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A GEOMETRICAL CHARACTERIZATION OF BANACH SPACES WITH THE RADON-NIKODYM PROPERTY

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ABSTRACT. A characterization of Banach spaces having the Radon-Nikodym property is obtained in terms of a convexity requirement on all bounded subsets. In addition a Radon-Nikodym theorem, utilizing this convexity property, is given for the Bochner integral and it is easily shown that this theorem is equivalent to the Phillips-Metivier Radon-Nikodym theorem as well as all the standard Radon-Nikodym theorems for the Bochner integral.

1. **Introduction.** Rieffel [9] proved a Radon-Nikodym theorem for the Bochner integral, using techniques established in [8], in an attempt to establish the Radon-Nikodym theorem of Phillips [7] and Metivier [5]. He was unable to establish it in the nonseparable case, the result depending upon a proof that every convex weakly compact set in a B-space is dentable. This circle of ideas was not closed until Troyanski [10] proved that a Banach space with a weakly compact fundamental subset is isomorphic to a locally uniformly convex Banach space. This is, as would be expected, much deeper than necessary and a simpler proof will be indicated in §2.

The obvious characterization of Banach spaces with the Radon-Nikodym property would seem to be that every bounded subset must be dentable. In §3 it is demonstrated that a characterization is that every bounded subset must be σ-dentable, where σ-dentability is a dentable type condition which is strictly weaker than dentability. It is however an open question if dentable and σ-dentable coincide in Banach spaces having the Radon-Nikodym property.

2. **Dentability and σ-dentability with application to Phillip's Radon-Nikodym theorem.** The following notation will be observed in the remainder of this paper. B will denote a Banach space and if $D \subset B$ then $c(D)$ and $\bar{c}(D)$ will denote the convex hull of D and the closed convex hull of D, respectively. The open and closed spheres of radius r about $x \in B$ will be $S_r(x)$ and $\bar{S}_r(x)$. If (X,Σ,μ) is a totally finite positive measure space then $\Sigma^+ = \{E \in \Sigma: \mu(E) > 0\}$ and for a B-valued measure m on Σ, the average range of m over $E \in \Sigma^+$ with respect to μ is $A_E(m) = \{m(F)/\mu(F): F \subset E, F \in \Sigma^+\}$.

Definition 2.1. A set $D \subset B$ is *σ-convex* iff for every sequence $\{a_i\}_{i=1}^\infty$, $a_i \geq 0$, $\sum_{i=1}^\infty a_i = 1$, and for every sequence $\{d_i\}_{i=1}^\infty \subset D$ such that $\sum_{i=1}^\infty a_i d_i$ converges, we have $\sum_{i=1}^\infty a_i d_i \in D$.

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The *σ-convex hull* of $D \subset B$ is given by

$$\sigma(D) = \left\{ \sum_{i=1}^{\infty} a_i d_i : a_i \geq 0, \sum_{i=1}^{\infty} a_i = 1, \text{ and } \sum_{i=1}^{\infty} a_i d_i \text{ converges} \right\}.$$

If D is bounded then the infinite convex sums in $\sigma(D)$ always exist. We may also assume that the constants $a_i > 0$. In addition we always have the following relations:

$$D \subset c(D) \subset \sigma(D) \subset \bar{c}(D)$$

where the inclusions may be strict.

We now recall the definition of dentable and introduce the concept of σ-dentable.

Definition 2.2. A set $D \subset B$ is *dentable* [*σ-dentable*] iff for each $\varepsilon > 0$ there exists $d \in D$, such that

$$d \notin \bar{c}(D \sim S_\varepsilon(d)) \qquad [d \notin \sigma(D \sim S_\varepsilon(d))].$$

If D is not dentable [σ-dentable] then any number $\varepsilon > 0$ such that for all $d \in D$, $d \in \bar{c}(D \sim S_\varepsilon(d))$ $[d \in \sigma(D \sim S_\varepsilon(d))]$ is called a *dentable limit* [*σ-dentable limit*] for the set D.

The following lemma is immediate.

