A GEOMETRICAL CHARACTERIZATION OF BANACH SPACES WITH THE RADON-NIKODYM PROPERTY

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ABSTRACT. A characterization of Banach spaces having the Radon-Nikodym property is obtained in terms of a convexity requirement on all bounded subsets. In addition a Radon-Nikodym theorem, utilizing this convexity property, is given for the Bochner integral and it is easily shown that this theorem is equivalent to the Phillips-Metivier Radon-Nikodym theorem as well as all the standard Radon-Nikodym theorems for the Bochner integral.

1. Introduction. Rieffel [9] proved a Radon-Nikodym theorem for the Bochner integral, using techniques established in [8], in an attempt to establish the Radon-Nikodym theorem of Phillips [7] and Metivier [5]. He was unable to establish it in the nonseparable case, the result depending upon a proof that every convex weakly compact set in a $B$-space is dentable. This circle of ideas was not closed until Troyanski [10] proved that a Banach space with a weakly compact fundamental subset is isomorphic to a locally uniformly convex Banach space. This is, as would be expected, much deeper than necessary and a simpler proof will be indicated in §2.

The obvious characterization of Banach spaces with the Radon-Nikodym property would seem to be that every bounded subset must be dentable. In §3 it is demonstrated that a characterization is that every bounded subset must be $\sigma$-dentable, where $\sigma$-dentability is a dentable type condition which is strictly weaker than dentability. It is however an open question if dentable and $\sigma$-dentable coincide in Banach spaces having the Radon-Nikodym property.

2. Dentability and $\sigma$-dentability with application to Phillip's Radon-Nikodym theorem. The following notation will be observed in the remainder of this paper. $B$ will denote a Banach space and if $D \subset B$ then $c(D)$ and $c(D)$ will denote the convex hull of $D$ and the closed convex hull of $D$, respectively. The open and closed spheres of radius $r$ about $x \in B$ will be $S_r(x)$ and $\bar{S}_r(x)$. If $(X, \Sigma, \mu)$ is a totally finite positive measure space then $\Sigma^+ = \{E \in \Sigma: \mu(E) > 0\}$ and for a $B$-valued measure $m$ on $\Sigma$, the average range of $m$ over $E \in \Sigma^+$ with respect to $\mu$ is $A_E(m) = \{m(F)/\mu(F): F \subset E, F \in \Sigma^+\}$.

**Definition 2.1.** A set $D \subset B$ is $\sigma$-convex iff for every sequence $\{a_i\}_{i=1}^\infty$, $a_i \geq 0$, $\sum_{i=1}^\infty a_i = 1$, and for every sequence $\{d_i\}_{i=1}^\infty \subset D$ such that $\sum_{i=1}^\infty a_id_i$ converges, we have $\sum_{i=1}^\infty a_id_i \in D$.

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The $\sigma$-convex hull of $D \subseteq B$ is given by

$$\sigma(D) = \left\{ \sum_{i=1}^{\infty} a_i d_i : a_i \geq 0, \sum_{i=1}^{\infty} a_i = 1, \text{ and } \sum_{i=1}^{\infty} a_i d_i \text{ converges} \right\}.$$ 

If $D$ is bounded then the infinite convex sums in $\sigma(D)$ always exist. We may also assume that the constants $a_i > 0$. In addition we always have the following relations:

$$D \subseteq c(D) \subseteq \sigma(D) \subseteq \varepsilon(D)$$

where the inclusions may be strict. 

We now recall the definition of dentable and introduce the concept of $\sigma$-dentable.

**Definition 2.2.** A set $D \subseteq B$ is dentable [$\sigma$-dentable] iff for each $\epsilon > 0$ there exists $d \in D$, such that

$$d \in \varepsilon(D - S_\epsilon(d)) \quad [d \in \sigma(D - S_\epsilon(d))].$$

If $D$ is not dentable [$\sigma$-dentable] then any number $\epsilon > 0$ such that for all $d \in D$, $d \in \varepsilon(D - S_\epsilon(d))$ [$d \in \sigma(D - S_\epsilon(d))$] is called a dentable limit [$\sigma$-dentable limit] for the set $D$.

The following lemma is immediate.

