}^{\infty} D^*_r(x - t) \Phi_{m_r}^*(x - t)
\]

(14)

\[
= \sum_{r=0}^{\nu-1} D^*_r(x - t) \Phi_{m_r}^*(x - t)
\]

\[
= \sum_{r=0}^{\nu-1} m_r (\chi_{m_r}(x - t))^\sim \left( \sum_{j=0}^{a_r-1} \chi_{m_r}(x - t) \right).
\]

For \( 0 \leq r \leq \nu - 1 \), \( \chi_{m_r} \in X_{r+1} \subset X_r \subset X_s \). Recall that \( G_s \) is the annihilator of \( X_s \). Thus for any \( t \in \omega \) and \( 0 \leq r \leq \nu - 1 \), we have

\[
\chi_{m_r}(x - t) = \chi_{m_r} \left( \sum_{i=0}^{r-1} (c_i - b_i) x_i \right) \chi_{m_r}(g_s(x)) \chi_{m_r}(g_s(t))
\]

Hence \( \chi_{m_r}(x - t) \) is constant as \( t \) ranges over \( \omega \) for \( 0 \leq r \leq \nu - 1 \). By (14) it follows that \( D^*_s(x - t) \) is constant as \( t \) ranges over \( \omega \). This completes the proof of (13).
Plancherel's formula. In this section we deal with the completeness of the system $X$ on $G$ and $X/X_s$ on $G_s$ by using probabilistic methods. Let $B$ denote the class of Borel sets, that is, the sigma-algebra generated by the compact sets in $G$. Let $F_s$ denote the sigma-algebra generated by the cosets of $G_s$ in $G$. If $F$ denotes the sigma-algebra generated by $\bigcup_{i=0}^{\infty} F_i$, then $F = B$ [9, Lemma 3.2]. Let $x \in \omega = \sum_{i=0}^{\infty} b_i x_i + G$. Then

$$S_m f(x) = \int_G f(t) D_m(x - t) d\mu(t)$$

$$= m_s \int_{x+G_s} f(t) d\mu(t)$$

$$= \mu(\omega)^{-1} \int_{\omega} f(t) d\mu(t).$$

It follows that

$$S_m f = E(f \mid F_s)$$

where $E(f \mid K)$ denotes the conditional expectation of $f$ with respect to the sigma-algebra $K$ [13, p. 90]. Since $F = B$, the martingale convergence theorem [8, Theorem 3.1] implies $S_m f \to f$ a.e. as $s \to \infty$. The completeness of $X$ on $G$ now follows since any function, $f \in L^1(G)$, which has all vanishing coefficients, must satisfy $f(x) = 0$ a.e.

The completeness of $X/X_s$ on $G_s$ follows by an identical argument and normalization of the Haar measure on $G_s$. A simple translation argument shows that $X/X_s$ is a complete orthonormal system on any coset of $G_s$ in $G$ with respect to the normalized measure $m_s \mu$.

We now have the following version of Plancherel's formula: Let $f \in L^2(G)$ and let $\omega$ be any interval. Then

$$\sum_{n(\omega)^{-1}}^{\infty} |c_{n(\omega)}(\omega)|^2 = \mu(\omega)^{-1} \int_{\omega} |f(t)|^2 d\mu(t).$$

The martingale maximal function. In place of the Hardy-Littlewood maximal function, we use a probabilistic analogue, the martingale maximal function. Let $f \in L^1(G)$ and define

$$E \ast f(x) = \sup_{r \geq 0} |E(f \mid F_r)(x)| = \sup_{r \geq 0} |S_m f(x)|.$$

Then the martingale maximal theorem states that if $1 < q \leq \infty$,

$$\|E \ast f\|_q \leq A_q \|f\|_q,$$

where $A_q$ depends only on $q$ [11, Theorem 6, p. 91]. Furthermore, we have $A_q = O(q/(q-1)) = O(1)$ as $q \to \infty$ [11, Lemma 2, p. 93].
Paley's theorem. The result proved in this section, Paley's theorem, states that the \( n \)th partial sum operators are bounded, uniformly in \( n \), from \( L^q(G) \) into itself for \( 1 < q < \infty \). That is, there exists a constant \( A_q \) depending only on \( q \) such that for \( n \geq 1 \) and \( f \in L^q(G) \), \( 1 < q < \infty \),

\[
\|S_n f\|_q \leq A_q \|f\|_q.
\]

We begin the proof by making several reductions. By considering \( f^+ \) and \( f^- \) separately, we may assume \( f \) is nonnegative. Since \( S_n f = \chi_n S_n^* (f \chi_n) \), it suffices to prove the result for \( S_n^* \). Since \( S_n^* = \chi_n S_n (f \chi_n) \), we have

\[
\|S_n^*\|_2 \leq \|f\|_2.
\]

To obtain the result for \( 1 < q < 2 \), it suffices, by the Marcinkiewicz interpolation theorem [16, p. 112, vol. 2], to prove \( S_n^* \) has weak type \((1, 1)\) independent of \( n \). That is, for any \( \lambda > 0 \),

\[
\mu\{x \in G : |S_n^* f(x)| > \lambda\} \leq A \lambda^{-1} \|f\|_1.
\]

A standard duality argument, which we delay until the end of this section, then yields the result for \( q > 2 \).

To prove (20), we use a Calderón-Zygmund decomposition [2]. Let \( \lambda > 0 \) be fixed. We may assume \( \|f\|_1 < \lambda \). Let

\[
\Omega_1 = \left\{ \omega : \omega = b_0 x_0 + G_1, \mu(\omega)^{-1} \int_\omega f(t) d\mu(t) > \lambda \right\},
\]

\[
\Omega_2 = \left\{ \omega : \omega = b_1 x_1 + G_2, \omega \subset \Omega_1, \mu(\omega)^{-1} \int_\omega f(t) d\mu(t) > \lambda \right\}.
\]

In general, let

\[
\Omega_j = \left\{ \omega : \omega = \sum_{i=0}^{j-1} b_i x_i + G_j, \omega \subset \bigcup_{i=1}^{j-1} \Omega_i, \mu(\omega)^{-1} \int_\omega f(t) d\mu(t) > \lambda \right\}.
\]

We obtain a sequence \( \{\Omega_j\}_{j=1}^\infty \) and set \( \Omega = \bigcup_{j=1}^\infty \Omega_j \). Define

\[
g(x) = \mu(\omega)^{-1} \int_\omega f(t) d\mu(t) \quad \text{if } x \in \omega, \omega \in \Omega,
\]

\[
g(x) = f(x) \quad \text{if } x \notin \omega, \omega \in \Omega,
\]

and let \( b = f - g \). Then

\[
\mu\{x \in G : |S^* f(x)| > \lambda\} \leq \mu\{x \in G : |S^* g(x)| > \lambda/2\}
\]

\[+ \mu\{x \in G : |S^* b(x)| > \lambda/2\}.
\]

We show that each of these expressions is dominated by \( A \lambda^{-1} \|f\|_1 \). We begin with the estimate for \( g \) which readily follows from the inequality \( \|g\|_2^2 \leq A \lambda \|f\|_1 \). We
note that this estimate relies heavily on the bound of the $p_j$'s. It follows from the martingale convergence theorem that $g(t) \leq \lambda$ for almost all $t$ outside $\Omega$. We have

$$
\int_G (g(t))^2 \, d\mu(t) = \sum_{\omega \in \Omega} \int_\omega (g(t))^2 \, d\mu(t) + \sum_{\omega \in \Omega} \int_\omega (g(t))^2 \, d\mu(t) \\
\leq \sum_{\omega \in \Omega} \lambda \int_\omega f(t) \, d\mu(t) + \sum_{\omega \in \Omega} \int_\omega (g(t))^2 \, d\mu(t).
$$

Using (6), we obtain

$$
\sum_{\omega \in \Omega} \int_\omega (g(t))^2 \, d\mu(t) = \sum_{\omega \in \Omega} \int_\omega g(t) (\mu(\omega)^{-1} \int_\omega f(s) \, d\mu(s)) \, d\mu(t) \\
\leq \sum_{\omega \in \Omega} \int_\omega g(t) \left( \frac{\mu(\omega^*)}{\mu(\omega)} \right) \left( \mu(\omega^*)^{-1} \int_\omega f(s) \, d\mu(s) \right) \, d\mu(t) \\
\leq p\lambda \sum_{\omega \in \Omega} \int_\omega g(t) \, d\mu(t) \\
= p\lambda \sum_{\omega \in \Omega} \int_\omega f(t) \, d\mu(t).
$$

