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THE LAW OF THE ITERATED LOGARITHM FOR BROWNIAN MOTION IN A BANACH SPACE

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ABSTRACT. Strassen's version of the law of the iterated logarithm is proved for Brownian motion in a real separable Banach space. We apply this result to obtain the law of the iterated logarithm for a sequence of independent Gaussian random variables with values in a Banach space and to obtain Strassen's result.

Introduction. Let H denote a real separable Hilbert space with norm $\|\cdot\|_H$ and assume $\|\cdot\|_B$ is a measurable norm on H in the sense of [2]. Then there exists a constant $M > 0$ such that $\|x\|_B \leq M\|x\|_H$ for all $x \in H$, and if B is the completion of H in $\|\cdot\|_B$ it follows that B is a real separable Banach space. We will view H as a subspace of B and since $\|\cdot\|_B$ is weaker than $\|\cdot\|_H$ on H it follows that B^*, the topological dual of B, can be continuously injected into H^*, the topological dual of H. We call (H, B) an *abstract Wiener space.*

For $t > 0$, let m_t denote the canonical Gaussian cylinder set measure on H with variance parameter t and let μ_t $(t > 0)$ denote the Borel probability measure on B induced by m_t $(t > 0)$. We call μ_t the *Wiener measure on B generated by H with variance parameter t.*

Let Ω_B denote the space of continuous functions w from $[0, \infty)$ into B such that $w(0) = 0$, and let $\mathcal{G}$ be the σ-field of Ω_B generated by the functions $w \to w(t)$. Then there is a unique probability measure P on $\mathcal{G}$ such that if $0 = t_0 < t_1 < \cdots < t_n$ then $w(t_j) - w(t_{j-1})$ $(j = 1, \ldots, n)$ are independent and $w(t_j) - w(t_{j-1})$ has distribution $\mu_{t_j - t_{j-1}}$ on B. In particular, the stochastic process W_t defined on $(\Omega_B, \mathcal{G}, P)$ by $W_t(w) = w(t)$ has stationary independent Gaussian increments with transition probabilities $P_t(x, A) = \mu((A - x)/\sqrt{t})$ for $t > 0$. We call it *Brownian motion in B.* For a more detailed discussion see [2].

It is known from [2] that if B is an arbitrary real separable Banach space, then there exists a dense subset H of B which is a real separable Hilbert space and the given norm on B is a measurable norm on H. Hence any real separable Banach space can be used in the setup we described above. We also know from [5] or from [6] and [1] that if μ is any mean zero Gaussian probability measure on the Borel subsets of a real separable Banach space B, then there exists a real separable Hilbert space H which is a subset of B, the given norm on B is a

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measurable norm on H, $\mu(M) = 1$ (M is the closure of H in B), and μ is the Wiener measure on M generated by H with variance parameter 1. Furthermore, H is unique as a subset of B since it is precisely the set of vectors such that μ translated by such a vector yields a measure equivalent to μ. However, the main point to be realized is that given any real separable Banach space B, or B and a mean zero Gaussian measure μ on B, we can construct a Brownian motion on B as indicated above. Further, in the case μ is given on B we see that $\mu = \mu_1$ and that if M is a proper subspace of B then our Brownian motion is, with probability one, in the closed subspace M satisfying $\mu_1(M) = 1$.

Let C_B denote the continuous functions on $[0, 1]$ into B which vanish at zero. Then C_B is a Banach space in the norm $\|f\|_{C_B} = \sup_{0\leq t\leq 1} \|f(t)\|_B$.

Lemma 1. (a) *If B is a real separable Banach space, then C_B is a real separable Banach space in the norm $\|\cdot\|_{C_B}$.*

(b) *The minimal sigma-algebra $\mathcal{B}$ making the mappings $f \to f(t)$ measurable consists of the Borel subsets of C_B.*

(c) *Brownian motion on B induces a probability measure P on $(C_B, \mathcal{B})$ which is a mean zero Gaussian measure, i.e. every linear functional in C_B^* has a Gaussian distribution with mean zero.*

Proof. (a) Let $t_j = j/2^N, j = 0, 1, \ldots, 2^N$. Let $\{x_n\}$ be a dense subset of B. Let S_N denote the subset of C_B consisting of functions which are linear on each of the subintervals $[t_{j-1}, t_j]$ with values at t_j in $\{x_n\}$. Then $\bigcup_{N=1}^{\infty} S_N$ is a countable dense subset of C_B.

(b) Since C_B is separable it suffices to prove that if $f_0 \in C_B$ and $\epsilon > 0$, then $U = \{f: \|f - f_0\|_{C_B} \leq \epsilon\}$ is a set in $\mathcal{B}$. Let $I_N = \{f: \sup_{1\leq j\leq 2^N} \|f(t_j) - f_0(t_j)\|_{C_B} \leq \epsilon\}$ for $N = 1, 2, \ldots$ and $\{t_j\}$ as in (a). Then $U = \bigcap_{N=1}^{\infty} I_N$.