**Lemma 2.1.** If $D \subseteq B$ is dentable then it is $\sigma$-dentable.

**Example.** By considering the following subset of $L'(X, \Sigma, \mu)$ where $(X, \Sigma, \mu)$ is a nonatomic, finite, positive measure space with $\mu(X) = 1$, we can see that $\sigma$-dentable is a strictly weaker concept than dentable.

Let $P$ be the positive cone in $L'(X, \Sigma, \mu)$ and $U_1$ be the unit cell [$U_1 = \{ f : \|f\| = 1 \}$] in $L'(X, \Sigma, \mu)$. Then if $D = \bigcup_{\theta < \epsilon} e^{i\theta} P \cap U_1$ $\cup \{1\}$ it is easy to establish that the constant function 1 is a $\sigma$-denting point for $D$ [i.e. $\forall \epsilon > 0$, 1 is the appropriate element of $D$] and yet $D$ is not dentable.

In order to prove dentability or $\sigma$-dentability of a set it is often possible to reduce the problem to the consideration of countable sets.

**Lemma 2.2.** If $D \subseteq B$ has the property that every countable subset is dentable ($\sigma$-dentable) then $D$ is dentable ($\sigma$-dentable).

**Proof.** The proof of the $\sigma$-dentable assertion is entirely analogous to that of the dentable case and thus we will only prove the dentable assertion.

Suppose $D$ is not dentable. Then there exists $\epsilon > 0$ such that $\epsilon$ is a dentable limit for $D$. Now for each $x \in D$ there exists a countable set $A_x \subset D - S_\epsilon(x)$ such that $x \in \varepsilon(A_x)$.

Define by induction a sequence $\{A_n\}$ of subsets as follows. Pick any $x \in D$ and set $A_1 = \{x\}$. Given $A_{n-1}$ let $A_n = \bigcup\{A_x : x \in A_{n-1}\}$. Thus the set $A = \bigcup_{n=1}^{\infty} A_n \subset D$ is countable and is clearly not dentable and hence the lemma is established.
Theorem 2.1. If $K \subset B$ is a relatively weakly compact set, then it is dentable.

Proof. By Lemma 2.2 we need only consider the case when $B$ is separable. In this case the argument given by Rieffel [9, p. 76] or the argument by Namioka [6, p. 150] can be used to obtain the result.

An elementary proof of this fact can be obtained in the following manner.

It suffices to assume that $K$ is a convex weakly compact set since Rieffel [9] showed that if $\mathcal{E}(D)$ is dentable then $D$ is dentable.

Then by Lemma 2.2 it suffices to assume that $B$ is separable. Suppose $\varepsilon > 0$ and let $A$ be the set of extreme points of $K$. By the Krein-Milman theorem, $A \neq \emptyset$. Let $(x_i)_{i=1}^{\infty}$ be a dense subset in $B$; then since $A^\omega$ is weakly compact and since

$$
\mathcal{A}^\omega = \bigcup_{i=1}^{\infty} \mathcal{A}^\omega \cap [x_i + \mathcal{S}_{\varepsilon/2}(0)],
$$

there exists at least one $i$ and a weak convex neighborhood $N$ such that $\mathcal{A}^\omega \cap [x_i + \mathcal{S}_{\varepsilon/2}(0)]$ contains $N \cap \mathcal{A}^\omega$. This follows since $\mathcal{A}^\omega$ is a Baire space and since $\mathcal{S}_{\varepsilon/2}(0) = \mathcal{S}_{\varepsilon}(0)$.

Thus there exists $x \in A$ such that $x$ is in the interior of $N$ and the diameter of $N \cap \mathcal{A}^\omega$ is bounded by $\varepsilon/2$.