Hence

$$
\int_G (g(t))^2 \, d\mu(t) \leq \lambda \sum_{\omega \in \Omega} \int_\omega f(t) \, d\mu(t) + p\lambda \sum_{\omega \in \Omega} \int_\omega f(t) \, d\mu(t) \\
\leq p\lambda \int_G f(t) \, d\mu(t).
$$

The estimate for $g$ now follows:

$$
\mu\{x \in G: |S^*_n g(x)| > \lambda/2\} \leq 4\lambda^{-2} \|g\|_2^2 \leq (4\lambda^{-2})(p\lambda \|f\|_1) = 4p\lambda^{-1} \|f\|_1.
$$

To prove $\mu\{x \in G: |S^*_n b(x)| > \lambda/2\} \leq A\lambda^{-1} \|f\|_1$, we write

$$
\mu\{x \in G: |S^*_n b(x)| > \lambda/2\} \\
\leq \mu\{x \in G: |S^*_n b(x)| > \lambda/2, x \not\in \omega \in \Omega\} \\
+ \mu\{x \in G: |S^*_n b(x)| > \lambda/2, x \in \omega \in \Omega\} \\
\leq \mu\{x \in G: |S^*_n b(x)| > \lambda/2, x \not\in \omega \in \Omega\} + \sum_{\omega \in \Omega} \mu(\omega).
$$

It suffices to prove

(21) \quad $x \not\in \omega \in \Omega$ implies $S^*_n b(x) = 0$

and

(22) \quad $\sum_{\omega \in \Omega} \mu(\omega) \leq A\lambda^{-1} \|f\|_1$. 

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To prove (21), we note that \( \int b(t) \, dp(t) = 0 \) for each \( \omega \in \Omega \). We write

\[
S^*_n b(x) = \sum_{\omega \in \Omega} b(t) D^*_n(x - t) \, dp(t).
\]

For \( x \in \omega \), (13) implies \( D^*_n(x - t) \) is constant as \( t \) ranges over \( \omega \). Since \( b \) has a vanishing integral on each \( \omega \), it follows that \( S^*_n b(x) = 0 \) for \( x \in \omega \in \Omega \), and (21) is proved. To prove (22), recall that each \( \omega \in \Omega \) satisfies \( \mu(\omega)^{-1} \int f(t) \, dp(t) > \lambda \). Thus

\[
\sum_{\omega \in \Omega} \mu(\omega) < \lambda^{-1} \sum_{\omega \in \Omega} \int f(t) \, dp(t) \leq \lambda^{-1} \|f\|_1,
\]

and (22) is proved.

We finally extend the result for \( q > 2 \) by a duality argument. At the same time, we shall obtain an estimate of the operator norm, \( \|S^*_n\|_q \), from \( L^q \) into itself, as \( q \) tends to infinity. By the Marcinkiewicz interpolation theorem [16, p. 112, vol. 2] there exists a constant \( A \) independent of \( n \) such that, for \( 1 < q < 2 \), \( \|S^*_n\|_q \leq A(q/(q - 1)) \). Let \( q' > 2 \) satisfy \( q^{-1} + q'^{-1} = 1 \). Then

\[
\|S^*_n f\|_{q'} = \sup_{h \in L^q(G); \|h\|_q \leq 1} \left\| \int_G S^*_n f(x) h(x) \, dp(x) \right\|
\leq \sup_{h \in L^q(G); \|h\|_q \leq A(q/(q - 1))} \left\| \int_G f(x) h(x) \, dp(x) \right\|
\leq A(q/(q - 1)) \|f\|_{q'}
\leq Aq' \|f\|_{q'}.
\]

Hence \( \|S^*_n\|_q \leq Aq' \). That is

\[
(23) \quad \|S^*_n\|_q = O(q') \quad \text{as } q' \to \infty
\]

with bound independent of \( n \). This completes the proof of Paley's theorem.

III. THE \( L^2 \) RESULT

Introduction and basic results. The main result of this work is the following

**Theorem.** Let \( f \in L^2(G) \). Then \( S_nf \) converges to \( f \) almost everywhere as \( n \) tends to infinity.

As in the case of Paley's theorem, we make several reductions of the proof. Let \( Mf \) be defined by \( Mf(x) = \sup_{n \geq 0} |S_nf(x)| \) for \( x \in G \). Then it suffices to prove that, for every \( \lambda > 0 \),

\[
(24) \quad \mu(x \in G : Mf(x) > \lambda) \leq A \lambda^{-2/2},
\]
where \( \mathcal{A} \) is independent of \( f \) and \( \lambda \). To see this, let \( \{\varepsilon_k\}_{k=1}^{\infty} \) be a positive sequence decreasing to zero, and let \( \{R_k\}_{k=1}^{\infty} \) be a sequence of finite linear combinations of characters such that \( \|f - R_k\|_2^2 \leq \varepsilon_k^2 \). Then assuming (24), we have

\[
\begin{align*}
\mu\left\{ x \in G : \limsup_{n \to \infty} |S_n f(x) - f(x)| > \varepsilon_k \right\} \\
\leq \mu\left\{ x \in G : \limsup_{n \to \infty} |S_n (f - R_k)(x)| > \varepsilon_k/3 \right\} \\
+ \mu\left\{ x \in G : \limsup_{n \to \infty} |S_n R_k(x) - R_k(x)| > \varepsilon_k/3 \right\} \\
+ \mu\{ x \in G : |R_k(x) - f(x)| > \varepsilon_k/3 \} \\
\leq \mu\{ x \in G : M(f - R_k)(x) > \varepsilon_k/3 \} \\
+ \mu\{ x \in G : |R_k(x) - f(x)| > \varepsilon_k/3 \} \\
\leq 3A\varepsilon_k^2 \|f - R_k\|_2^2 + 3\varepsilon_k^2 \|f - R_k\|_2^2 \\
\leq A\varepsilon_k.
\end{align*}
\]

For each positive integer \( N \) let \( M_N f(x) = \max_{1 \leq n \leq m_N} |S_n f(x)| \). For each \( \lambda > 0 \), we define an exceptional set \( E(\lambda, N, f) \) such that

\[
(25) \quad \mu(E(\lambda, N, f)) \leq A_1 \lambda^{-2} \|f\|_2^2,
\]

and

\[
(26) \quad x \notin E(\lambda, N, f) \implies M_N f(x) \leq A_2 \lambda,
\]

where \( A_1 \) and \( A_2 \) are two positive constants which do not depend on \( N, \lambda, \) or \( f \). Since

\[
\{ x \in G : Mf(x) > \lambda \} = \{ x \in G : M(A_2 f) > A_2 \lambda \} \\
\subseteq \bigcup_{N=1}^{\infty} E(\lambda, N, A_2 f),
\]

(25) and (26) imply

\[
\begin{align*}
\mu\{ x \in G : Mf(x) > \lambda \} &\leq \mu\left\{ \bigcup_{N=1}^{\infty} E(\lambda, N, A_2 f) \right\} \\
&= \lim_{N \to \infty} \mu\{ E(\lambda, N, A_2 f) \} \\
&\leq A_1 \lambda^{-2} \|A_2 f\|_2^2 \\
&= A_1 A_2^2 \lambda^{-2} \|f\|_2^2.
\end{align*}
\]

Thus it suffices to prove (25) and (26) for \( \lambda, N, \) and \( f \) fixed. From this point on we shall write \( E(\lambda, N, f) \) simply as \( E \). We may also assume \( \|f\|_2 < \lambda \).
The exceptional set $E$ will consist of two basic parts, $E_1$ and $E_2$. $E_1$ will be made up of certain intervals $\omega$, and it will be easy to show

$$\mu(E_1) \leq A \lambda^2 \|f\|_2^2. \quad (27)$$

$E_2$ will be more complicated. We shall define a sequence, $\{A_j^\ast\}_{j=1}^\infty$, of collections of pairs $(n(\omega^\ast), \omega^\ast)$, where $n$ is a positive integer. For each pair $(n(\omega^\ast), \omega^\ast) \in A_j^\ast$, we define an exceptional subset $V(n(\omega^\ast), \omega^\ast, j)$ such that

$$\mu(V(n(\omega^\ast), \omega^\ast, j)) \leq p^{-3/2} \mu(\omega^\ast). \quad (28)$$

