

RELATIVE PROJECTIVITY, THE RADICAL AND COMPLETE REDUCIBILITY IN MODULAR GROUP ALGEBRAS

BY

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ABSTRACT. If $H \leq G$ and every G -module is H -projective then (G, H) is a projective pairing. If $\text{Rad } FG \subseteq (\text{Rad } FH)FG$ then (G, H) is said to have property ρ . A third property considered is that for each irreducible H -module the induced G -module be completely reducible. It is shown that these three are equivalent properties in many interesting cases. Also examples are given to show that they are, in general, independent of each other.

Let H be a subgroup of a finite group G and F a field of characteristic $p > 0$. The H -projective FG -modules were studied by Hochschild [6] and D. G. Higman [5]. Green [4] used this concept to classify indecomposable FG -modules through the theory of vertices and sources. In an earlier paper [7] we characterized the class of these subgroups H of G for which every FG -module is H -projective. In this case we call (G, H) a projective pairing. As was shown in the last part of [7], a certain property (property ρ : See Definition 3) of radicals of group algebras of G and H is closely connected with this concept. As it turns out, another property—the complete reducibility of induced modules, induced from an irreducible FH -module—is almost equivalent to the two properties mentioned above. In this paper we give some sufficient conditions under which the three properties mentioned above are equivalent. We also show by means of an example that in general projective pairing does not imply property ρ .

2. Preliminary definitions and results. Let F be a field of characteristic $p > 0$ and G a finite group such that p divides $|G|$, the order of G . All modules under consideration will be right modules, finitely generated over F . Let \mathfrak{M} be an FG -module, FG being the group algebra of G over F , and let H be a subgroup of G . By \mathfrak{M}_H we denote \mathfrak{M} considered as FH -module in a natural way. Similarly, if \mathfrak{N} is a FH -module, the induced FG -module is denoted by $\mathfrak{N}^G = \mathfrak{N} \otimes_{FH} FG$, where \otimes denotes the tensor product. For standard properties of induced modules we refer to Curtis and Reiner [3], or Green [4].

Definition 1. Let H be a subgroup of G . An FG -module \mathfrak{M} is said to be H -projective if every exact sequence $0 \rightarrow \mathfrak{N} \rightarrow L \rightarrow \mathfrak{M} \rightarrow 0$ of FG -modules for which

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the associated sequence of restrictions $0 \rightarrow \mathfrak{N}_H \rightarrow L_H \rightarrow \mathfrak{M}_H \rightarrow 0$ splits over FH is itself split over FG .

Definition 2. Let $H < G$. We say that (G, H) is a projective pairing if every exact sequence $0 \rightarrow \mathfrak{N} \rightarrow L \rightarrow \mathfrak{M} \rightarrow 0$ of FG -modules for which the exact sequence $0 \rightarrow \mathfrak{N}_H \rightarrow L_H \rightarrow \mathfrak{M}_H \rightarrow 0$ splits over FH is itself split over FG .

It follows immediately that (G, H) is a projective pairing if and only if every FG -module is H -projective.

Definition 3. Let $G = \bigcup_{i=1}^k Hx_i$, $x_1 = 1$, be a coset decomposition of G over H . Treating FG as free module over FH with basis $\{x_1, x_2, \dots, x_k\}$ we say that (G, H) has property ρ over F if, whenever $\sum_i p_i x_i \in \text{Rad } FG$, $p_i \in FH$, then each $p_i \in \text{Rad } FH$, where Rad denotes the Jacobson radical of the ring concerned.

In [9] it is shown that property ρ is independent of the choice of coset representatives and that it is a transitive property in the sense that if $K \subseteq H \subseteq G$ and the pairs (G, H) and (H, K) have property ρ then (G, K) has property ρ .

It is evident that property ρ is equivalent to the requirement $\text{Rad } FG \subseteq (\text{Rad } FH)FG$.

The following criteria of H -projectivity are due to Higman [5] and Green [4]:

(2.1) The following statements for an FG -module \mathfrak{M} are equivalent:

- (i) \mathfrak{M} is H -projective.
- (ii) \mathfrak{M} is a component (i.e. is isomorphic to a direct summand of $(\mathfrak{M}_H)^G$).
- (iii) There exists an FH -module \mathfrak{N} such that \mathfrak{M} is a component of \mathfrak{N}^G .
- (iv) There exists an FH -endomorphism η of \mathfrak{M} such that $\sum_i x_i^{-1} \eta x_i = 1_{\mathfrak{M}}$ where $\{x_i\}$ are the coset representatives of G over H and $1_{\mathfrak{M}}$ is identity mapping over \mathfrak{M} .

We list the following two results of [7] which we will use frequently:

(2.2) If $K \subseteq H \subseteq G$ and (G, H) and (H, K) are projective pairings, then (G, K) is a projective pairing.