(c) P is the probability measure on $(C_B, \mathcal{B})$ such that if $0 = t_0 < t_1 < \cdots < t_n \leq 1$ then $f(t_j) - f(t_{j-1})$ ($j = 1, \ldots, n$) are independent and $f(t_j) - f(t_{j-1})$ has distribution $\mu_{t_j - t_{j-1}}$ on B. We now must show P is a mean zero Gaussian measure on C_B. Let $f^* \in C_B^*$ and let $X_1, \ldots, X_n$ be independent random variables with values in C_B and the same distribution as P. Then $X_1 + \cdots + X_n/\sqrt{n}$ has distribution P since for each $t \in [0, 1]$ the law of the map $f \to f(t)$ is the convolution of $\mu_{t/n}$ n times yielding μ_t. Hence the distribution of f^* has the same distribution as

$$f^*(X_1 + \cdots + X_n/\sqrt{n}) = f^*(X_1) + \cdots + f^*(X_n)/\sqrt{n}$$

and we see by [4, p. 166] that f^* is strictly stable with characteristic exponent 2 and this implies f^* has a Gaussian distribution with mean zero.

Our main result is a law of the iterated logarithm for Brownian motion in a Banach space as described prior to Lemma 1. This may be regarded as a general synthesis of the two log log law which follows.

I. (Strassen [10]). Let Ω_k denote the set of continuous maps from $[0, \infty)$ into real k-dimensional space ($\mathbf{R}^k$) which vanish at zero, and let C_k denote the space of continuous maps vanishing at zero and mapping $[0, 1]$ into $\mathbf{R}^k$ endowed with the supremum of the Euclidean norm for $\mathbf{R}^k$. If $W(t) = (W_1(t), \ldots, W_k(t))$, $0 \leq t < \infty$, is a version of the k-dimensional Brownian motion with sample paths in Ω_k, then the sequence of random functions

$$\zeta_n(t) = W(nt)/(2n \log \log n)^{1/2} \qquad (0 \leq t \leq 1, n \geq 3)$$

satisfies the following log log law:

$\{\zeta_n, n \geq 3\} \subseteq C_k$ and with probability one converges in C_k

to a compact set K_k of C_k and clusters at every point of K_k.

Here K_k denotes those $f = (f_1, \ldots, f_k) \in C_k$ such that f is coordinatewise absolutely continuous with respect to Lebesgue measure on $[0, 1]$, and satisfies $\sum_{i=1}^k \int_0^1 [df_i(s)/ds]^2 ds \leq 1$. By saying $\{\zeta_n : n \geq 3\}$ converges to K_k we mean that for every $\epsilon > 0$ the sequence is eventually within an ϵ-neighborhood of K_k and since K_k is compact this implies that with probability one $\{\zeta_n : n \geq 3\}$ is relatively norm compact in C_k.

II. (LePage [8]). Suppose B is a real separable Banach space and μ is a mean zero Gaussian measure on the Borel subsets of B. If $X_1, X_2, \ldots$ are independent identically distributed B-valued random vectors with distribution μ, then the sequence

$$\xi_n = X_1 + \cdots + X_n/(2n \log \log n)^{1/2} \qquad (n \geq 3)$$

almost surely converges in B-norm to a closed set $K \subseteq B$ and clusters at every point of K, where K is the unit ball of the reproducing kernel Hilbert space defined on $B^* \times B^*$ by μ.

The set K_k of Strassen's result may be identified as the unit ball of the reproducing kernel Hilbert space of the kernel defined for $0 \leq s, t \leq 1$, $0 \leq i, j \leq k$ by

$$E\,(w_i(s)w_j(t)) = \min(s, t)\delta_{ij}. \tag{1}$$

This suggests that I may be extended to B-valued Brownian motion using the methods of II. As it turns out, the resulting Theorem 1 of §4 contains both I and II as special cases, and is obtained in a self-contained manner independent of I and II.

2. **Some properties of Brownian motion on** B. Here we provide some basic lemmas. The content of Lemma 2 is found in [7] and can also be expressed in slightly different terms using [6].

Lemma 2. *For* (B, μ) *as in* II, *let* $\mathcal{L}$ *be the closure of* B^* *in* $L_2(B, \mu)$. *For each* $L \in \mathcal{L}$ *the convergent Bochner integral* $x_L = \int_B L(x)x\,\mu(dx) \in B$ *exists. H*

$= \{x_L: L \in \mathcal{L}\} \subseteq B$ is a real separable Hilbert space isometrically isomorphic to $\mathcal{L}$ under the inner product $(x_{L_1}, x_{L_2}) = \int_B L_1(x) L_2(x)\,\mu(dx)$. On H, $\|\cdot\|_B \leq \|\mu\|\,\|\cdot\|_H$ where $\|\mu\|^2 = \int_B \|x\|_B^2 \mu(dx) < \infty$. If $y^* \in B^*$ and $y = \int_B y^*(x) x\,\mu(dx)$, then $(y, x)_H = y^*(x)$ for every $x \in H$. If $\{x_j^*: j \geq 1\} \subseteq B^*$ is a complete orthonormal sequence for $\mathcal{L}$ and $\{x_j: j \geq 1\} \subseteq H$ is the set of images $x_j = \int_B x_j^*(x) x\,\mu(dx)$ $(j \geq 1)$, then $\sum_{j=1}^k x_j^*(x) x_j \to x$ as $k \to \infty$, everywhere on H in the sense of the H-norm and almost everywhere on B in the B-norm. The closure $\overline{H}$ of H in B is the topological support of μ on B and if elements of B are interpreted as (evaluation) functions on B^*, H may be interpreted as the reproducing kernel Hilbert space of μ.