Lemma 2.1. *If $D \subset B$ is dentable then it is σ-dentable.*

Example. By considering the following subset of $L^1(X, \Sigma, \mu)$ where (X,Σ,μ) is a nonatomic, finite, positive measure space with $\mu(X) = 1$, we can see that σ-dentable is a strictly weaker concept than dentable.

Let P be the positive cone in $L^1(X, \Sigma, \mu)$ and U_1 be the unit cell $[U_1 = \{f\colon \|f\| = 1\}]$ in $L^1(X, \Sigma, \mu)$. Then if $D = [\bigcup_{0<\theta<\pi} e^{i\theta} P \cap U_1] \cup \{1\}$ it is easy to establish that the constant function 1 is a σ-denting point for D [i.e. $\forall\ \varepsilon > 0$, 1 is the appropriate element of D] and yet D is not dentable.

In order to prove dentability or σ-dentability of a set it is often possible to reduce the problem to the consideration of countable sets.

Lemma 2.2. *If $D \subset B$ has the property that every countable subset is dentable (σ-dentable) then D is dentable (σ-dentable).*

Proof. The proof of the σ-dentable assertion is entirely analogous to that of the dentable case and thus we will only prove the dentable assertion.

Suppose D is not dentable. Then there exists $\varepsilon > 0$ such that ε is a dentable limit for D. Now for each $x \in D$ there exists a countable set $A_x \subset D \sim S_\varepsilon(x)$ such that $x \in \bar{c}(A_x)$.

Define by induction a sequence $\{A_n\}$ of subsets as follows. Pick any $z \in D$ and set $A_1 = \{z\}$. Given A_{n-1} let $A_n = \bigcup\{A_x : x \in A_{n-1}\}$. Thus the set $A = \bigcup_{n=1}^{\infty} A_n \subset D$ is countable and is clearly not dentable and hence the lemma is established.

Theorem 2.1. *If $K \subset B$ is a relatively weakly compact set, then it is dentable.*

Proof. By Lemma 2.2 we need only consider the case when B is separable. In this case the argument given by Rieffel [9, p. 76] or the argument by Namioka [6, p. 150] can be used to obtain the result.

An elementary proof of this fact can be obtained in the following manner.

It suffices to assume that K is a convex weakly compact set since Rieffel [9] showed that if $\bar{c}(D)$ is dentable then D is dentable.

Then by Lemma 2.2 it suffices to assume that B is separable. Suppose $\varepsilon > 0$ and let A be the set of extreme points of K. By the Kreĭn-Milman theorem, $A \neq \varnothing$. Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset in B; then since $\bar{A}^w$ is weakly compact and since

$$\bar{A}^w = \bigcup_{i=1}^{\infty} \bar{A}^w \cap [x_i + \bar{S}_{\varepsilon/2}(0)],$$

there exists at least one i and a weak convex neighborhood N such that $\bar{A}^w \cap [x_i + \bar{S}_{\varepsilon/2}(0)]$ contains $N \cap \bar{A}^w$. This follows since $\bar{A}^w$ is a Baire space and since $\bar{S}_{\varepsilon/2}(0) = \bar{S}^w_{\varepsilon/2}(0)$.

Thus there exists $x \in A$ such that x is in the interior of N and the diameter of $N \cap \bar{A}^w$ is bounded by $\varepsilon/2$.

Let $K_1 = \bar{c}(K \sim N)$, $K_2 = \bar{c}(N \cap A)$. K_1 and K_2 are both weakly compact, convex, and disjoint. Thus

$$\begin{aligned} c(K_1 \cup K_2) &= \bar{c}(K_1 \cup K_2) \\ &= \{\lambda x_1 + (1-\lambda)x_2 : 0 \le \lambda \le 1, x_1 \in K_1, x_2 \in K_2\}. \end{aligned}$$

The diameters of K_1 and K_2 have the following bounds: $\delta(K_2) \le \varepsilon/2$ and if $d = \delta(K) < \infty$, $\delta(K_i) \le d$. Assume $d \neq 0$. Let $C = \{\lambda x_1 + (1-\lambda)x_2 : x_1 \in K_1, x_2 \in K_2, \varepsilon/4d \le \lambda \le 1\}$. Thus $C \supset K_1$ and C is weakly compact. Suppose $y_1, y_2 \in K \sim C$. Then

$$y_\iota = \lambda_\iota x_1^\iota + (1-\lambda_\iota)x_2^\iota, \qquad 0 \le \lambda_\iota < \varepsilon/4d,\ x_1^\iota \in K_1,\ x_2^\iota \in K_2,\ \iota = 1, 2.$$