(2.3) (G, H) is a projective pairing if and only if H contains a p -Sylow subgroup of G .

Definition 4. We define the following classes of subgroups of G :

$\mathcal{P} = \mathcal{P}(G, F) = \{H < G \mid (G, H) \text{ is a projective pairing over } F\}$,

$\mathcal{R} = \mathcal{R}(G, F) = \{H < G \mid (G, H) \text{ has property } \rho \text{ over } F\}$,

$\mathcal{C} = \mathcal{C}(G, F) = \{H < G \mid \text{for each irreducible } FH\text{-module } \mathfrak{N}, \text{ the induced } FG\text{-module } \mathfrak{N}^G \text{ is completely reducible}\}$.

Note that here $H < G$ means H is a proper subgroup of G , $\{1\}$ being considered as proper subgroup.

Also the following two results of D. A. R. Wallace ([12], [13]) will be important to our discussion:

(2.4) If $|G| = p^a m$, $(p, m) = 1$ then $\dim_F (\text{Rad } FG) \geq p^a - 1$; the equality

holds if and only if $G = PM$, $M \triangleleft G$, $P \cap M = 1$ is a Frobenius group with kernel M and complement P , a p -Sylow subgroup of G ; p being the characteristic of the base field F .

(2.5) Let G be nonabelian and $p \mid |G|$. Then $\text{Rad } FG$ is central if and only if $G'P$ is a Frobenius group with kernel G' , the derived group of G and complement a p -Sylow subgroup P of G .

We will also use the result about the dimension of the radical of FG due to Brauer and Nesbitt [2]:

$$(2.6) \quad \dim_F(\text{Rad } FG) = |G| - \sum_{k=1}^l f_k^2,$$

where f_1, \dots, f_l are the degrees of absolutely irreducible p -modular representations occurring in the regular representation of G .

3. Relation between projective pairing property ρ and the complete reducibility of induced modules for normal subgroups. As was realized by Sinha [9] and as we shall see later, the three properties under consideration are closely related to each other. In fact, they turn out to be equivalent to each other in many cases. In this section we give a proof of their equivalence for normal subgroups. The following theorem was proved by Sinha [9]. We state it and refer to the above paper for the proof.

Theorem 1. *Let H be a subgroup of G . If for each irreducible FH -module \mathfrak{N} , the induced FG -module \mathfrak{N}^G is completely reducible then (G, H) has property ρ . Conversely if $H \triangleleft G$ and (G, H) has property ρ then \mathfrak{N}^G is completely reducible for every irreducible FH -module \mathfrak{N} .*

Remark. In particular, we always have $\mathcal{C} \subseteq \mathcal{R}$.
We now prove the following theorem.

Theorem 2. *Let $H \triangleleft G$. Then the following statements are equivalent:*

- (i) H contains a p -Sylow subgroup of G .
- (ii) (G, H) is a projective pairing.
- (iii) (G, H) has property ρ .
- (iv) For every irreducible FH -module \mathfrak{N} , \mathfrak{N}^G is completely irreducible.

Proof. The equivalence of (i) and (ii) was proved by Khatri and Sinha [7] and (iv) implies (iii) by Sinha [9]. (Note that normality of H in G is irrelevant for these implications.) We now prove that (ii) \implies (iv) and (iii) \implies (i).

(ii) \implies (iv). Suppose that $H \triangleleft G$ and (G, H) is a projective pairing. Let $G = \bigcup_i Hx_i$, $x_1 = 1$, be a coset decomposition of G over H . We observe that, for $b \in H$, $x_i b = \phi_i(b)x_i$ for all i , where each $\phi_i(b)$ induces an automorphism of the

normal subgroup H . We can extend each ϕ_i to an automorphism of FH by linearity.

Now let \mathfrak{N} be an irreducible FH -module. Then $\mathfrak{N}^G = \bigoplus \sum_i \mathfrak{N} \otimes x_i$ (direct sum) where each $\mathfrak{N} \otimes x_i$ is an irreducible FH -module. Thus \mathfrak{N}^G is completely reducible over FH . Hence given any exact sequence of FG -modules

$$0 \rightarrow \mathfrak{M} \rightarrow \mathfrak{N}^G \rightarrow L \rightarrow 0,$$

the sequence splits over FH . Then by projective pairing of (G, H) , the sequence splits over FG as well. Hence \mathfrak{N}^G is completely reducible over FG .

Since \mathfrak{N} was arbitrary, H satisfies (iv).