Proof. That H is separable follows from [6] and the remainder is given in [7].

By [1, Theorems 2 and 3] and the fact that $\|\cdot\|_B \leq M\|\cdot\|_H$ we have that $\|\cdot\|_B$ is a measurable norm in the sense of Gross [2] and hence $(H, \overline{H})$ is an abstract Wiener space.

Lemma 3. *Let B be a real separable Banach space with norm $\|\cdot\|_B$. Let H be a subspace of B which is a real Hilbert space in the norm $\|\cdot\|_H$ and assume $\|\cdot\|_B$ is a measurable norm on H. Let K denote the unit ball of H, i.e. $K = \{x \in H: \|x\|_H \leq 1\}$. Then K is a compact subset of B.*

Proof. First we show K is a closed subset of B. Let $\{y_n\} \subseteq K$ and assume $\lim_n y_n = y \in B$ in the norm $\|\cdot\|_B$. Now $\{y_n\} \subseteq K$ implies there is a subsequence $\{y_{n_j}\}$ such that $\{y_{n_j}\}$ converges weakly in H to $z \in H$. Now $\|z\|_H \leq 1$ by the uniform boundedness principle, and since $\{y_{n_j}\}$ also converges to y in $\|\cdot\|_B$ we have $\{y_{n_j}\}$ converging weakly to y and to z in B because B^* can be viewed as a subset of H^*. That is, since $\|\cdot\|_B$ is a measurable norm on H we have a constant M such that $\|\cdot\|_B \leq M\|\cdot\|_H$, and hence B^* can be continuously injected into H^*. Now B^* separating points of B implies $y = z$, and hence $z \in K$ implies $y \in K$. Hence K is closed in B.

Now we show K is compact in B. To do this we note that since $\|\cdot\|_B$ is a measurable norm on H we can construct a second measurable norm on H as in [2], call it $\|\cdot\|_1$, such that for $r > 0$, $V_r = \{x \in H: \|x\|_1 \leq r\}$ has a compact closure in B. Now $\|\cdot\|_1$ measurable on H implies there exists an $M > 0$ such that $\|x\|_1 \leq M\|x\|_H$ for all $x \in H$, and hence $K \subseteq \{x \in H: \|x\|_1 \leq M\}$. Thus K has compact closure in B and since K is closed we have K compact.

There are three separable Banach spaces, each with a mean zero Gaussian measure situated on its Borel subsets, which figure in our analysis. Of these, (C_B, P) and (B, μ) have already been introduced. The third is (C, ν) where C is the space of real-valued continuous maps on $[0, 1]$ which vanish at zero (with the supremum norm) and ν is Wiener measure. Each of (C_B, P), (C, ν), (B, μ) satisfies the hypotheses of Lemma 2. Let $\mathcal{H} \subseteq C_B$, $H_0 \subseteq C$, $H \subseteq B$ denote the respective Hilbert spaces given for each of these spaces by Lemma 2, and let $\mathcal{K}$, K_0, K be the respective unit balls of these spaces. Then Lemma 3 applies to $\mathcal{K}$, K_0 and K.

Using Lemma 2 one may prove the following familiar characterization of H_0: $\phi \in H_0$ iff $\phi(0) = 0$, ϕ is absolutely continuous with respect to Lebesgue measure on $[0, 1]$ and $\int_0^1 [(d/dt)\phi(t)]^2 dt < \infty$. The inner product on H_0 is

$$(\phi_1, \phi_2)_{H_0} = \int_0^1 \frac{d}{dt}\phi_1(t)\frac{d}{dt}\phi_2(t)\,dt.$$

Our next result enables us to interpret $\mathcal{H}$ as a denumerable direct sum of copies of H_0.

Lemma 4. *$\mathcal{H}$ has the following characterization in terms of any set $\{x_j^*: j \geq 1\} \subseteq B^*$ such that $\{x_j: j \geq 1\}$ is a complete orthonormal set for H: $f \in \mathcal{H}$ iff $f(0) = 0, f(t) \in H$ for each $t \in [0, 1]$, each $x_j^*(f) \in H_0$, and*

$$\sum_j \int_0^1 [(d/dt)x_j^*(f)(t)]^2 dt < \infty.$$

The inner product on $\mathcal{H}$ is given by

$$(f_1, f_2)_{\mathcal{H}} = \sum_j \int_0^1 \frac{d}{dt}x_j^* f_1(t)\frac{d}{dt}x_j^* f_2(t)\,dt \quad \text{for } f_1, f_2 \in \mathcal{H}.$$

In somewhat greater detail we have the following:

(a) *$\mathcal{H} = H_0 \otimes H$ (the tensor product).*

(b) *If $x^* \in B^*$ and $f \in \mathcal{H}$ then $x^*f \in H_0$ and, for every $\phi \in H_0$, $(x^*f, \phi)_{H_0} = (f, \phi x)_{\mathcal{H}}$, where $(x^*f)(t) = x^*(f(t))$, $t \in [0, 1]$. $\|x^*f\|_{H_0} \leq \|f\|_{\mathcal{H}}\|x\|_H$.*

(c) *If $f \in \mathcal{H}$ and $t \in [0, 1]$ then $f(t) \in H$ and, for every $x^* \in B^*$, $(f(t), x)_H = (f, \min(t, \cdot)x)_{\mathcal{H}}$. $\|f(t)\|_H \leq \|f\|_{\mathcal{H}}\sqrt{t}$.*