Thus

$$\begin{aligned} \|y_1 - y_2\| &\le |\lambda_1| \|x_1^1 - x_2^1\| + \|x_2^1 - x_2^2\| + |\lambda_2| \|x_1^2 - x_2^2\| \\ &< (\varepsilon/4d) \cdot d + \varepsilon/2 + (\varepsilon/4d) \cdot d = \varepsilon. \end{aligned}$$

Thus if $N_1 = N \sim C$, N_1 is weakly open, $x \in N_1$, and the diameter of $N_1 \cap K$ is less than ε. Thus $x \notin \overline{K \sim S_\varepsilon(x)}^w$ since $S_\varepsilon(x) \supset N_1 \cap K$. Thus since x is an extreme point of K, $x \notin \bar{c}(K \sim S_\varepsilon(x))$ and K is dentable.

The following theorem is due to Rieffel [9, Theorem 1, p. 71] and is obtained by replacing dentable with σ-dentable, the proof remaining essentially the same. We include a proof using the locally small average range Radon-Nikodym theorem [4, Theorem 3.1] in the spirit of the simple equivalence of all Radon-Nikodym theorems for the Bochner integral.

Theorem 2.2. *Let (X,Σ,μ) be a totally finite positive measure space and let B be a Banach space. Let m be a B-valued measure on Σ. Then there is a B-valued Bochner integrable function f on X such that $m(E) = \int_E f d\mu$ for all $E \in \Sigma$, iff*

(i) *m is μ-continuous,*

(ii) $|m|(X) < \infty$,

(iii) *m has locally σ-dentable average range, that is, given $E \in \Sigma^+$, there exists $F \subset E$, $F \in \Sigma^+$, such that $A_F(m)$ is σ-dentable.*

Proof. ($\Rightarrow$) This is immediate from Theorem 1, Rieffel [9, p. 71] and Lemma 2.1.

($\Leftarrow$) Let $E \in \Sigma^+$ and $\varepsilon > 0$ be given. Then there exists $E_d \subset E$, $E_d \in \Sigma^+$, such that $A_{E_d}(m)$ is σ-dentable. Thus choose $b \in A_{E_d}(m)$ such that $b \notin \sigma(A_{E_d}(m) - S_\varepsilon(b))$. Suppose $b = m(F_0)/\mu(F_0)$, $F_0 \subset E_d$, $F_0 \in \Sigma^+$. Then by Theorem 3.1 and its corollary [4, p. 16], if $b \in A(F_0,\varepsilon) = \{r \in B: \|m(A) - r\mu(A)\| \leq \varepsilon\mu(A), \forall A \subset F_0, A \in \Sigma^+\}$ we are done. So suppose $b \notin A(F_0,\varepsilon)$.

Claim. *There exists $F \subset F_0$, $F \in \Sigma^+$, such that $b \in A(F,\varepsilon)$.*

Proof. Suppose not. Then the property that $\|m(\tilde{E})/\mu(\tilde{E}) - b\| > \varepsilon$ is a local null difference property and hence by the exhaustion principle [4, Lemma 1.1, p. 2] $F_0 = \bigcup_{i=1}^{\infty} E_i$ where $m(E_i)/\mu(E_i) \in A_{F_0}(m) \sim S_\varepsilon(b) \subset A_{E_d}(m) \sim S_\varepsilon(b)$, but $m(F_0)/\mu(F_0) = \sum_{i=1}^{\infty} (\mu(E_i)/\mu(F_0))m(E_i)/\mu(E_i) \in \sigma(A_{E_d}(m) \sim S_\varepsilon(b))$ and this yields a contradiction.

Thus there must exist $F \subset F_0 \subset E$, $F \in \Sigma^+$, such that $b \in A(F,\varepsilon)$ and by Theorem 3.1 and its corollary [4, p. 16] we have the desired conclusion.