Let $K_1 = \mathcal{E}(K \sim N)$, $K_2 = \mathcal{E}(N \cap A)$. $K_1$ and $K_2$ are both weakly compact, convex, and disjoint. Thus

$$
c(K_1 \cup K_2) = \mathcal{E}(K_1 \cup K_2)
$$

$$
= \{\lambda x_1 + (1 - \lambda)x_2: 0 \leq \lambda \leq 1, x_1 \in K_1, x_2 \in K_2\}.
$$

The diameters of $K_1$ and $K_2$ have the following bounds: $\delta(K_2) \leq \varepsilon/2$ and if $d = \delta(K) < \infty$, $\delta(K_i) \leq d$. Assume $d \neq 0$. Let $C = \{\lambda x_1 + (1 - \lambda)x_2: x_1 \in K_1, x_2 \in K_2, \varepsilon/4d < \lambda \leq 1\}$. Thus $C \supset K_1$ and $C$ is weakly compact. Suppose $y_1, y_2 \in K \sim C$. Then

$$
y_i = \lambda_i x_1 + (1 - \lambda_i)x_2^i, \quad 0 \leq \lambda_i < \varepsilon/4d, x_1^i \in K_1, x_2^i \in K_2, i = 1, 2.
$$

Thus

$$
\|y_1 - y_2\| \leq |\lambda_1| \|x_1 - x_1^2\| + \|x_1 - x_2\| + |\lambda_2| \|x_1^i - x_2^i\| + \varepsilon/4d \cdot d + \varepsilon/2 + (\varepsilon/4d) \cdot d = \varepsilon.
$$

Thus if $N_1 = N \sim C$, $N_1$ is weakly open, $x \in N_1$, and the diameter of $N_1 \cap K$ is less than $\varepsilon$. Thus $x \notin K \sim S_{\varepsilon}(x)$ since $S_{\varepsilon}(x) \supset N_1 \cap K$. Thus since $x$ is an extreme point of $K$, $x \notin \mathcal{E}(K \sim S_{\varepsilon}(x))$ and $K$ is dentable.

The following theorem is due to Rieffel [9, Theorem 1, p. 71] and is obtained by replacing dentable with $\varepsilon$-dentable, the proof remaining essentially the same. We include a proof using the locally small average range Radon-Nikodym
Theorem 2.2. Let \((X, \Sigma, \mu)\) be a totally finite positive measure space and let \(B\) be a Banach space. Let \(m\) be a \(B\)-valued measure on \(\Sigma\). Then there is a \(B\)-valued Bochner integrable function \(f\) on \(X\) such that \(m(E) = \int_E f \, d\mu\) for all \(E \in \Sigma\), iff
\begin{enumerate}[(i)]  \item \(m\) is \(\mu\)-continuous,  \item \(|m(X)| < \infty\),  \item \(m\) has locally \(\sigma\)-dentable average range, that is, given \(E \in \Sigma^+\), there exists \(F \subset E, F \in \Sigma^+\), such that \(A_F(m)\) is \(\sigma\)-dentable. \end{enumerate}

Proof. \((\Rightarrow)\) This is immediate from Theorem 1, Rieffel [9, p. 71] and Lemma 2.1.

\((\Leftarrow)\) Let \(E \in \Sigma^+\) and \(\epsilon > 0\) be given. Then there exists \(E_d \subset E, E_d \in \Sigma^+\), such that \(A_{E_d}(m)\) is \(\sigma\)-dentable. Thus choose \(b \in A_{E_d}(m)\) such that \(b \notin \sigma(A_{E_d}(m) - S_\nu(b))\). Suppose \(b = m(F_0) / \mu(F_0), F_0 \subset E_d, F_0 \in \Sigma^+\). Then by Theorem 3.1 and its corollary [4, p. 16], if \(b \in A(F_0, \epsilon) = \{r \in B : ||m(A) - r \mu(A)|| \leq \epsilon \mu(A), \forall A \subset F_0, A \in \Sigma^+\}\) we are done. So suppose \(b \notin A(F_0, \epsilon)\).

Claim. There exists \(F \subset F_0, F \in \Sigma^+\), such that \(b \in A(F, \epsilon)\).

Proof. Suppose not. Then the property that \(||m(E) / \mu(E) - b|| > \epsilon\) is a local null difference property and hence by the exhaustion principle [4, Lemma 1.1, p. 2] \(F_0 = \bigcup^n_{i=1} E_i\) where \(m(E_i) / \mu(E_i) \in A_{E_i}(m) \sim S_\nu(b) \subset A_{E_i}(m) \sim S_\nu(b)\), but \(m(F_0) / \mu(F_0) = \sum^n_{i=1} (\mu(E_i) / \mu(F_0)) m(E_i) / \mu(E_i) \in \sigma(A_{E_i}(m) \sim S_\nu(b))\) and this yields a contradiction.