By using Plancherel's formula (16), we shall prove that

$$\sum_{\lambda_j^o} \mu(\omega^\ast) \leq A p^{3/2} \lambda^2 \|f\|_2^2, \quad (29)$$

where the sum is taken over all pairs $(n(\omega^\ast), \omega^\ast) \in A_j^\ast$. Setting

$$E_2 = \bigcup_{j=1}^\infty \bigcup_{\lambda_j^o} V(n(\omega^\ast), \omega^\ast, j),$$

(28) and (29) imply

$$\mu(E_2) \leq A \left( \sum_{j=1}^\infty p^{-j} \right) \lambda^2 \|f\|_2^2. \quad (30)$$

Combining (27) and (30), we have

$$\mu(E) \leq A \lambda^2 \|f\|_2^2. \quad (31)$$

For certain pairs $(n(\omega^\ast), \omega^\ast) \in A_j^\ast$, we define a partition of $\omega^\ast$, $\Pi(n(\omega^\ast), \omega^\ast, j)$, where the elements of the partition are intervals. If $x \notin E$ and $\bar{\omega}^\ast$ denotes the partition element which contains $x$, we obtain the estimate

$$|S_{n(\omega^\ast)}f(x) - S_{\bar{\omega}^\ast}f(x)| \leq p^{-j/2} \lambda. \quad (32)$$

If the partition $\Pi(n(\bar{\omega}^\ast), \bar{\omega}^\ast, j')$ were defined for some $j' < j$, we could repeat the above argument and find $\bar{\omega}^\ast$ such that $x \in \bar{\omega}^\ast$ and

$$|S_{n(\bar{\omega}^\ast)}f(x) - S_{\bar{\omega}^\ast}f(x)| \leq p^{-j'/2} \lambda.$$

Summing over all such estimates would show that for $x \notin E$, $|S_n f(x)| \leq (\sum_{j=1}^\infty p^{-j/2}) \lambda$, and we would be done. However, since $\Pi(n(\bar{\omega}^\ast), \bar{\omega}^\ast, j')$ may not be defined, we must change from $(n(\bar{\omega}^\ast), \omega^\ast)$ to a new pair $(\bar{n}(\bar{\omega}^\ast), \bar{\omega}^\ast)$ and make the appropriate estimates. After this modification, we shall be able to prove that if $x \notin E$, $|S_n f(x)| \leq A \lambda$ where $A$ is a constant which depends only on $p$.

**Selected pairs $A_j$ and $A_j^\ast$.** Let $\omega \in G/G_s$, $1 \leq s \leq N$, and consider the collection of pairs $\{(n(\omega), \omega) : 1 \leq n \leq m_N\}$. For each pair set

$$\Delta(n(\omega), \omega) = \max\{|c_{n(\omega)}(\bar{\omega})| : \bar{\omega} \supseteq \omega^\ast, \bar{n}(\omega) = n(\omega)\}. \quad (33)$$
Let $\Lambda_j$ denote the collection of pairs $(n(\omega), \omega)$ which satisfy

\[ |c_{n(\omega)}(\omega)| \geq p^{-j}\lambda, \]

and for which one of the following conditions holds:

\[ \omega^* = G \quad \text{and} \quad |c_{n(\omega)}(\omega)| < p^{-j+1}\lambda, \]

\[ \omega^* \neq G \quad \text{and} \quad \Delta(n(\omega), \omega) < p^{-j}\lambda. \]

To estimate $\sum \mu(\omega)$, where the sum is taken over all pairs $(n(\omega), \omega) \in \Lambda_j$, we use a collection of "polynomials", $P_j(x; \omega)$. Let

\[ P_j(x; \omega) = \sum_{(n(\omega), \omega) \in \Lambda_j} c_{n(\omega)}(\omega)X_{n(\omega)}(x). \]

Suppose $\omega \in G/G_s$, $s > 1$. Then

\[ \int f(t) \, d\mu(t) - \sum_{(n(\omega), \omega) \in \Lambda_j} c_{n(\omega)}(\omega)X_{n(\omega)}(t) \int d\mu(t) \]

\[ = \int f(t) \, d\mu(t) - \sum_{(n(\omega), \omega) \in \Lambda_j} c_{n(\omega)}(\omega)X_{n(\omega)}(t) \int d\mu(t) \]

\[ - 2 \text{Re} \left( \int f(t) \sum_{(n(\omega), \omega) \in \Lambda_j} c_{n(\omega)}(\omega)X_{n(\omega)}(t) \, d\mu(t) \right) \]

\[ + 2 \text{Re} \left( \int P_j(t; \omega^*) \sum_{(n(\omega), \omega) \in \Lambda_j} c_{n(\omega)}(\omega)X_{n(\omega)}(t) \, d\mu(t) \right) \]

\[ + \int \sum_{(n(\omega), \omega) \in \Lambda_j} c_{n(\omega)}(\omega)X_{n(\omega)}(t) \, d\mu(t). \]

To see that the third integral in (38) is zero, consider a single term of the product, $c_{n(\omega)}(\omega)c_{n(\omega)}(\omega)X_{n(\omega)}(t)X_{n(\omega)}(t)$. By (37) we have $\omega^* \supset \omega^* \text{ and } (n(\omega), \omega) \in \Lambda_j$. Hence, (34) implies

\[ |c_{n(\omega)}(\omega)| \geq p^{-j}\lambda. \]

By the ordering on $X/X_s$, $X_{n(\omega)}$ and $X_{n(\omega)}$ are orthogonal on $\omega$ unless

\[ \overline{n}(\omega) = n(\omega). \]

Consequently, (33), (39), and (40) imply

\[ \Delta(n(\omega), \omega) \geq p^{-j}\lambda. \]

But $(n(\omega), \omega) \in \Lambda_j$, and $\omega^* \neq G$ since $\omega \in G/G_s$, $s > 1$, By (36),

\[ \Delta(n(\omega), \omega) < p^{-j}\lambda. \]
(41) and (42) are a contradiction, and so the third integral of (38) vanishes. Applying Plancherel’s formula (16) to the last integral of (38), we have

\[
\int_\omega \left| \sum_{(n(\omega),u) \in \Lambda_j} c_{n(\omega)}(\omega) \chi_{n(\omega)}(t) \right|^2 d\mu(t) = \mu(\omega) \sum_{(n(\omega),u) \in \Lambda_j} |c_{n(\omega)}(\omega)|^2.
\]

Dropping the third integral in (38) and using (43) we obtain

\[
\int_\omega |f(t) - P_j(t; \omega)|^2 d\mu(t)
\]

\[
= \int_\omega |f(t) - P_j(t; \omega^*)|^2 d\mu(t)
\]

\[
- 2 \text{ Re} \left\{ \sum_{(n(\omega),u) \in \Lambda_j} \overline{c_{n(\omega)}(\omega)} \int_\omega f(t) \chi_{n(\omega)}(t) d\mu(t) \right\}
\]

\[
+ \mu(\omega) \sum_{(n(\omega),u) \in \Lambda_j} |c_{n(\omega)}(\omega)|^2
\]

\[
= \int_\omega |f(t) - P_j(t; \omega^*)|^2 d\mu(t) - 2 \mu(\omega) \sum_{(n(\omega),u) \in \Lambda_j} |c_{n(\omega)}(\omega)|^2
\]

\[
= \int_\omega |f(t) - P_j(t; \omega^*)|^2 d\mu(t) - \mu(\omega) \sum_{(n(\omega),u) \in \Lambda_j} |c_{n(\omega)}(\omega)|^2.
\]

Summing (44) over all $\omega \in G/G_1$, we obtain

\[
\sum_{\omega \in G/G_1} \int_\omega |f(t) - P_j(t; \omega)|^2 d\mu(t)
\]

\[
= \sum_{\omega \in G/G_1} \int_\omega |f(t) - P_j(t; \omega^*)|^2 d\mu(t)
\]

\[
- \sum_{\omega \in G/G_1} \sum_{(n(\omega),u) \in \Lambda_j} \mu(\omega) |c_{n(\omega)}(\omega)|^2
\]

\[
= \sum_{\omega \in G/G_{r+1}} \int_\omega |f(t) - P_j(t; \omega)|^2 d\mu(t)
\]

\[
- \sum_{\omega \in G/G_r} \sum_{(n(\omega),u) \in \Lambda_j} \mu(\omega) |c_{n(\omega)}(\omega)|^2.
\]