(d) *For $\{x_j^*: j \geq 1\} \subseteq B^*$ and $\{x_j: j \geq 1\} \subseteq H$ as above, $\sum_{j=1}^k x_j^*(f)x_j \to f$ as $k \to \infty$ everywhere on $\mathcal{H}$ in the sense of $\mathcal{H}$ norm and almost everywhere on C_B in the sense of C_B norm. That is, if P is the Gaussian measure induced on the Borel subsets of C_B by Brownian motion on B, then with P-probability one for $f \in C_B$*

$$\sup_{0 \leq t \leq 1} \left\| f(t) - \sum_{j=1}^k x_j^*(f(t))x_j \right\|_B \to 0 \quad \text{as } k \to \infty,$$

and the law of $x_j^* f(t)$ $(j \geq 1)$ is that of mutually independent one dimensional Brownian motions normalized as usual.

Proof. For each $t \in [0, 1]$, $x^* \in B^*$, let $\Lambda_{t,x^*}(f) = x^*(f(t))$ for $f \in C_B$. Then $\Lambda_{t,x^*} \in C_{B^*}$ and these functionals separate points of C_B. To prove $\mathcal{H} = H_0 \otimes H$, suppose $t \in [0, 1]$, $x^* \in B^*$. We first show that the element of C_B defined by $\min\{t, \cdot\}x$ is the Bochner integral $\int_{C_B} \Lambda_{t,x^*}(f)fP(df)$ and therefore by Lemma 2 applied to (C_B, P) we have $\min(t, \cdot)x \in \mathcal{H}$. We proceed by evaluation of the two expressions. If $s \in [0, 1]$, $y^* \in B^*$, then

$$\Lambda_{s,y^*}(\min(t, \cdot)x) = \min(t, s)y^*(x) = \min(t, s)(y, x)_H \tag{2}$$

by Lemma 2. Now

$$
\begin{aligned}
(3)\qquad \Lambda_{s,y^*}\left(\int_{C_B} \Lambda_{t,x^*}(f) f P(df)\right) &= \int_{C_B} x^*(f(t)) y^*(f(s)) P(df) \\
&= \int_{C_B} x^*(f(\min(t,s))) y^*(f(\min(t,s))) P(df)
\end{aligned}
$$

by independence of increments, and using the stationarity of the increments of Brownian motion on B we have (3) equal to

$$
(4)\qquad \min(t,s)\int_{C_B} x^*(f(1)) y^*(f(1)) P(df) = \min(t,s)(y,x)_H
$$

since $f \to f(1)$ induces the measure $\mu = \mu_1$ on B. Combining (2) and (4) we have $\min(t,\cdot)x \in \mathcal{H}$. From (3) and (4) we have the factorization

$$
\begin{aligned}
(\min(s,\cdot)y, \min(t,\cdot)x)_{\mathcal{H}} &= \min(s,t)(x,y)_H \\
&= (\min(s,\cdot), \min(t,\cdot))_{H_0}(x,y)_H.
\end{aligned}
$$

This proves $\mathcal{H} = H_0 \otimes H$ provided the elements $\{\min(t,\cdot)x\colon 0 \le t \le 1, x^* \in B^*\}$ can be shown to span $\mathcal{H}$ (for a discussion of tensor products of reproducing kernel Hilbert spaces see [9]). To see this, suppose $f_0 \in \mathcal{H}$ and $(f_0, \min(t,\cdot)x)_{\mathcal{H}} = 0$ for all $t \in [0,1]$, $x^* \in B^*$. By Lemma 2 there is an element L_0 belonging to the closure of the subspace of $L^2(C_B, P)$ spanned by C_B^* for which $f_0 = \int_{C_B} L_0(f) f P(df)$. Then

$$
\Lambda_{t,x^*}(f_0) = \int_{C_B} L(f)\Lambda_{t,x^*}(f) P(df) = (f_0, \min(t,\cdot)x)_{\mathcal{H}} = 0
$$

for all t and x^*. Hence $f_0 = 0$ in C_B and in $\mathcal{H}$. This completes the proof of $\mathcal{H} = H_0 \otimes H$.

To prove (b) suppose $x^* \in B^*$, $f \in \mathcal{H}$, $\phi \in H_0$. If f is of the specialform $f = \phi_1 x_1$ with $\phi_1^* \in C^*$, $x_1^* \in B^*$, then $x^* f = \phi_1 x^*(x_1) \in H_0$ and$(f, \phi x)_{\mathcal{H}} = (\phi_1 x_1, \phi x)_{\mathcal{H}} = (x_1, x)_H(\phi_1, \phi)_{H_0} = x^*(x_1)(\phi_1, \phi)_{H_0} = (x^*(\phi_1 x_1), \phi)_{H_0} = (x^* f, \phi)_{H_0}$. Thus for every f expressible as a finite sum of elements of the form $\phi_1 x_1$ we have $x^* f \in H_0$ and $(x^* f, \phi)_{H_0} = (f, \phi x)$. To extend this to all $f \in \mathcal{H}$ we need for f of the type of sum just considered the inequality $\|x^* f\|_{H_0} \le \|f\|_{\mathcal{H}} \cdot \|x\|_H$. To prove this note that if $\{\phi_j^*\colon j \ge 1\} \subseteq C^*$ and $\{\phi_j\colon j \ge 1\} \subseteq H_0$ is complete and orthonormal for H_0 then for f of the above type,