Thus there must exist \(F \subset F_0 \subset E, F \in \Sigma^+\), such that \(b \in A(F, \epsilon)\) and by Theorem 3.1 and its corollary [4, p. 16] we have the desired conclusion.

Corollary [Phillips]. Let \((X, \Sigma, \mu)\) be a totally finite positive measure space and let \(B\) be a Banach space. Let \(m\) be a \(B\)-valued measure on \(\Sigma\). Then there is a \(B\)-valued Bochner integrable function \(f\) on \(X\), such that \(m(E) = \int_E f \, d\mu\), for all \(E \in \Sigma\), iff
\begin{enumerate}[(i)]  \item \(m\) is \(\mu\)-continuous,  \item \(|m(X)| < \infty\), and  \item \(m\) has locally relatively weakly compact average range. \end{enumerate}

Proof. \((\Rightarrow)\) This follows from Rieffel [8, p. 466].

\((\Leftarrow)\) If \(m\) has locally relatively weakly compact average range then, by Theorem 2.1, \(m\) has locally dentable average range.

3. A geometric characterization of Banach spaces with the Radon-Nikodym property. The concept of \(\sigma\)-dentability allows us to obtain a relatively simple characterization of Banach spaces with the Radon-Nikodym property using Theorem 2.2.

Definition. A Banach space \(B\) has the Radon-Nikodym property (R-N property) iff for any totally finite positive measure space \((X, \Sigma, \mu)\) and any \(B\)-
valued $\mu$-continuous measure $m$ on $\Sigma$, with $|m|(X) < \infty$, there exists $f \in L^p_1(\Sigma, \mu, \mu)$ such that $m(E) = \int_E f \, d\mu$ for all $E \in \Sigma$.

**Definition.** A Banach space $B$ is said to be a $\sigma$-dentable space iff every bounded set $K \subset B$ is $\sigma$-dentable.

It should be emphasized that it is not known if a $\sigma$-dentable space need have all of its bounded subsets dentable.

**Theorem 3.1.** A Banach space $B$ has the Radon-Nikodym property iff $B$ is a $\sigma$-dentable space.

**Proof.** ($\Leftarrow$) If $B$ is a $\sigma$-dentable space then Theorem 2.2 immediately implies that $B$ has the R-N property because any $B$-valued, $\mu$-continuous measure of finite variation has locally bounded average range.

($\Rightarrow$) Suppose $B$ is not a $\sigma$-dentable space. Then there exists a bounded subset $K \subset B$ such that $K$ is not $\sigma$-dentable. We will construct two regular measures $m$ and $\mu$ which negate the Radon-Nikodym property.

Since $K$ is bounded and not $\sigma$-dentable we can choose $\epsilon, N$ such that

(i) $\epsilon$ is a $\sigma$-dentable limit for $K$, and

(ii) $K \subset S_N(0)$.

Let $X = [0,1)$ and choose an increasing sequence of infinite partitions $(\pi_n)_{n=1}^\infty$ of $X$ such that the following conditions are satisfied:

(i) $\pi_n = \{A^n_z\}_{z \in N^n}$ where each $A^n_z = [a^n_z, b^n_z)$.

(ii) For each $n, z \in N^n$, $A^n_z = \cup_{i=1}^\infty A^n_{(z,i)}$ where we consider $(z,i) \in N^{n+1}$.

(iii) For each $n, z \in N^n$, $b^n_{(z,i)} = a^n_{(z,i+1)}$. Thus the decomposition of each half open interval $A^n_z$ proceeds from left to right.

We now define a ring of subsets $\mathcal{R}$ of $X$. Let $\mathcal{R} = \{A \cup B : A$ is a finite union of $A^*_i$'s and $B$ is a finite union of sets of the form $\cup_{j=1}^\infty A^*_{(z,j)} = A^*_z \sim \cup_{j=1}^\infty A^*_{(z,j)}\}.$

We consider both $\emptyset$ and $X$ to be elements of $\mathcal{R}$. We will now define $\mu$ and $m$ on $\mathcal{R}$ and extend to regular countably additive measures on $\sigma(\mathcal{R})$, the $\sigma$-algebra generated by $\mathcal{R}$. $\sigma(\mathcal{R})$ consists of the Borel subsets of $[0,1)$.