We repeat the above argument, beginning with (38), to the first term on the right-hand side of (45). We continue this procedure until we obtain after a finite number of steps

\[
\sum_{\omega \in G/G_r} \int_\omega |f(t) - P_j(t; \omega)|^2 d\mu(t)
\]

\[
= \sum_{\omega \in G/G_1} \int_\omega |f(t) - P_j(t; \omega^*)|^2 d\mu(t)
\]

\[
- \sum_{r=1} \sum_{\omega \in G/G_r} \sum_{(n(\omega),u) \in \Lambda_j} \mu(\omega) |c_{n(\omega)}(\omega)|^2.
\]

If $\omega \in G/G_1$, $P_j(t; \omega^*) = 0$ for all $t$. Setting $s = N$ in (46), we obtain
$$0 \leq \sum_{\omega \in \mathcal{G}} \int_\omega |f(t) - P(t; \omega)|^2 d\mu(t)$$

$$= \|f\|_2^2 - \sum_{(n(\omega), \omega) \in \Lambda} \mu(\omega)|c_{n(\omega)}(\omega)|^2.$$ Consequently

$$(47) \quad \sum_{(n(\omega), \omega) \in \Lambda} \mu(\omega)|c_{n(\omega)}(\omega)|^2 \leq \|f\|_2^2.$$ If \((n(\omega), \omega) \in \Lambda\), we have, by (34), \(|c_{n(\omega)}(\omega)| \geq p^{-1} \lambda\). Therefore (47) implies

$$(48) \quad \sum_{(n(\omega), \omega) \in \Lambda} \mu(\omega) \leq p^2 \lambda^{-2} \|f\|_2^2.$$ We now define \(\Lambda'_*\) as the collection of pairs \(\{(n(\omega^*), \omega^*): (n(\omega), \omega) \in \Lambda\}\). Note that for each pair in \(\Lambda\), there are at most \(p\) pairs in \(\Lambda'_*\). This fact, (6) and (48) imply

$$(49) \quad \sum_{\Lambda'_*} \mu(\omega^*) \leq p^2 \sum_{\Lambda_j} \mu(\omega) \leq p^{2+\gamma} \lambda^{-2} \|f\|_2^2,$$ where the sums in (49) are taken over all pairs in \(\Lambda'_*\) and \(\Lambda_j\) respectively. Estimate (49) will be used later to estimate \(\mu(E_2)\).

The set \(E_1\). At this point we must define \(E_1\), the first part of the exceptional set \(E\). Let

$$\mathcal{E}_1 = \left\{ \omega: \mu(\omega)^{-1} \int_\omega |f(t)|^2 d\mu(t) > \lambda^2 \right\} \quad \text{and}$$

$$(50) \quad E_1 = \bigcup_{\omega \in \mathcal{E}_1} \{x \in G: x \in \omega^*\}.$$ Using (6) and (50) we obtain

$$(51) \quad \mu(E_1) \leq p \sum_{\omega \in \mathcal{E}_1} \mu(\omega) \leq p \lambda^{-2} \sum_{\omega \in \mathcal{E}_1} \int_\omega |f(t)|^2 d\mu(t)$$

$$\leq A \lambda^{-2} \|f\|_2^2.$$ Now suppose \(\omega^* \subset E_1\). Then if \(\overline{\omega}\) is such that \(\overline{\omega} = \omega^*\), we have \(\overline{\omega} \subset \overline{E_1}\). Consequently, for any \(n\), we have

$$(52) \quad |c_{n(\overline{\omega})}(\overline{\omega})| = \mu(\overline{\omega})^{-1} \left| \int_\omega f(t) \overline{c_{n(\overline{\omega})}(t)} d\mu(t) \right|$$

$$\leq \mu(\overline{\omega})^{-1} \int_\omega |f(t)| d\mu(t)$$

$$\leq \mu(\overline{\omega})^{-1} \left( \int_\omega |f(t)|^2 d\mu(t) \right)^{1/2} (\mu(\overline{\omega}))^{1/2}$$

$$\leq \mu(\overline{\omega})^{-1/2} (\lambda^2 \mu(\overline{\omega}))^{1/2} = \lambda.$$ It follows that if \(\omega^* \subset E_1\),
The partitions $\Pi(n(\omega^*), \omega^*, j)$. In this section we define a partition $\Pi(n(\omega^*), \omega^*, j)$ for each pair $(n(\omega^*), \omega^*) \in \Lambda^*$ such that $\omega^* \subset E_i$. If $\omega^* \in G/G_s, 0 \leq s < N$, the elements of the partition $\Pi(n(\omega^*), \omega^*, j)$ will be cosets in $G/G_s$ where $s < r \leq N$.

At this point we must make a small technical adjustment. If $\omega^* = G$, by (35) a pair $(n(\omega^*), \omega^*) = (n, \omega^*)$ may belong to more than one $\Lambda^*_j$. If this is so, we delete $(n, \omega^*)$ from all $\Lambda^*_j$ except the one with minimal $j$.

Suppose $\omega^* \subset E_i$ and $(n(\omega^*), \omega^*) \in \Lambda^*_j$. Then we show

$$C_{n(\omega^*)}(\omega^*) < p^{-j+1} \lambda.$$ 

Consider $\omega$ such that $\omega^* = \omega^*$ and $|c_{n(\omega)}(\omega)| > 0$. Since $\omega^* \subset E_i$, we have $|c_{n(\omega)}(\omega)| < \lambda$ so there exists $j > 1$ such that $p^{-j} \lambda < |c_{n(\omega)}(\omega)| < p^{-j+1} \lambda$. If $\omega^* = G$, (34) and (35) imply $(n(\omega^*), \omega^*) \in \Lambda_j$. By the above deletion it follows that $j > j$. Therefore

$$C_{n(\omega^*)}(\omega^*) = \max_{\omega^* \in \omega^*} |c_{n(\omega)}(\omega)| < p^{-j+1} \lambda \leq p^{-j+1} \lambda$$

and (54) is true. If $\omega^* \neq G$ and $\omega = \sum_{i=0}^{s-1} b_i x_i + G_s, s > 1$, we have by (4)(iii) and (7) applied to $(X/G_{s-1}, G_{s-1})$,

$$|c_{n(\omega)}(\omega)| = \mu(\omega)^{-1} \left| \int_{\omega} f(t) \chi_{n(\omega)}(t) \, d\mu(t) \right|$$

$$= \mu(\omega^*)^{-1} p_s^{-1} \left| \int_{\omega^*} f(t) \chi_{n(\omega)}(t) \sum_{t=0}^{s-1} x_{n(t)} \left( \sum_{t=0}^{s-1} b_i x_i - \tau \right) \, d\mu(t) \right|$$

$$= \mu(\omega^*)^{-1} \sum_{\substack{r=0 \ldots s-1 \vdots 1}} \left| \int_{\omega^*} f(t) \chi_{n(t)}(t) \chi_{n(t)}(t) \, d\mu(t) \right|$$

$$\leq \mu(\omega^*)^{-1} \left| \sum_{t=0}^{s-1} \mu(\omega^*)^{-1} \int_{\omega^*} f(t) \chi_{n(t)}(t) \chi_{n(t)}(t) \, d\mu(t) \right|.$$ 

Since $(n(\omega^*), \omega^*) \in \Lambda^*_j$, there exists $\omega$ with $\omega^* = \omega^*$ and $(n(\omega), \omega) \in \Lambda_j$. If $n(\omega^*) = p_{s-1} + n(\omega)$, $n(\omega) = n(\omega)$, we have $n(\omega) = n(\omega) = n(\omega)$. Since $\omega^* \neq G$, (36) implies

$$\mu(\omega^*)^{-1} \left| \int_{\omega^*} f(t) \chi_{n(\omega)}(t) \, d\mu(t) \right| \leq \Delta(n(\omega), \omega) < p^{-j} \lambda.$$ 

Combining (55) and (56), we obtain
\[ |c_m(\tilde{\omega})| < \sum_{j=0}^{p-1} p^{-j} \lambda \leq p^{-j+1} \lambda. \]

Since \( \tilde{\omega} \) was any interval with \( \omega^* = \omega^* \), (57) implies
\[ c_{n(\omega^*)}(\omega^*) = \max_{\tilde{\omega} \subseteq \omega^*} |c_m(\tilde{\omega})| < p^{-j+1} \lambda \]
and (54) is true if \( \omega^* \neq G \). This establishes (54).