$$
\|x^* f\|_{H_0}^2 = \sum_j (x^* f, \phi_j)_{H_0}^2 = \sum_j (f, \phi_j x)^2 \le \|f\|_{\mathcal{H}}^2 \|x\|_H^2
$$

since $\{\phi_j x\colon j \ge 1\} \subseteq \mathcal{H}$ are orthogonal in the tensor product and each have norm squared equal to $\|x\|_H^2$. If $f \in \mathcal{H}$ then there exists $f_n \in \mathcal{H}$ of the above type tending to f in $\mathcal{H}$. Now $\mathcal{H} \subseteq C_B$ and $x^* \in B^*$ implies $x^*(f_n(t)) \to x^*(f(t))$ as $n \to \infty$ since $f_n \to f$ in $\mathcal{H}$ implies $f_n \to f$ in C_B by Lemma 2. Further, by the above inequality $x^* f_n$ converges in H_0 as $n \to \infty$. Combining the last two statements we have $x^* f \in H_0$ and $x^* f_n \to x^* f$ in H_0. Finally,

$$(x^*f, \phi)_{H_0} = \lim_n (x^*f_n, \phi)_{H_0} = \lim_n (f_n, x)_{\mathcal{H}} = (f, \phi x)_{\mathcal{H}}$$

and

$$\|x^*f\|_{H_0}^2 = \lim_n \|x^*f_n\|_{H_0}^2 \le \lim_n \|f_n\|_{\mathcal{H}}^2 \|x\|_H^2 = \|f\|_{\mathcal{H}}^2 \|x\|_H^2.$$

The proof of (c) is analogous to (b). Suppose $f \in \mathcal{H}$ and $t \in [0, 1]$. If f is of the form $\phi_1 x_1$ for some $\phi_1^* \in C^*$, $x_1^* \in B^*$ then $f(t) = \phi_1(t)x_1 \in H$ and if $x \in H$ then from Lemma 2 $(f(t), x)_H = x^*f(t) = (f, \min(t, \cdot)x)_{\mathcal{H}}$ from part (a). If f is a finite sum of elements of the above type then $f(t) \in H$, $(f(t), x)_H = (f, \min(t, \cdot\, x))_{\mathcal{H}}$ and

$$\|f(t)\|_H^2 = \sum_j (f(t), x_j)_H^2 = \sum_j (f, \min(t, \cdot)x_j)_{\mathcal{H}}^2 \le \|f\|_{\mathcal{H}}^2 \|\min(t, \cdot)\|_{H_0}^2 = t\|f\|_{\mathcal{H}}^2.$$

Suppose f_n are such finite sums and $f_n \to f$ in $\mathcal{H}$. Then $f_n(t)$ converges to $f(t)$ in B and $x^*f_n(t) \to x^*f(t)$ for each $x^* \in B^*$. By the previous inequality $f_n(t)$ converges in H, and hence $f(t) \in H$ and $f_n(t) \to f(t)$ in H. By passage to the limit we get $(f(t), x)_H = (f, \min(t, \cdot)x)_{\mathcal{H}}$ and $\|f(t)\|_H \le \|f\|_{\mathcal{H}}\sqrt{t}$.

To prove (d) assume $\{x_j^* : j \ge 1\} \subseteq B^*$ and $\{x_j : j \ge 1\} \subseteq H$ is a complete orthonormal set for H. Likewise suppose $\{\phi_n^* : n \ge 1\} \subseteq C^*$ and $\{\phi_n : n \ge 1\} \subseteq H_0$ is complete and orthonormal for H_0. Then from (a) $\{\phi_n x_j : n \ge 1, j \ge 1\} \subseteq \mathcal{H}$ is complete and orthonormal for $\mathcal{H}$. For arbitrary $n \ge 1, j \ge 1$ we see that for every $\phi \in H_0$, $x \in H$, $\phi_n^* x_j^*(\phi x) = \phi_n^*(\phi) x_j^*(x)$ by linearity. From Lemma 2 we have $\phi_n^* x_j^*(\phi x) = (\phi_n, \phi)_{H_0}(x_j, x)_h$ and hence $\phi_n^* x_j^*$ yields $\phi_n x_j$ by Bochner integration on C_B. Then everywhere on $\mathcal{H}$ and in $\mathcal{H}$ norm we have $f = \sum_{n,j} (f, \phi_n x_j)_{\mathcal{H}} \phi_n x_j = \sum_{n,j} \phi_n^*(x_j^*(f))\phi_n x_j$. For each $j \ge 1$ the series may be summed on n in $\mathcal{H}$ obtaining

$$f = \sum_j \left(\sum_n (f, \phi_n x_j)_{\mathcal{H}} \phi_n \right) x_j = \sum_j \left(\sum_n (x_j^* f, \phi_n)_{H_0} \phi_n \right) x_j = \sum_j x_j^*(f) x_j.$$

The argument may be repeated almost everywhere on C_B in C_B norm and hence $f = \sum_j x_j^*(f) x_j$ almost everywhere on C_B.