Corollary [Phillips]. *Let (X,Σ,μ) be a totally finite positive measure space and let B be a Banach space. Let m be a B-valued measure on Σ. Then there is a B-valued Bochner integrable function f on X, such that $m(E) = \int_E f d\mu$, for all $E \in \Sigma$, iff*

(i) *m is μ-continuous,*

(ii) $|m|(X) < \infty$, *and*

(iii) *m has locally relatively weakly compact average range.*

Proof. ($\Rightarrow$) This follows from Rieffel [8, p. 466].

($\Leftarrow$) If m has locally relatively weakly compact average range then, by Theorem 2.1, m has locally dentable average range.

3. **A geometric characterization of Banach spaces with the Radon-Nikodym property.** The concept of σ-dentability allows us to obtain a relatively simple characterization of Banach spaces with the Radon-Nikodym property using Theorem 2.2.

Definition. A Banach space B has the Radon-Nikodym property (R-N property) iff for any totally finite positive measure space (X,Σ,μ) and any B-

valued μ-continuous measure m on Σ, with $|m|(X) < \infty$, there exists $f \in L^1_B(X, \Sigma, \mu)$ such that $m(E) = \int_E f d\mu$ for all $E \in \Sigma$.

Definition. A Banach space B is said to be a σ-dentable space iff every bounded set $K \subset B$ is σ-dentable.

It should be emphasized that it is not known if a σ-dentable space need have all of its bounded subsets dentable.

Theorem 3.1. *A Banach space B has the Radon-Nikodym property iff B is a σ-dentable space.*

Proof. ($\Leftarrow$) If B is a σ-dentable space then Theorem 2.2 immediately implies that B has the R-N property because any B-valued, μ-continuous measure of finite variation has locally bounded average range.

($\Rightarrow$) Suppose B is not a σ-dentable space. Then there exists a bounded subset $K \subset B$ such that K is not σ-dentable. We will construct two regular measures m and μ which negate the Radon-Nikodym property.

Since K is bounded and not σ-dentable we can choose ε, N such that

(i) ε is a σ-dentable limit for K, and

(ii) $K \subset S_N(0)$.

Let $X = [0, 1)$ and choose an increasing sequence of infinite partitions $\{\pi_n\}_{n=1}^{\infty}$ of X such that the following conditions are satisfied:

(i) $\pi_n = \{A^n_z\}_{z \in N^n}$ where each $A^n_z = [a^n_z, b^n_z)$.

(ii) For each n, $z \in N^n$, $A^n_z = \bigcup_{i=1}^{\infty} A^{n+1}_{(z,i)}$ where we consider $(z, i) \in N^{n+1}$.

(iii) For each n, $z \in N^n$, $b^{n+1}_{(z,i)} = a^{n+1}_{(z,i+1)}$. Thus the decomposition of each half open interval A^n_z proceeds from left to right.

We now define a ring of subsets $\mathcal{R}$ of X. Let $\mathcal{R} = \{A \cup B : A$ is a finite union of A^k_z's and B is a finite union of sets of the form $\bigcup_{i=m}^{\infty} A^{n+1}_{(z,i)} = A^n_z \sim \bigcup_{i=1}^{m-1} A^{n+1}_{(z,i)}\}$.

We consider both $\varnothing$ and X to be elements of $\mathcal{R}$. We will now define μ and m on $\mathcal{R}$ and extend to regular countably additive measures on $\sigma(\mathcal{R})$, the σ-algebra generated by $\mathcal{R}$. $\sigma(\mathcal{R})$ consists of the Borel subsets of $[0, 1)$.