Let \( (n(\omega^*), \omega^*) \in \Lambda^*, \omega^* \subseteq E_1 \), and \( \omega^* \subseteq G/G_t \). We define the partition \( \Pi(n(\omega^*), \omega^*, j) \) as follows: Let

\[ \Omega_1(n(\omega^*), \omega^*, j) = \{ \omega \in G/G_{j+1} : \omega \subseteq \omega^*, c_{n(\omega^*)}(\omega) \geq p^{-j+1} \lambda \}, \]
\[ \Omega_2(n(\omega^*), \omega^*, j) = \{ \omega \in G/G_{j+2} : \omega \subseteq \omega^* \setminus \Omega_1(n(\omega^*), \omega^*, j), c_{n(\omega^*)}(\omega) \geq p^{-j+1} \lambda \}. \]

In general, if \( 1 \leq i < N-s \), let
\[ \Omega_i(n(\omega^*), \omega^*, j) = \{ \omega \in G/G_{i+1} : \omega \subseteq \omega^* \setminus \bigcup_{r=1}^{i-1} \Omega_r(n(\omega^*), \omega^*, j), c_{n(\omega^*)}(\omega) \geq p^{-j+1} \lambda \}. \]

Finally, let
\[ \Omega_{N-s}(n(\omega^*), \omega^*, j) = \{ \omega \in G/G_N : \omega \subseteq \omega^* \setminus \bigcup_{r=1}^{N-s-1} \Omega_r(n(\omega^*), \omega^*, j) \}. \]

Then \( \bigcup_{r=1}^{N-s} \Omega_r(n(\omega^*), \omega^*, j) \) forms a partition of \( \omega^* \), \( \Pi(n(\omega^*), \omega^*, j) \) with the following properties:

(i) \( \tilde{\omega} \subseteq \omega^* \) for each \( \tilde{\omega} \in \Pi(n(\omega^*), \omega^*, j) \);

(ii) if \( \tilde{\omega} \subseteq \omega \subseteq \omega^* \) and \( \tilde{\omega} \in \Pi(n(\omega^*), \omega^*, j) \), \( |c_{n(\omega^*)}(\tilde{\omega})| < p^{-j+1} \lambda \):

(iii) if \( \tilde{\omega} \in \Pi(n(\omega^*), \omega^*, j) \) and \( \tilde{\omega} \in G/G_N \), \( s < N \), then \( |c_{n(\omega^*)}(\tilde{\omega})| \geq p^{-j+1} \lambda \) for at least one \( \tilde{\omega} \) such that \( \omega^* = \tilde{\omega} \).

To see (58)(i) note that each \( \tilde{\omega} \in \Pi(n(\omega^*), \omega^*, j) \) must satisfy \( c_{n(\omega^*)}(\tilde{\omega}) \geq p^{-j+1} \lambda \), and by (54) this cannot be satisfied if \( \tilde{\omega} = \omega^* \). To see (58)(ii) we note that if \( |c_{n(\omega^*)}(\tilde{\omega})| \geq p^{-j+1} \lambda \), \( c_{n(\omega^*)}(\omega^*) \geq p^{-j+1} \lambda \) and so there exists a largest interval \( \hat{\omega}^* \) such that \( c_{n(\omega^*)}(\hat{\omega}^*) \geq p^{-j+1} \lambda \), \( \hat{\omega}^* \subseteq \omega^* \subseteq \omega^* \). Then \( \hat{\omega}^* \in \Pi(n(\omega^*), \omega^*, j) \). But then \( \tilde{\omega} \subseteq \hat{\omega} \) which is impossible since \( \tilde{\omega} \in \Pi(n(\omega^*), \omega^*, j) \). Thus (58)(iii) holds. (58)(iii) is clear from the construction of \( \Pi(n(\omega^*), \omega^*, j) \).

The basic estimate. Let \( (n(\omega^*), \omega^*) \in \Lambda^*, \omega^* \subseteq G/G_t \), and \( \omega^* \subseteq E_1 \). Then the partition \( \Pi(n(\omega^*), \omega^*, j) \) is defined. Let \( \tilde{\omega} \) satisfy \( \tilde{\omega} \subseteq \omega \subseteq \omega^* \) where \( \omega \) is any element of \( \Pi(n(\omega^*), \omega^*, j) \). Then \( \tilde{\omega} \) is a union of elements \( \omega' \in \Pi(n(\omega^*), \omega^*, j) \).

This follows from the fact that given any two cosets, either they are disjoint or
one contains the other. Our aim is to estimate $S_{n(\omega^*),f}(x) - S_{n(\omega),f}(x)$ where $x \in \omega'$. We define

$$h(t) = 0 \quad \text{if } t \notin \omega^*, \quad (59)$$

$$= \mu(\omega)^{-1} \int_{\omega} f(t)\chi_{m(\omega)}(t) \, dp(t) \quad \text{if } t \in \omega \in \Pi(n(\omega^*), \omega^*, j).$$

Note that if $t \in \omega \in \Pi(n(\omega^*), \omega^*, j)$, $h(t) = c_{m(\omega)}(\omega)$. Consequently, by (58)(ii), we have

$$||h||_\infty \leq p^{-j+1}\lambda. \quad (60)$$

If $\omega \in G/G_j$, $s' \geq s$, we have by (9)

$$S_{n(\omega^*),f}(x) - S_{n(\omega),f}(x) = \int_{\omega} f(t)\{D_{n(\omega^*)}(x-t) - D_{n(\omega)}(x-t)\} \, dp(t)$$

$$= \int_{\omega} f(t) \left\{ \chi_{n(\omega^*)}(x-t) \left( \sum_{r=2}^{s'} D_m(x-t) \Phi_{m_r} \right) - \chi_{n(\omega)}(x-t) \left( \sum_{r=2}^{s'} D_m(x-t) \Phi_{m_r} \right) \right\} \, dp(t). \quad (61)$$

By (7) both sums vanish if $x-t \notin G_j$ or equivalently if $t \notin \omega^*$. Now

$$\chi_{n(\omega^*)} = \left( \prod_{r=2}^{s'-1} \varphi_{m_r} \right) \chi_{n(\omega)}.$$

where by (1)(v), $\varphi_{m_r} \in X_r$ for $s \leq r \leq s' - 1$. By (7) the second sum vanishes unless $x-t \in G_j$. Consequently, $\chi_{n(\omega^*)}$ and $\chi_{n(\omega)}$ agree whenever the second sum does not vanish. Using the facts, we write (61) as

$$S_{n(\omega^*),f}(x) - S_{n(\omega),f}(x)$$

$$= \sum_{\omega' \in \Pi(n(\omega^*), \omega^*, j)} \int_{\omega} f(t)\chi_{n(\omega^*)}(t) \left( \sum_{r=2}^{s'-1} D_m(x-t) \Phi_{m_r} \right) \, dp(t) \quad (62)$$

where the sum is taken over all $\omega' \in \Pi(n(\omega^*), \omega^*, j)$. For each $\omega'$,

$$\sum_{r=2}^{s'-1} D_m(x-t) \Phi_{m_r}$$

is constant as $t$ ranges over $\omega'$. To see this, first consider the case $x \in \omega'$. The result follows by applying (13) with $n' = \sum_{r=2}^{s'-1} \alpha_r m_r$. In the case $x \in \omega'$, we have $x-t \in G_j$ as $t$ ranges over $\omega'$. With $n' = \sum_{r=2}^{s'-1} \alpha_r m_r$, we have by (12),
\[
\sum_{r=0}^{s-1} \Phi_m \Phi_m = \sum_{r=0}^{s-1} \sum_{k=m+1}^{r} \chi_k.
\]

In particular, \(\sum_{r=0}^{s-1} \Phi_m \Phi_m\) is a sum of characters \(\{\chi_k\}\) with \(k < m\). Hence by (4)(i) \(\sum_{r=0}^{s-1} \Phi_m \Phi_m\) is a sum of characters from \(X_r\). Since \(x - t \in G_r\) as \(t\) ranges over \(\omega\), the result holds. By (59) it follows that for each \(\omega' \in \Pi(n(\omega),\omega^*,j)\), we may replace \(f(t)\chi_{\omega^*}(t)\) by \(h(t)\) in (62). Using this fact, (12), and (59), we obtain

\[
S_{n(\omega)}f(x) = S_{n(\omega)}f(x) - S_{n(\omega)}f(x) = \sum_{x \in \omega^*} \int_{G} h(t) \left( \sum_{r=0}^{s-1} D_m(x - t) \Phi_m \Phi_m(x - t) \right) d\mu(t)
\]

(63)

The last equality follows from (12). It now follows from (15) that

\[
|S_{n(\omega)}f(x) - S_{n(\omega)}f(x)| \leq E^*(S^*_{n(\omega)}h)(x)
\]

where \(E^*\) denotes the martingale maximal function.