Using the explicit description of H_0 given previous to the present lemma the characterization of $\mathcal{H}$ with which we began the statement of the lemma follows easily from the above series representation.

Since P is a mean zero Gaussian measure on C_B^1 it follows easily from (3) and (4) (since the joint distributions are all Gaussian) that $x_j^* f(t)$ $(j \ge 1)$ are independent one dimensional Brownian motions.

For every $\epsilon > 0$ let $\mathcal{K}_\epsilon$ denote the open ϵ-neighborhood of $\mathcal{K}$ in C_B.

Lemma 5. *For each* $\epsilon > 0$, *there exists* $r > 1$ *such that*

$$P\{f \in C_B : f/\sqrt{2 \log \log s} \notin \mathcal{K}_\epsilon\} \le \exp(-r^2 \log \log s)$$

for all sufficiently large s.

Proof. This result can be proved just as Proposition 1 of [8] is obtained.

Lemma 6. *If $\epsilon > 0$ one may choose $c > 1$ sufficiently close to one so that for every $f \in \Omega_B$ the statements $[c^n] \leq s \leq [c^{n+1}]$ and $f([c^{n+1}]\,\cdot)/\{2[c^{n+1}]\log\log[c^{n+1}]\}^{1/2} \in \mathcal{K}_\epsilon$ together imply $f(s\,\cdot)/\sqrt{2s\log\log s} \in \mathcal{K}_{2\epsilon}$ for all sufficiently large n.*

Proof. Suppose $\epsilon > 0$ and choose $c > 1$ so that for all sufficiently large n, $\gamma_n\epsilon + (\gamma_n - 1)\|P\| < 2\epsilon$ where

$$\gamma_n = \left(\frac{[c^{n+1}]\log\log[c^{n+1}]}{[c^n]\log\log[c^n]}\right)^{1/2}.$$

This is possible because $[c^{n+1}] < c^2[c^n]$ for all large n. If $h \in \mathcal{K}, f \in C_B$,

$$\|f([c^{n+1}]\,\cdot)/(2[c^{n+1}]\log\log[c^{n+1}])^{1/2} - h(\cdot)\|_{C_B} < \epsilon,$$

then

$$\left\|f(s\,\cdot)/(2[c^{n+1}]\log\log[c^{n+1}])^{1/2} - h\left(\frac{s\,\cdot}{[c^{n+1}]}\right)\right\|_{C_B} < \epsilon$$

and

$$\left\|h\left(\frac{s\,\cdot}{[c^{n+1}]}\right)\right\|_{\mathcal{H}} \leq \frac{s}{[c^{n+1}]}\|h\|_{\mathcal{H}} \leq 1.$$

Hence $h((s/[c^{n+1}])\,\cdot) \in \mathcal{K}$ and

$$\begin{aligned}
&\left\|f(s\,\cdot)/(2s\log\log s)^{1/2} - h\left(\frac{s\,\cdot}{[c^{n+1}]}\right)\right\|_{C_B} \\
&\quad\leq \left\|f(s\,\cdot)/(2s\log\log s)^{1/2} - \left(\frac{[c^{n+1}]\log\log[c^{n+1}]}{s\log\log s}\right)^{1/2} h\left(\frac{s\,\cdot}{[c^{n+1}]}\right)\right\|_{C_B} \\
&\qquad + \|P\|\left\|\left(\frac{[c^{n+1}]\log\log[c^{n+1}]}{s\log\log s}\right)^{1/2} h\left(\frac{s\,\cdot}{[c^{n+1}]}\right) - h\left(\frac{s\,\cdot}{[c^{n+1}]}\right)\right\|_{\mathcal{H}} \\
&\quad\leq \gamma_n\epsilon + (\gamma_n - 1)\|P\| < 2\epsilon
\end{aligned}$$

if n is sufficiently large.

For what follows we assume $\{x_j^*: j \geq 1\} \subseteq B^*$ and $\{x_j: j \geq 1\} \subseteq H$ is complete and orthonormal for H. For each $k \geq 1$ and $f \in C_B$ let $f^{(k)} = \sum_{j=1}^k x_j^*(f)x_j$.

Lemma 7. *For each $\epsilon > 0$ and $r > 1$ there exists k sufficiently large so that*

$$P(f \in C_B: \|f - f^{(k)}\|_{C_B} \geq \epsilon\sqrt{2\log\log s}) \leq \exp(-r^2\log\log s) \tag{5}$$

for all sufficiently large s.

Proof. By (d) of Lemma 4 this result follows just as in Lemma 4 of [8].

The main theorem and some corollaries. Our basic theorem is the following and implies the result of Strassen mentioned in I and that of LePage in II.

Theorem 1. *Let* $\{W(t): 0 \leq t < \infty\}$ *be Brownian motion on* B *and for each* $t \in [0,1]$, $s \geq 3$, *let*

$$\zeta_s(t) = W(st)/\sqrt{2s \log \log s}\,.$$

Then the net $\{\zeta_s: s \geq 3\}$ *is a subset of* C_B *and with probability one converges in* C_B *to the compact set* $\mathcal{K}$ *and clusters at every point of* $\mathcal{K}$, *where* $\mathcal{K}$ *is the unit ball of the reproducing kernel Hilbert space (equivalently,* $\mathcal{K}$ *is the unit ball of the Hilbert subspace of* C_B *which generates* P).