Define μ and m by induction on the sequence of partitions. Let $\mu(\varnothing) = 0$, $m(\varnothing) = 0$, $\mu(X) = 1$, $m(X) = k$ where k is any element of K. Suppose μ and m are defined on the elements of π_n such that $m(A^n_z)/\mu(A^n_z) = k^n_z \in K$ for each $A^n_z \in \pi_n$. Then since K is not σ-dentable, $k^n_z = \sum_{i=1}^{\infty} \alpha^{n+1}_{(z,i)} k^{n+1}_{(z,i)}$, $\alpha^{n+1}_{(z,i)} > 0$, $\sum_{i=1}^{\infty} \alpha^{n+1}_{(z,i)} = 1$ and $\{k^{n+1}_{(z,i)}\}_{i=1}^{\infty} \subset K \sim S_\varepsilon(k^n_z)$. We now define $\mu(A^{n+1}_{(z,i)}) = \alpha^{n+1}_{(z,i)} \mu(A^n_z)$ and $m(A^{n+1}_{(z,i)}) = \mu(A^{n+1}_{(z,i)}) k^{n+1}_{(z,i)}$. Let $\pi = \{A \subset X : A \in \pi_n$ for some $n\}$.

Thus m and μ are defined on each π_n and hence can be extended by finite additivity to all of $\mathcal{R}$.

Notice that the diameter of the average range of m over each A^n_z is at least ε. This fact, after extension to $\sigma(\mathcal{R})$, will yield the contradiction.

Notice also that the construction yields a "horizontal" countable additivity, that is,

$$\mu(A_z^n) = \sum_{i=1}^{\infty} \mu(A_{(z,i)}^{n+1}) \qquad \text{and}$$

$$m(A_z^n) = \sum_{i=1}^{\infty} m(A_{(z,i)}^{n+1}) \qquad \text{for all } n \in N, z \in N^n.$$

Claim 1. *μ can be extended to a Borel measure on $[0, 1)$ and hence is regular and countably additive.*

Proof. It suffices to show that μ is regular on π relative to $\mathscr{R}$ since it is then regular on $\mathscr{R}$ and hence has an extension to a Borel measure on $[0, 1)$.

Let $\varepsilon > 0$ be arbitrary and $A \in \pi$. Then using the "horizontal" countable additivity there exists $\{A_i\}_{i=1}^{n} \subset \pi$ such that

$$\left|\mu(A) - \sum_{i=1}^{m} \mu(A_i)\right| < \varepsilon.$$

Thus we have

$$\bigcup_{i=1}^{m} A_i \subset \overline{\bigcup_{i=1}^{m} A_i} \subset A,$$

$\overline{\bigcup_{i=1}^{m} A_i}$ is compact and hence μ is inner regular on A.

Suppose $A = [a, b)$. Then by choosing the tail end of the decomposition of the preceding interval, we can find a sequence $\{A_i\}_{i=1}^{\infty}$ such that $\mu(\bigcup_{i=1}^{\infty} A_i) < \varepsilon$, $\bigcup_{i=1}^{\infty} A_i \in \mathscr{R}$, and $(A \cup [\bigcup_{i=1}^{\infty} A_i])' \supset A$, where D' is the interior of D. Thus μ is outer regular on A and hence μ is regular on all of π.

Claim 2. *m can be extended to a Borel measure on $[0, 1)$ such that $\|m(A)\| \leq N\mu(A)$ for all $A \in \sigma(\mathscr{R})$. Thus the extension is countably additive and regular.*

Proof. Since μ is regular and dominates m we can apply Theorem 1 [1, p. 62] of Dinculeanu which implies that m has a countably additive extension of finite variation such that m remains dominated by μ and m is regular.

Claim 3. *m is not an indefinite integral with respect to μ.*

Proof. It suffices to show that, for $B \in \sigma(\mathscr{R})$, the average range of m over B, $A_B(m)$, has diameter not less that $\varepsilon/2$. This sufficiency follows from Theorem 3.1 and its corollary [4, p. 16].

Let $B \in \sigma(\mathscr{R})$. Now by the regularity of μ and m on $\sigma(\mathscr{R})$ we can choose a compact C and an open O such that (i) $C \subset B \subset O$, and (ii) $\mu(O - C) < (\varepsilon/16N)\mu(B)$.

Now those elements in $\mathscr{R}$ of the form $A_{(z,i)}^n \cup [\bigcup_{i=m}^{\infty} A_{(z,i-1)}^n]$ form a base of the topology in $[0, 1)$ and hence by the compactness of C and the openness of O we can find a finite number of these which cover C and are contained in O. Thus there exists a disjoint sequence $\{A_i\}_{i=1}^{\infty} \subset \pi$ such that $C \subset \bigcup_{i=1}^{\infty} A_i \subset O$.