The set \(E_2\). We are now in position to define \(E_2\). For each pair \((n(\omega^*),\omega^*) \in \Lambda^*_\omega\) with \(\omega^* \subseteq E_1\), we define the subset

\[
V(n(\omega^*),\omega^*, j) = \{x \in \omega^* : E^*(S^*_{n(\omega)}h)(x) > p^{-j/2}\lambda\},
\]

where \(h\) is defined on \(\omega^*\) as in (59). Applying (17) and (18), each with \(q = 6\), and (60), we obtain

\[
\mu(V(n(\omega^*),\omega^*, j)) \leq (p^{-j/2}\lambda)^{-6} \|E^*(S^*_{n(\omega)}h)\|_{L^6}(p^{-j+1}\lambda)^{6} \mu(\omega^*)
\]

(66)

\[
\leq A_6(p^{-j/2}\lambda)^{-6} \mu(\omega^*)
\]

where \(\Lambda^*_\omega\) is the union of \(\Lambda_{j}^*\) for all \(j \geq 1\). Then set \(E_2 = \bigcup_{j=1}^{\infty} E_2^j\). Using (49) and (66) we obtain

\[
\leq A^j(p^{-j}\lambda)^{-6} \mu(\omega^*)
\]
We now set $E = E(\lambda, N, f) = E_1 \cup E_2$. Inequalities (51) and (67) imply

$$\mu(E) \leq A \lambda^{-2} ||f||^2.$$

Changing of pairs. Let $\omega^* \subseteq E$ satisfy $p^{-j} \lambda \leq C_{n(\omega^*)}(\omega^*)$. We show that there exist $\tilde{n}, \tilde{\omega}^*$ and $\tilde{j}$ such that

(i) $\tilde{n}(\tilde{\omega}) = n(\omega)$ where $\tilde{\omega}^* = \omega^*$;

(ii) $\tilde{\omega}^* \supset \omega^*$;

(iii) $1 \leq \tilde{j} \leq j$;

(iv) $(\tilde{n}(\tilde{\omega}^*), \tilde{\omega}^*) \in \Lambda^*.$

If $(n(\omega^*), \omega^*) \in \Lambda^*$, the result is obvious by setting $\tilde{n} = n, \tilde{j} = j, \tilde{\omega}^* = \omega^*$. We may therefore assume $(n(\omega^*), \omega^*) \notin \Lambda^*$. We first consider the case when $\omega^* = G$. Since $\omega^* \subseteq E$, (53) implies $C_{n(\omega^*)}(\omega^*) < \lambda$. Hence there exists $\tilde{j}$ with $1 \leq \tilde{j} \leq j$ such that

$$p^{-j} \lambda \leq C_{n(\omega^*)}(\omega^*) < p^{-j+1} \lambda.$$

Then there exists $\tilde{\omega}$ with $\tilde{\omega}^* = \omega^*$ such that

$$p^{-j} \lambda < |c_{n(\tilde{\omega})}(\tilde{\omega})| < p^{-j+1} \lambda.$$

By (35), $(n(\tilde{\omega}), \tilde{\omega}) \in \Lambda$. From (70) it follows that $\tilde{j} = \min\{j : (n(\tilde{\omega}), \tilde{\omega}) \in \Lambda, \tilde{\omega}^* = \omega^* \}$. Thus $(n(\omega^*), \omega^*) \notin \Lambda^*.$ (Recall the deletion.) We now consider the case when $\omega^* \neq G$. Since $p^{-j} \lambda \leq C_{n(\omega^*)}(\omega^*)$ and $(n(\omega^*), \omega^*) \notin \Lambda^*$, there must exist $\tilde{\omega}$ with $\tilde{\omega}^* = \omega^*$ and

$$\Delta(n(\tilde{\omega}), \tilde{\omega}) \geq p^{-j} \lambda.$$

(33) and (71) imply there exist $\omega'$ with $\omega' \supset \omega^*$ and $n'$ with $n'(\tilde{\omega}) = n(\tilde{\omega})$ such that

$$|c_{n'(\omega')}(\omega')| \geq p^{-j} \lambda.$$

Consequently,

$$C_{n'(\omega')}(\omega') \geq p^{-j} \lambda.$$
If \((n'(\omega^*), \omega^*) \in \Lambda^*_j\), we stop and set \(j = j\), \(\omega^* = \omega^*\), and \(\bar{n} = n'\). If \((n'(\omega^*), \omega^*) \not\in \Lambda^*_j\), we repeat the above argument and find \(n'', \omega''\) and \(j''\) such that \(\omega'' \supset \omega^*\), \(n''(\omega') = n'(\omega')\) and \(C_{n''(\omega'), n''(\omega''*)} \geq p^{-j}\lambda\). Note that \(n''(\omega') = n'(\omega')\) implies \(n''(\omega) = n'(\omega) = n(\omega)\). If \((n''(\omega^*), \omega^*) \in \Lambda^*_j\), we stop as before. Otherwise we continue until we reach a pair \((n_0(\omega^*_0), \omega^*_0)\) in \(\Lambda^*_j\) or reach \(\omega^*_0 = G\). If \(\omega^*_0 = G\), \(p^{-j}\lambda \leq C_{n_0(\omega^*_0), \omega^*_0} < \lambda\) since \(\omega^*_0 \subset E_2\). Hence there exists \(j_0\) such that \(1 \leq j_0 \leq j\) and

\[ p^{-j_0}\lambda \leq C_{n_0(\omega^*_0), \omega^*_0} < p^{-j_0+1}\lambda. \]

The argument of the preceding paragraph now implies \((n_0(\omega^*_0), \omega^*_0) \in \Lambda^*_j\). Setting \(\bar{n} = n_0, \omega^* = \omega^*_0\), and \(j = j_0\), we obtain \(\bar{n}, \omega^*,\) and \(j\) as in (69).

Thus given any \(\omega^* \subset E\) and \(C_{n(\omega), \omega^*} \geq p^{-j}\lambda\), there exist \(\bar{n}, \omega^*,\) and \(j\) which satisfy (69) such that \(j\) is minimal. It now follows that, for any \(\omega\) such that \(\omega^* \subset \omega \subset \omega^*\),

\[ C_{\bar{n}(\omega), \omega} < p^{-j+1}\lambda. \tag{73} \]

If (73) were false, the above argument applied to \((\bar{n}(\omega), \omega)\) would contradict the minimality of \(j\).

An additional estimate. An additional estimate is required because of the above change of pairs. Let \((n(\omega^*), \omega^*) \in \Lambda^*_j, \omega^* \subset E, \omega^* \in G/G_1\), so that \(\Pi(n(\omega^*), \omega^*, j)\) is defined. Let \(\omega^*_1\) be a partition element. We wish to estimate \(S_{n(\omega^*_1)} f(x) - S_{n(\omega)} f(x)\) where \(x \in \omega_1\). We have

\[ |S_{n(\omega^*_1)} f(x) - S_{n(\omega)} f(x)| \]

\[ = \left| \int_G f(t) (D_{n(\omega^*_1)}(x - t) - D_{n(\omega)}(x - t)) d\mu(t) \right| \]

\[ = \left| \int_G f(t) \left\{ \chi_{n(\omega^*_1)}(x - t) \sum_{r=1}^\infty D_{n(\omega^*_1)}(x - t) \Phi_{n, \omega^*_1}(x - t) \right. \right. \]

\[ - \chi_{n(\omega)}(x - t) \sum_{r=1}^\infty D_{n(\omega)}(x - t) \Phi_{n, \omega}(x - t) \left. \left. \right\} d\mu(t) \right|. \tag{74} \]