Proof. That $\mathcal{K}$ is compact in C_B follows from Lemma 3 by applying the lemma to C_B, $\mathcal{H}$, and $\|\cdot\|_{C_B}$. For every $\epsilon > 0$, there exists $r > 1$ such that

$$\begin{aligned}\Pr(\zeta_s \notin \mathcal{K}_\epsilon) &= P(f \in C_B: \|f - \mathcal{K}\|_{C_B} \geq \epsilon\sqrt{2 \log \log s}) \\ &\leq \exp(-r^2 \log \log s)\end{aligned}$$

for all sufficiently large s by Lemma 5. Hence by the Borel-Cantelli lemma for $c > 1$ there is a set A of probability one such that the sequence $\zeta_{[c^n]} \in \mathcal{K}_\epsilon$ for all but finitely many n. Therefore by Lemma 6 $\zeta_s \in \mathcal{K}_{2\epsilon}$ for all s sufficiently large on the set A. Letting ϵ converge to zero through a countable set we have

$$\Pr\{\zeta_s \to \mathcal{K} \text{ as } s \to \infty \text{ in } C_B\} = 1.$$

To prove $\mathcal{K}$ is almost surely the set of cluster points of $\{\zeta_s: s \geq 3\}$ it suffices by the separability of $\mathcal{K}$ to prove that if $h \in \mathcal{K}$, $\|h\|_{\mathcal{H}} < 1$ and $\epsilon > 0$ there is a $c > 1$ so that with probability one $\|\zeta_{[c^n]} - h\|_{C_B} < \epsilon$ for infinitely many n. By Lemma 7 choose $r > 1$ and k sufficiently large so that (5) holds with ϵ replaced by $\epsilon/3$ for all sufficiently large s. By Lemma 4(d) choose k large enough so that $\|h - h^{(k)}\|_{C_B} < \epsilon/3$. Then for every $c > 1$, applying these estimates and the Borel-Cantelli lemma, we have with probability one that

$$\begin{aligned}\|\zeta_{[c^n]} - h\|_{C_B} &\leq 2\epsilon/3 + \|\zeta^{(k)}_{[c^n]} - h^{(k)}\|_{C_B} \\ &\leq 2\epsilon/3 + \|P\| \sup_{0 \leq t \leq 1} \|\zeta^{(k)}_{[c^n]}(t) - h^{(k)}(t)\|_H\end{aligned}$$

for all sufficiently large n.

It now suffices to show that with probability one

$$\sup_{0 \leq t \leq 1} \|\zeta^{(k)}_{[c^n]}(t) - h^{(k)}(t)\|_H < \epsilon/3\|P\| \tag{6}$$

for infinitely many n. Our argument follows an idea due to Strassen.

Let $m \geq 2$ be an integer, $0 < \delta < 1$, and assume $Z_{j,[c^n]}$ and h_j $(j = 1, \ldots, k)$ are the jth-coordinates of $\zeta_{[c^n]}{}^{(k)}$ and $h^{(k)}$. We define the event

$$A_n = \{w\colon |(Z_{j,[c^n]}(w, i/m) - Z_{j,[c^n]}(w, i - 1/m)) - (h_j(i/m) - h_j(i - 1/m))| < \delta/k \text{ for } i = 2, \ldots, m \text{ and } j = 1, \ldots, k\}.$$

Then

$$\Pr(A_n) \geq \prod_{i=2}^{m} \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi}} \int_{a_{ij}}^{b_{ij}} e^{-s^2/2} ds$$

where (letting LL denote log log)

$$a_{ij} = |h_j(i/m) - h_j(i - 1/m)|\sqrt{2m\mathrm{LL}[c^n]},$$

$$b_{ij} = (|h_j(i/m) - h_j(i - 1/m)| + \delta/k)\sqrt{2m\mathrm{LL}[c^n]},$$

for $i = 2, \ldots, m; j = 1, \ldots, k$. Using the estimate

$$\int_a^b \exp(-s^2/2)\, ds \geq \frac{\exp(-a^2/2)}{b}(1 - \exp(-(b^2 - a^2))/2) \quad \text{for } 0 \leq a < b$$

we have a constant $\gamma > 0$ such that

$$\Pr(A_n) \geq \gamma \prod_{i=2}^{m} \prod_{j=1}^{k} \frac{\exp(-a_{ij}^2/2)}{b_{ij}}$$

for all n sufficiently large (because $0 \leq a_{ij} < b_{ij}$ implies $b_{ij}^2 - a_{ij}^2 \geq (b_{ij} - a_{ij})^2 \geq (\delta^2/k^2)2m\mathrm{LL}[c^n]$). Hence there is a constant $\gamma_1 > 0$,

$$\Pr(A_n) \geq \frac{\gamma_1 \exp\left\{\sum_{i=2}^{m} \sum_{j=1}^{k} (h_j(i/m) - h_j(i - 1/m))^2 m\mathrm{LL}[c^n]\right\}}{(2m\mathrm{LL}[c^n])^{(m-1)k/2}}$$

$$\geq \frac{\gamma_1 \exp\{-\|h^k\|_{\mathscr{H}} \cdot \mathrm{LL}[c^n]\}}{(2m\mathrm{LL}[c^n])^{mk/2}},$$

and since $\theta = \|h^k\|_{\mathscr{H}} < 1$ we have

$$\Pr(A_n) \geq \frac{\gamma_1}{(\log[c^n])^\theta (2m\mathrm{LL}[c^n])^{mk/2}}.$$

Hence for $c = m$ we have $A_1, A_2, \ldots$ independent and

$$\Pr(A_n) \geq \frac{\gamma_1}{(n \log m)^\theta (2m(\log n + \mathrm{LL}m))^{mk/2}} \geq \frac{\gamma_2}{n^\theta (\log n)^{mk/2}}$$

for all n sufficiently large.