Define $\mu$ and $m$ by induction on the sequence of partitions. Let $\mu(\emptyset) = 0, m(\emptyset) = 0, \mu(X) = 1, m(X) = k$ where $k$ is any element of $K$. Suppose $\mu$ and $m$ are defined on the elements of $\pi_n$ such that $m(A^*_z) / \mu(A^*_z) = k^*_z \in K$ for each $A^*_z \in \pi_n$. Then since $K$ is not $\sigma$-dentable, $k^*_z = \sum_{i=1}^\infty \alpha^*_z(i) k^*_{(z,i)}$, $\alpha^*_z(i) > 0$, $\sum_{i=1}^\infty \alpha^*_z(i) = 1$ and $\{k^*_{(z,i)}\}_{i=1}^\infty \subset K \sim S_1(k^*_z)$. We now define $\mu(A^*_z) = \alpha^*_z(1) \mu(A^*_z)$ and $m(A^*_z) = \mu(A^*_z) k^*_{(z,1)}$. Let $\pi = \{A \subset X : A \in \pi_n$ for some $n\}.$

Thus $m$ and $\mu$ are defined on each $\pi_n$ and hence can be extended by finite additivity to all of $\mathcal{R}$.

Notice that the diameter of the average range of $m$ over each $A^*_z$ is at least $\epsilon$. This fact, after extension to $\sigma(\mathcal{R})$, will yield the contradiction.

Notice also that the construction yields a "horizontal" countable additivity, that is,
Claim 1. \( \mu \) can be extended to a Borel measure on \([0,1)\) and hence is regular and countably additive.

**Proof.** It suffices to show that \( \mu \) is regular on \( \pi \) relative to \( \mathcal{R} \) since it is then regular on \( \mathcal{R} \) and hence has an extension to a Borel measure on \([0,1)\).

Let \( \epsilon > 0 \) be arbitrary and \( A \in \pi \). Then using the "horizontal" countable additivity there exists \( \{A_i\}_{i=1}^n \subset \pi \) such that

\[
\left| \mu(A) - \sum_{i=1}^n \mu(A_i) \right| < \epsilon.
\]

Thus we have

\[
\bigcup_{i=1}^n A_i \subset A,
\]

\( \bigcup_{i=1}^n A_i \) is compact and hence \( \mu \) is inner regular on \( A \).

Suppose \( A = [a, b) \). Then by choosing the tail end of the decomposition of the preceding interval, we can find a sequence \( \{A_i\}_{i=1}^\infty \) such that \( \mu(\bigcup_{i=1}^\infty A_i) < \epsilon \), \( \bigcup_{i=1}^\infty A_i \in \mathcal{R} \), and \( (A \cup [\bigcup_{i=1}^\infty A_i])' \supset A \), where \( D' \) is the interior of \( D \). Thus \( \mu \) is outer regular on \( A \) and hence \( \mu \) is regular on all of \( \pi \).

Claim 2. \( m \) can be extended to a Borel measure on \([0,1)\) such that \( \|m(A)\| \leq N\mu(A) \) for all \( A \in \sigma(\mathcal{R}) \). Thus the extension is countably additive and regular.

**Proof.** Since \( \mu \) is regular and dominates \( m \) we can apply Theorem 1 [1, p. 62] of Dinucleanu which implies that \( m \) has a countably additive extension of finite variation such that \( m \) remains dominated by \( \mu \) and \( m \) is regular.

Claim 3. \( m \) is not an indefinite integral with respect to \( \mu \).

**Proof.** It suffices to show that, for \( B \in \sigma(\mathcal{R}) \), the average range of \( m \) over \( B \), \( A_B(m) \), has diameter not less than \( \epsilon/2 \). This sufficiency follows from Theorem 3.1 and its corollary [4, p. 16].

Let \( B \in \sigma(\mathcal{R}) \). Now by the regularity of \( \mu \) and \( m \) on \( \sigma(\mathcal{R}) \) we can choose a compact \( C \) and an open \( O \) such that (i) \( C \subset B \subset O \), and (ii) \( \mu(O - C) < (\epsilon/16 N)\mu(B) \).