(iii) \Rightarrow (i). Suppose that $H \triangleleft G$ and (G, H) has property ρ . Let \mathfrak{M} be an irreducible FG -module. By Clifford's theorem we have

$$\mathfrak{M}_H = \mathfrak{N}_1 \oplus \mathfrak{N}_2 \oplus \dots \oplus \mathfrak{N}_r,$$

where each \mathfrak{N}_i is an irreducible FH -module. Consider \mathfrak{N}_1 . We have $\text{Hom}_{FH}(\mathfrak{N}_1, \mathfrak{M}_H) \cong \text{Hom}_{FG}(\mathfrak{N}_1^G, \mathfrak{M})$ (see, for example, the proof of (43.14) in [3]). Now the left-hand side of this equation is different from zero as \mathfrak{N}_1 is a direct summand of \mathfrak{M}_H , and so is the right side. Thus \mathfrak{M} is a composition factor of \mathfrak{N}_1^G . But by Theorem 1, \mathfrak{N}_1^G is completely reducible over FG . Therefore \mathfrak{M} is a component of \mathfrak{N}_1^G and so \mathfrak{M} is H -projective ((2.1) (iii)).

As \mathfrak{M} was arbitrary, we infer that every irreducible FG -module is H -projective. In particular, the trivial FG -module F is H -projective. Hence the vertex of F , namely a p -Sylow subgroup of G , is contained in H (Green [4]). Thus H satisfies (i).

This completes the proof of the theorem.

Corollary. *Let $H \triangleleft G$. Then a necessary and sufficient condition for (G, H) to be a projective pairing is that every irreducible FG -module is H -projective.*

To get the case on p -groups out of the way, we state the following theorem whose proof is easy, and hence omitted.

Theorem 3. *Let G be a p -group and F a field of characteristic p . Then each of the classes \mathcal{P} , \mathcal{R} and \mathcal{C} is vacuous over F .*

4. **Special type of Frobenius groups.** In this section we show that if G is a p -nilpotent Frobenius group with kernel the normal p -complement, then the three classes \mathcal{P} , \mathcal{R} , \mathcal{C} coincide. For groups of this type, Wallace's results [12], [13] are important to our investigation.

The following lemma, essentially due to Wallace [13], determines the radical of such a group.

Lemma 1. *Let $G = PM$, $M \triangleleft G$, $P \cap M = 1$, be a Frobenius group with kernel M and complement a p -Sylow subgroup P of G . Then over a field F of characteristic p ,*

$$\text{Rad } FG = \left(\sum_{x \in M} x \right) \cdot \text{Rad } FP = \text{Rad } FP \cdot \left(\sum_{x \in M} x \right).$$

Proof. By Wallace's result (2.4) we have $\dim_F \text{Rad } FG = |P| - 1$. Let I be the subspace of FG spanned by $\{(\sum_{x \in M} x)(g - 1) \mid g \in P\}$. Clearly $\dim_F I = |P| - 1$. Since $M \triangleleft G$, we have

$$y \left(\sum_{x \in M} x \right) = \sum_{x \in M} x = \left(\sum_{x \in M} x \right) y, \quad b \left(\sum_{x \in M} x \right) = \left(\sum_{x \in M} x \right) b, \quad y \in M, b \in P.$$

Therefore for $b, g \in P$ and $y \in M$, we have

$$by \left[\left(\sum_{x \in M} x \right) (g - 1) \right] = \left(\sum_{x \in M} x \right) b(g - 1) = \left(\sum_{x \in M} x \right) [(bg - 1) - (b - 1)]$$

and

$$\begin{aligned} \left[\left(\sum_{x \in M} x \right) (g - 1) \right] by &= \left[(g - 1) \left(\sum_{x \in M} x \right) \right] by = (g - 1)b \left(\sum_{x \in M} x \right) y \\ &= [(gb - 1) - (b - 1)] \left(\sum_{x \in M} x \right) = \left(\sum_{x \in M} x \right) [(gb - 1) - (b - 1)]. \end{aligned}$$

Thus I is an ideal in FG . But it is well known that $\text{Rad } FP = \langle g - 1 \mid g \in P \rangle$, hence $I = (\sum_{x \in M} x) \text{Rad } FP$ is a nilpotent ideal of dimension $|P| - 1$. This proves the lemma.

Theorem 4. *Let p be an odd prime and P be a p -Sylow subgroup of G . If $G = PM$, $M \triangleleft G$, $P \cap M = 1$, is a Frobenius group with kernel M , then the classes \mathcal{P} , \mathcal{R} and \mathcal{C} are equivalent.*

Proof. First we observe that $G' = M$ (Wallace [13]) so that by Wallace's result (2.5), $\text{Rad } FG \subseteq Z(FG)$.

Our scheme is to prove the inclusions $\mathcal{P} \subseteq \mathcal{C}$ and $\mathcal{R} \subseteq \mathcal{P}$, since $\mathcal{C} \subseteq \mathcal{R}$ always by Theorem 1.