Now $\theta < 1$ implies $\sum_{n=1}^{\infty} \Pr(A_n) = \infty$ so by Borel-Cantelli $\Pr(\limsup_n A_n) = 1$.

Using the fact that the H-norm and the B-norm are equivalent on the finite dimensional subspace of B generated by $\{x_1, \ldots, x_k\}$ we have by the first part of the proof that with probability one $\zeta_{[c^n]}{}^{(k)}(t)$ is eventually within δ of $\mathcal{K}^{(k)}$. Here $\mathcal{K}^{(k)}$ is the subset of $\mathcal{K}$ consisting of functions of the form $\sum_{j=1}^{k} x_j^*(w(t))x_j$. Hence with probability one

$$\|\zeta_{[c^n]}^{(k)}(s) - \zeta_{[c^n]}^{(k)}(t)\|_H \leq \sqrt{|t-s|} + \delta \tag{7}$$

for all $0 \leq s, t \leq 1$ and all n sufficiently large. Now if $y \in C_B[0, 1]$ and y satisfies

(a) $\|y^{(k)}(t) - y^{(k)}(s)\|_H \leq \sqrt{|t-s|} + \delta \ (0 \leq s, t \leq 1)$,

(b) $|(y_j(i/m) - y_j((i-1)/m)) - (h_j(i/m) - h_j((i-1)/m))| < \delta/k$

for all $j = 1, \ldots, k$, $2 \leq i \leq m$ where y_j is the jth coordinate of y, then

$$\sup_{0 \leq t \leq 1} \|y^{(k)}(t) - h^k(t)\|_H < \epsilon/3\|P\|$$

provided m is sufficiently large and δ is sufficiently small. Using the definition of A_n and (7) we see that with probability one (6) holds for infinitely many n. This concludes the proof.

The next corollary follows immediately from Theorem 1.

Corollary 1. *Let θ be a continuous function on C_B into a Hausdorff topological space Y and assume the notation of Theorem 1. Then with probability one $\{\theta \circ \zeta_s : s \geq 3\}$ converges to the compact set $\theta(\mathcal{K})$ and clusters at each point of $\theta(\mathcal{K})$.*

Corollary 2. *If $\{W(t): 0 \leq t < \infty\}$ is Brownian motion on B, then*

$$\Pr\left(\overline{\lim_{s \to \infty}} \frac{\|W(s)\|_B}{\sqrt{2s \log \log s}} = \sup_{x \in K} \|x\|_B\right) = 1.$$

Proof. Since $\|W(s)\|_B/\sqrt{2s \log \log s} = \|\zeta_s(1)\|_B$ this result follows from Corollary 1 with $\theta(f) = \|f(1)\|_B$ and by showing that $\sup_{f \in \mathcal{K}} \|f(1)\|_B = \sup_{x \in K} \|x\|_B$. Now if $f \in \mathcal{K}$, then by Lemma 4(c) $\|f(1)\|_H \leq \|f\|_{\mathcal{H}} \leq 1$ and hence $f(1) \in K \subseteq H$. On the other hand, if $x \in K$ we can set $f(t) = tx$ and then $\|f\|_{\mathcal{H}}^2 = (x, x)_H \leq 1$. Hence $f(t) = tx \in \mathcal{K}$ and $\theta(f) = \|x\|_B$. By combining the above we have

$$\sup_{f \in \mathcal{K}} \|f(1)\|_B = \sup_{x \in K} \|x\|_B,$$

and the proof is complete.

For the following recall the statements I and II of the introduction.

Corollary 3. I *holds.*

Proof. If $B = \mathbf{R}^k$ then $H = B$, $\mathcal{K} = K_k$ and the result in I follows immediately.

Corollary 4. II *holds.*

Proof. Construct Brownian motion in B, call it $\{W(t): 0 \leq t < \infty\}$, such that $\mu_1 = \mu$. Let $\theta(f) = f(1)$ for $f \in C_B$. Using the stationary independent increments of $\{W(t)\}$ it follows that the joint distributions of $\{\xi_n: n \geq 3\}$ are identical to those of $\{\theta(\zeta_n): n \geq 3\}$ where ζ_n is as in Theorem 1. Hence with probability one $\{\xi_n: n \geq 3\}$ converges to the set $\theta(\mathcal{K})$ and clusters at each point of $\theta(\mathcal{K})$ by Corollary 1. Now by the argument given in Corollary 2 $\theta(\mathcal{K}) = K$ and hence the proof is complete.

Remark. In view of Lemma 3 it follows that K is compact in B and hence Corollary 4 actually implies the sequence $\{\zeta_n: n \geq 3\}$ is relatively norm compact with probability one and that its limit points consist precisely of K (with probability one). This is slightly stronger than II. Finally II generalizes the law of the iterated logarithm of [3] to Gaussian random variables in B.

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