Now those elements in \( \mathcal{R} \) of the form \( A_{(i, j)}^*(\mathcal{R}) \cup [\bigcup_{i=m}^{\infty} A_{(i, j - 1)}^*] \) form a base of the topology in \([0,1)\) and hence by the compactness of \( C \) and the openness of \( O \) we can find a finite number of these which cover \( C \) and are contained in \( O \). Thus there exists a disjoint sequence \( \{A_i\}_{i=1}^\infty \subset \pi \) such that \( C \subset \bigcup_{i=1}^\infty A_i \subset O \).

Now there must exist at least one set \( A_i \) such that \( \mu(A_i \sim B)/\mu(A_i) < \epsilon/8 N = \delta \) since if not, we have
Thus choose $A_\alpha$ such that

$$\mu(A_\alpha \sim B)/\mu(A_\alpha) < \varepsilon/8N.$$  

Let $D = A_\alpha \cap B \in \sigma(\mathcal{F})$, then $D \subseteq B$ and $\mu(D) > 0$. Now by taking the next partition of $A_\alpha$ we get $A_\alpha = \bigcup_{i=1}^{\infty} C_k$ where the $\{C_k\}_{k=1}^{\infty} \subseteq \pi$ and are disjoint. Then there must exist a small $\varepsilon$ such that

$$\mu(C_n \sim B) < (\varepsilon/16N)\mu(C_n).$$

since if not $\mu(A_\alpha \sim B) = \mu(\bigcup_{i=1}^{\infty}(C_n \sim B)) \geq (\varepsilon/8N) \sum_{i=1}^{\infty} \mu(C_n)$

$$= (\varepsilon/8N)\mu(A_\alpha)$$

which contradicts $(\ast)$. Let $E = C_n \cap B$. Now from the construction of $m$ and $\mu$

$$\frac{m(A_\alpha)}{\mu(A_\alpha)} - \frac{m(C_n)}{\mu(C_n)} \geq \varepsilon.\]

In addition

$$\left\| \frac{m(D)}{\mu(D)} - \frac{m(A_\alpha)}{\mu(A_\alpha)} \right\| = \left\| \left(1 - \frac{\mu(D)}{\mu(A_\alpha)} \frac{m(D)}{\mu(D)} + \frac{\mu(A_\alpha \sim B)}{\mu(A_\alpha)} \frac{m(A_\alpha \sim B)}{\mu(A_\alpha \sim B)} \right) \right\|

\leq \frac{\mu(A_\alpha \sim B)}{\mu(A_\alpha)} \left\{ \left\| \frac{m(D)}{\mu(D)} \right\| + \left\| \frac{m(A_\alpha \sim B)}{\mu(A_\alpha \sim b)} \right\| \right\} \leq \frac{\varepsilon}{4}.$$ 

In the same manner we get $\|m(E)/\mu(E) - m(n)/\mu(n)\| < \varepsilon/4$. Thus

$$\left\| \frac{m(E)}{\mu(E)} - \frac{m(D)}{\mu(D)} \right\| \geq \frac{\varepsilon}{2}.$$ 

Thus the diameter of $A_\alpha(m)$ is not less than $\varepsilon/2$ for all $B \in \sigma(\mathcal{F})$ and hence $m$ is not an indefinite integral with respect to $\mu$.

Thus $B$ does not have the R-N property.

The following corollary is due to Uhl [11, Theorem 1, p. 2].

**Corollary.** If $B$ is a Banach space such that every closed separable subspace of $B$ is linearly homeomorphic to a subspace of a separable dual space, then $B$ has the Radon-Nikodym property.

**Proof.** Suppose $B$ satisfies the hypothesis of the corollary. Let $K$ be any bounded set in $B$ and $D$ any countable subset of $K$. Then the closed linear span
$[D]$ of $D$ is linearly homeomorphic to a subspace of a separable dual space. Since a linear homeomorphism maps $\sigma$-dentable sets into $\sigma$-dentable sets and since a separable dual space has the R-N property, $D$ is mapped into a $\sigma$-dentable set and hence is itself $\sigma$-dentable. Thus $K$ is $\sigma$-dentable and $B$ has the R-N property.

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