As before, both sums vanish if \(t \not\in \omega_1\), and \(\chi_{n(\omega^*_1)} = \chi_{n(\omega)}\), when the second sum fails to vanish. This allows us to write (74) as

\[ |S_{n(\omega^*_1)} f(x) - S_{n(\omega)} f(x)| \]

\[ = \left| \int_{\omega^*_1} f(t) \chi_{n(\omega^*_1)}(t) D_{n(\omega^*_1)}(x - t) \Phi_{n, \omega^*_1}(x - t) d\mu(t) \right| \]

\[ = \left| \int_{\omega^*_1} f(t) \chi_{n(\omega^*_1)}(t) m(t) \left( \sum_{k=p_{r+1}-1}^{p_{r+1}} \chi_{n(\omega^*_1)}(x - t) \right) d\mu(t) \right| \]

\[ \leq \sum_{k=p_{r+1}-1}^{p_{r+1}} |\mu(\omega^*_1)|^{-1} \left| \int_{\omega^*_1} f(t) \chi_{n(\omega^*_1)}(t) \Phi_{n, \omega^*_1}(x - t) d\mu(t) \right| \]

\[ \leq \sum_{k=p_{r+1}-1}^{p_{r+1}} |\mu(\omega^*_1)|^{-1} \left| \int_{\omega^*_1} f(t) \chi_{n(\omega^*_1)}(t) d\mu(t) \right|. \tag{75} \]
In the last line we made use of the fact that \( \chi_{km} \) is constant on cosets of \( G_{n+1} \). For each \( \omega_1 \) with \( \omega_1^\dagger = \omega_1 \) we have

\[
\mu(\omega_1)^{-1} \left| \int_{\omega_1} f(t) \chi_{km}(t) \, d\mu(t) \right| \leq C_{m+1}(\omega_1^\dagger).
\]

Thus we have

\[
(76) \quad |S_{m+1}(f(x) - S_m(f(x))| \leq p^2 C_{m+1}(\omega_1^\dagger).
\]

**Proof of \( L^2 \) result.** We now prove that if \( x \in E = E(\lambda, N, f) \), \( |S_n(f(x))| \leq A\lambda \), \( 1 \leq n \leq m_N \). Let \( \omega_0^* = G \). We may assume \( C_n(\omega_0^*) > 0 \). Then there exists \( j_0 \) such that \( p^{-j_0} \lambda \leq C_n(\omega_0^*) < p^{-j_0+1} \lambda \) since \( x \in E \) (see (53)). Then \( (n, \omega_0^*) \in \Lambda_{j_0}^* \) and the partition \( \Pi(n, \omega_0^*, j_0) \) is defined. Let \( \omega_1^\dagger \) denote the partition element such that \( x \in \omega_1 \). Then by (64) and (65), we have

\[
(77) \quad |S_{m+1}(f(x) - S_m(f(x))| \leq p^{-j_0/2} \lambda.
\]

If \( n(\omega_1^\dagger) = 0 \), we stop. Otherwise we continue with a typical step: \( n(\omega_1^\dagger) \neq 0 \) implies \( \omega_1^\dagger \not\in G/G_N \). Since \( \omega_1^\dagger \in \Pi(n(\omega_0^*), \omega_0^*, j_0) \) and \( x \in E \) (see (53)) we have \( p^{-j_0+1} \lambda \leq C_{m+1}(\omega_1^\dagger) \lambda \). Hence there exists \( j_1 \) such that, \( 1 \leq j_1 < j_0 \),

\[
p^{-j_1} \lambda \leq C_{m+1}(\omega_1^\dagger) \lambda < p^{-j_1+1} \lambda.
\]

By a change of pairs, we obtain \( \tilde{n}_1, \tilde{\omega}_1^\dagger, \tilde{j}_1 \) such that \( \tilde{n}_1(\omega_1) = n(\omega_1) \), \( \tilde{\omega}_1^\dagger \supset \omega_1^\dagger \), \( (\tilde{n}(\omega_1^\dagger), \tilde{\omega}_1^\dagger) \in \Lambda_{j_1}^* \), and \( \tilde{j}_1 \) is minimal. Then the partition \( \Pi(\tilde{n}_1(\omega_1^\dagger), \tilde{\omega}_1^\dagger, \tilde{j}_1) \) is defined. Let \( \omega_2^\dagger \) be the partition element such that \( x \in \omega_2 \). Since

\[
(78) \quad C_{n_2(\omega_2^\dagger)}(\omega_2^\dagger) = C_{m+1}(\omega_1^\dagger) < p^{-j_1+1}\lambda \leq p^{-j_1+1}\lambda,
\]

it follows that \( \omega_2^\dagger \supset \omega_1^\dagger \). Hence \( \tilde{n}_1(\omega_1) = n(\omega_1) \) implies \( \tilde{n}_1(\omega_2^\dagger) = n_1(\omega_2^\dagger) \). We have

\[
(79) \quad |S_{m+1}(f(x) - S_m(f(x))| \leq |S_{m+1}(f(x) - S_{m_2}(f(x))| + |S_{n_2}(f(x) - S_{n_1}(f(x))| + |S_{n_1}(f(x) - S_{n_2}(f(x))| \leq 2p^{-j_1/2} \lambda + |S_{m_2}(f(x) - S_{m}(f(x))|
\]

by (64) and (65). Now (76) and (78) imply

\[
(80) \quad |S_{m_2}(f(x) - S_{m}(f(x))| \leq p^2 C_{m+1}(\omega_1^\dagger) \leq p^{-j_1+3}\lambda.
\]

Combining (79) and (80), we have

\[
(81) \quad |S_{m_2}(f(x) - S_{m}(f(x))| \leq (2p^{-j_1/2} + p^{-j_1+3}) \lambda.
\]

Combining (77) and (81), we obtain
\[ |S_n f(x) - S_{n(\omega_2^*)} f(x)| \leq p^{-j_i/2} \lambda + \{2p^{-j_i/2} + p^{-j_i+3}\} \lambda. \]

If \( n(\omega_2^*) = 0 \), we stop. If \( n(\omega_2^*) \neq 0 \), we repeat the above step until we reach

\[ G = \omega_i^* \supset \omega_{i-1}^* \supset \cdots \supset \omega_{i-r}^* \]

with \( n(\omega_i^*) \neq 0 \), \( i = 1, 2, \ldots, r - 1 \), \( n(\omega_r^*) = 0 \), and \( j_0 > j_1 > j_2 > \cdots > j_r \geq 1 \)

\[ |S_{n(\omega_r^*)} f(x) - S_{n(\omega_{r+1}^*)} f(x)| < \{2p^{-j_i/2} + p^{-j_i+3}\} \lambda. \]

Then

\[ |S_n f(x)| \leq \sum_{i=0}^{r-1} |S_{n(\omega_i^*)} f(x) - S_{n(\omega_{i+1}^*)} f(x)| \]
\[ \leq 2\left( \sum_{j=1}^{\infty} p^{-j/2} \right) \lambda + p^3 \left( \sum_{j=1}^{\infty} p^{-j} \right) \lambda = A \lambda. \]

This completes the proof of the \( L^2 \) result.

IV. THE \( L^q \) RESULT

**Basic result.** To obtain the \( L^q \) result for \( 1 < q < 2 \), some properties of Lorentz spaces \( [6, \text{p.} 236] \) and an interpolation theorem of R. Hunt \( [5] \) reduce the problem to the following

**Basic result.** Let \( 1 < q < \infty, q \neq 2, \lambda > 0, \) and \( F \) be a measurable set in \( G \). Then there exists a constant \( A_q > 0, \) independent of \( \lambda \) and \( F \), such that

(82) \[ \mu \{ x \in G : M_I F(x) > \lambda \} \leq A_q \lambda^{-q} \mu(F) \]

where \( I_F \) is the characteristic function of \( F \).

Since the proof of the basic result follows the \( L^2 \) proof closely, we shall only indicate the necessary modifications. We shall borrow all the notation of the \( L^2 \) proof. We may also assume \( \mu(F) < \lambda^q \).