Now there must exist at least one set A_i such that $\mu(A_i \sim B)/\mu(A_i) < \varepsilon/8N = \delta$ since if not, we have

$$\begin{aligned}\mu(O \sim C) &\geq \mu\left(\bigcup_{i=1}^{\infty} A_i \sim B\right) = \sum_{i=1}^{\infty} \mu(A_i \sim B) \\ &\geq \delta \sum_{i=1}^{\infty} \mu(A_i) \geq \delta\mu(C)\left[1 - \frac{\varepsilon}{8N}\right] \\ &\geq (\varepsilon/16N)\mu(B) \qquad \Rightarrow\Leftarrow .\end{aligned}$$

Thus choose A_α such that

$$(*) \qquad \mu(A_\alpha \sim B)/\mu(A_\alpha) < \varepsilon/8N.$$

Let $D = A_\alpha \cap B \in \sigma(\mathscr{R})$, then $D \subset B$ and $\mu(D) > 0$. Now by taking the next partition of A_α we get $A_\alpha = \bigcup_{k=1}^{\infty} C_k$ where the $\{C_k\}_{k=1}^{\infty} \subset \pi$ and are disjoint. Then there must exist a small n such that

$$\mu(C_n \sim B) < (\varepsilon/8N)\mu(C_n)$$

since if not $\mu(A_\alpha \sim B) = \mu(\bigcup_{i=1}^{\infty}(C_n \sim B)) \geq (\varepsilon/8N)\sum_{i=1}^{\infty}\mu(C_n) = (\varepsilon/8N)\mu(A_\alpha)$ which contradicts $(*)$.

Let $E = C_n \cap B$. Now from the construction of m and μ

$$\left\|\frac{m(A_\alpha)}{\mu(A_\alpha)} - \frac{m(C_n)}{\mu(C_n)}\right\| \geq \varepsilon.$$

In addition

$$\begin{aligned}\left\|\frac{m(D)}{\mu(D)} - \frac{m(A_\alpha)}{\mu(A_\alpha)}\right\| &= \left\|\left(1 - \frac{\mu(D)}{\mu(A_\alpha)}\frac{m(D)}{\mu(D)} + \frac{\mu(A_\alpha \sim B)}{\mu(A_\alpha)}\frac{m(A_\alpha \sim B)}{\mu(A_\alpha \sim B)}\right)\right\| \\ &\leq \frac{\mu(A_\alpha \sim B)}{\mu(A_\alpha)}\left\{\left\|\frac{m(D)}{\mu(D)}\right\| + \left\|\frac{m(A_\alpha \sim B)}{\mu(A_\alpha \sim b)}\right\|\right\} < \frac{\varepsilon}{4}.\end{aligned}$$

In the same manner we get $\|m(E)/\mu(E) - m(C_n)/\mu(C_n)\| < \varepsilon/4$. Thus

$$\left\|\frac{m(E)}{\mu(E)} - \frac{m(D)}{\mu(D)}\right\| \geq \frac{\varepsilon}{2}.$$

Thus the diameter of $A_B(m)$ is not less than $\varepsilon/2$ for all $B \in \sigma(\mathscr{R})$ and hence m is not an indefinite integral with respect to μ.

Thus B does not have the R-N property.

The following corollary is due to Uhl [11, Theorem 1, p. 2].

Corollary. *If B is a Banach space such that every closed separable subspace of B is linearly homeomorphic to a subspace of a separable dual space, then B has the Radon-Nikodym property.*

Proof. Suppose B satisfies the hypothesis of the corollary. Let K be any bounded set in B and D any countable subset of K. Then the closed linear span

$[\overline{D}]$ of D is linearly homeomorphic to a subspace of a separable dual space. Since a linear homeomorphism maps σ-dentable sets into σ-dentable sets and since a separable dual space has the R-N property, D is mapped into a σ-dentable set and hence is itself σ-dentable. Thus K is σ-dentable and B has the R-N property.

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