**Proof of basic result.** We begin by defining

\[ E_1 = \left\{ \omega : \mu(\omega)^{-1} \int_0^1 I_F(t) d\mu(t) \geq \lambda^q \right\} \quad \text{and} \]
\[ E_i = \{ \omega^* : \omega \in E_i \}. \]

Then (6) and (83) imply

\[ \sum_{\omega^* \in E_1} \mu(\omega^*) \leq p \sum_{\omega \in E_1} \mu(\omega) \]
\[ \leq p \lambda^{-q} \sum_{\omega \in E_1} \int_0^1 I_F(t) d\mu(t) \]
\[ \leq p \lambda^{-q} \mu(F). \]
Let \( L = L_q = [2q^2/(q - 1)] + 1 \) where \([x]\) denotes the greatest integer not greater than \(x\). Then if \((n(\omega), \omega) \in \Lambda_j, \omega \subset E_j\), we have

\[
\lambda^{-2} \leq p^L \lambda^{-q}.
\]

To see this we consider the cases \(1 < q < 2\) and \(q > 2\) separately. If \(1 < q < 2\), we have

\[
p^{-1} \lambda \leq |c(n(\omega))| \leq \mu(\omega)^{-1} \int_0 I_F(t) d\mu(t) \leq \lambda^q.
\]

This yields

\[
\lambda^{1-q} \leq p^j.
\]

Now for \(1 < q < 2\),

\[
(q - 2)(1 - q)^{-1} \leq 2q^2(q - 1)^{-1} \leq L.
\]

From (86) and (87) we have

\[
\lambda^{q-2} = (\lambda^{1-q})^{q-2}(1-q)^{-1} \leq (p^j)^L = p^L
\]

which is (85). In the case \(q > 2\), we have

\[
p^{-1} \lambda \leq |c(n(\omega))| \leq \mu(\omega)^{-1} \int_0 I_F(t) d\mu(t) \leq \mu(\omega \cap F)/\mu(\omega) \leq 1.
\]

Thus \(\lambda \leq p^j\) and so \(\lambda^{q-2} \leq p^{(q-2)} \leq p^L\), since \(q - 2 \leq L\), which is (85). Hence (85) is established. Applying (85) to (49), we obtain

\[
\sum_{n(\omega)} \mu(\omega) \leq p^{2j+2} \lambda^{-2} \mu(F) = p^{2j+2} \lambda^{-q} (\lambda^{q-2}) \mu(F)
\]

\[
\leq p^{2j+2} \lambda^{-q} (p^L) \mu(F) = p^{2j+2+L} \lambda^{-q} \mu(F)
\]

\[
\leq p^{4L} \lambda^{-q} \mu(F).
\]

As before, we use (88) to estimate \(\mu(E_2)\).

The partitions \(\Pi(n(\omega^*), \omega^*, j)\), the basic estimate, and the changing of pairs are the same as in the \(L^2\) proof. We modify the set \(E_2\) somewhat to compensate for the above estimate. Consider the operator \(E^* (S_{n(\omega)^*})\) which is sublinear and has strong type \((q, q)\) for \(1 < q < \infty\). Recall that \(\|E^*\|_q = O(1)\) as \(q \to \infty\) and \(\|S_{n(\omega)^*}\|_q = O(q)\) as \(q \to \infty\) independent of \(n(\omega^*)\). Hence \(\|E^* S_{n(\omega)^*}\|_q = O(q)\) as \(q \to \infty\) independent of \(n(\omega^*)\). By extrapolation [16, p. 119, vol. 2], there exist positive constants \(A_1\) and \(A_2\) such that
\( \mu(x \in \omega^*: |E^* S_{\omega^*}^* h(x)| > A_1 \lambda) \leq \exp(-A_2 \lambda \|h\|_{\infty}) \mu(\omega^*). \)

For the moment, let \( R \) denote an absolute constant to be determined later. We define

\( \mathcal{V}(n(\omega^*), \omega^*, j) = \{x \in \omega^*: |E^* (S_{\omega^*}^* h)(x)| > A_1 C j L p^{-j+1} \lambda\}. \)

Then by (89) and (60), we have

\[
\mu(\mathcal{V}(n(\omega^*), \omega^*, j)) \leq \exp(-A_2 C j L p^{-j+1} \lambda \|h\|_{\infty}) \mu(\omega^*)
\]

\[
\leq \exp(-A_2 C j L) \mu(\omega^*).
\]

We now choose \( C \) such that \( A_2 C \geq 5 \log p \). Then from (91) we obtain

\[
\mu(\mathcal{V}(n(\omega^*), \omega^*, j)) \leq \exp(-A_2 C j L) \mu(\omega^*)
\]

\[
\leq \exp(-5 \log L p) \mu(\omega^*)
\]

(92)

Summing over \( \omega^* \) and using (88) and (92), we obtain

\[
\mu(E_j) \leq \sum_{\omega^*} \mu(\mathcal{V}(n(\omega^*), \omega^*, j))
\]

(93)

\[
\leq p^{-5 \log L} \sum_{\omega^*} \mu(\omega^*)
\]

\[
\leq (p^{-5 \log L}) (p^{4L \lambda^{-q}}) \mu(F)
\]

\[= p^{-jL} \lambda^{-q} \mu(F). \]

Summing (93) over all \( j \), we have

\[
\mu(E) \leq \sum_{j=1}^{\infty} \mu(E_j)
\]

\[
\leq \left( \sum_{j=1}^{\infty} p^{-jL} \right) \lambda^{-q} \mu(F) \leq \left( \sum_{j=1}^{\infty} p^{-j} \right) \lambda^{-q} \mu(F). \]

We finally consider \( x \notin E = E_1 \cup E_2 \). As before, we assume \( c_n(\omega_0^*) > 0 \) where \( \omega_0^* = G \). Then there exists \( j_0 \geq 1 \) with \( p^{-j_0} \lambda \leq c_n(\omega_0^*) < p^{-j_0+1} \lambda \) since \( \omega_0^* \notin E \). Let \( \omega_1^* \) denote the partition element such that \( x \in \omega_1 \). Then (64) and (90) imply

(94)

\[
|S_n f(x) - S_{\omega_1^*} f(x)| \leq A_2 C j p^{-j+1} \lambda,
\]

where \( f = I_F \). If \( n(\omega_1^*) = 0 \), we stop. Otherwise we continue with a typical step. Since \( \omega_1^* \notin G/G_N \), there exists \( j_1 \) with \( 1 \leq j_1 < j_0 \) such that

(95)

\[
p^{-j_1} \lambda \leq c_{n(\omega_1^*)} < p^{-j_1+1} \lambda.
\]
By a change of pairs, we obtain \( n_1, \hat{\omega}_1, \hat{j}_1 \) such that \( n(\omega_1) = n_1(\omega_1) = \hat{n}(\omega_1) \cap \omega_1 \), \( (\hat{n}(\hat{\omega}_1), \hat{\omega}_1) \in \Lambda^* \), and \( \hat{j}_1 \) is minimal. Then \( \Pi(n(\hat{\omega}_1), \hat{\omega}_1, \hat{j}_1) \) is defined. Let \( \omega_2 \) be the partition element such that \( x \in \omega_2 \). Then as before \( \omega_2 \subseteq \omega_1 \), and \( n_1(\omega_1) = n(\omega_1) \) implies \( n_1(\omega_2) = n(\omega_2) \). We have, by (64), (76), (90) and (95),

\[
|S_{n(\omega_2)}f(x) - S_{n(\omega_2)}f(x)| \\
\leq |S_{n(\omega_2)}f(x) - S_{n(\omega_1)}f(x)| + |S_{n(\omega_1)}f(x) - S_{n(\hat{\omega}_1)}f(x)| \\
+ |S_{n(\hat{\omega}_1)}f(x) - S_{n(\omega_2)}f(x)| \\
\leq 2A_2Cj_1L^p\lambda + |S_{n(\omega_2)}f(x) - S_{n(\omega_1)}f(x)| \\
\leq 2A_2Cj_1L^p\lambda + p^2C_{n(\omega_1)}(\omega_1) \\
\leq 2A_2Cj_1L^p\lambda + p^2\lambda
\]

since \( \hat{j}_1 \) is minimal. Combining estimates (94) and (96), we have

\[
|S_{n}f(x) - S_{n(\omega_2)}f(x)| \leq 2A_2CLj_0p^{-j_0+1} + j_1p^{-j_1+1} + p^{-j_1+3}\lambda.
\]

If \( n(\omega_2) = 0 \), we stop. If \( n(\omega_2) \neq 0 \), we repeat the above procedure until we reach

\[
G = \omega_0 \supseteq \omega_1 \supseteq \omega_2 \supseteq \cdots \supseteq \omega_r
\]

with \( n(\omega_i) \neq 0 \), \( i = 1, 2, \ldots, r-1 \), \( n(\omega_r) = 0 \) and \( j_0 > j_1 > j_2 > \cdots > j_r \geq 1 \), with

\[
|S_{n(\omega_r)}f(x) - S_{n(\omega_1)}f(x)| \leq 2A_2CLj_0p^3\lambda.
\]

Then

\[
|S_{n}f(x)| \leq \sum_{r=0}^{r-1} |S_{n(\omega_r)}f(x) - S_{n(\omega_{r+1})}f(x)| \\
\leq \left( 2A_2CL\left( \sum_{j=1}^{\infty} j^p \right) + p^3\left( \sum_{j=1}^{\infty} j^p \right) \right) \lambda.
\]

This establishes (82), the basic result, and completes the proof.

**Bibliography**


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