Suppose that (G, H) is a projective pairing. We wish to show $H \in \mathcal{C}$. In the factorization $G = PM$, pick a p -Sylow subgroup P which is contained in H so that $H = P(H \cap M)$, $H \cap M \triangleleft H$. Let

$$G = \bigcup_i Hx_i \text{ and } M = \bigcup_j (H \cap M)y_j$$

be coset decompositions of G over H and of M over $H \cap M$ respectively. By Lemma 1, a typical element τ of $\text{Rad } FG$ can be written as

$$r = a \left(\sum_{x \in M} x \right) = a \left(\sum_{b \in H \cap M} b \right) \left(\sum_j y_j \right), \text{ where } a \in \text{Rad } FP.$$

Let \mathfrak{N} be an irreducible FH -module. Since $\text{Rad } FG$ is central, we have for $r \in \text{Rad } FG$,

$$\begin{aligned} \mathfrak{N}^G \cdot r &= \left(\bigoplus_i \sum \mathfrak{N} \otimes x_i \right) r = \bigoplus_i \sum \mathfrak{N} \otimes x_i r = \bigoplus_i \sum \mathfrak{N} \otimes r x_i \\ &= \bigoplus_i \sum \mathfrak{N} \otimes \left[a \left(\sum_{b \in H \cap M} b \right) \left(\sum_j y_j \right) \right] x_i \\ &= \bigoplus_i \sum \mathfrak{N} \left[a \left(\sum_{b \in H \cap M} b \right) \right] \otimes \left(\sum_j y_j \right) x_i = 0, \end{aligned}$$

since $a(\sum_{b \in H \cap M} b) \in \text{Rad } FH$, H being a Frobenius group with kernel $H \cap M$ and \mathfrak{N} irreducible FH -module. Thus \mathfrak{N}^G is completely reducible over FG .

Since \mathfrak{N} was arbitrary, $H \in \mathcal{C}$.

$\mathcal{R} \subseteq \mathcal{P}$. Let H be a subgroup of G with (G, H) having property ρ . Then p divides $|H|$; for if not, then FH is semisimple and if $\sum_i p_i x_i$ is in $\text{Rad } FG$, $p_i \in FH$, $G = \bigcup_i Hx_i$, coset decomposition of G over H , then each $p_i \in \text{Rad } FH = (0)$ and this implies FG is semisimple, contrary to our hypothesis that p divides $|G|$. In the factorization $G = PM$, pick a p -Sylow subgroup P such that $P \cap H$ is a p -Sylow subgroup of H . Then $H = (H \cap P) \cdot (H \cap M)$ is a Frobenius group with kernel $H \cap M$ and complement $H \cap P$. Let

$$P = \bigcup_i (H \cap P)x_i \text{ and } M = \bigcup_j (H \cap M)y_j$$

be coset decompositions of P and M respectively. We first show that $G = \bigcup_{i,j} Hy_j x_i$ is a coset decomposition of G over H . To see this, observe that $[G:H] = [P:H \cap P][M:H \cap M]$. Suppose now that $Hy_j x_i = Hy_l x_k$; then

$$(x_i x_k^{-1})(x_k x_i^{-1} y_j x_i x_k^{-1} y_l^{-1}) \in H = (H \cap P)(H \cap M).$$

Therefore there exist $a \in H \cap P$ and $b \in H \cap M$ such that

$$(x_i x_k^{-1})(x_k x_i^{-1} y_j x_i x_k^{-1} y_l^{-1}) = ab$$

and this gives $a^{-1}(x_i x_k^{-1}) = b(x_k x_i^{-1} y_j x_i x_k^{-1} y_l^{-1})^{-1} \in P \cap M = \{1\}$. Hence $x_i x_k^{-1} \in H \cap P$ and $x_k x_i^{-1} y_j x_i x_k^{-1} y_l^{-1} \in H \cap M$. This gives $i = k$ and consequently $j = l$. Thus $\{Hy_j x_i\}$ are disjoint and a cardinality argument proves our assertion.

Suppose that (G, H) does not have a projective pairing. Then there exists $g \in P$ such that $g \notin H$. Write $g = b_p x_0$ where $b_p \in H \cap P$, $x_0 \neq 1$ is in $\{x_i\}$. Then by Lemma 1,

$$r = (g - 1) \left(\sum_{x \in M} x \right) \in \text{Rad } FG.$$

We now have

$$\begin{aligned} r &= (b_p x_0 - 1) \left(\sum_{x \in M} x \right) = b_p x_0 \left(\sum_{x \in M} x \right) - \left(\sum_{x \in M} x \right) \\ &= b_p \left(\sum_{x \in M} x \right) x_0 - \left(\sum_{x \in M} x \right), \text{ since } M \triangleleft G, \\ &= b_p \left(\sum_{b \in H \cap M} b \right) \left(\sum_j y_j \right) x_0 - \left(\sum_{b \in H \cap M} b \right) \left(\sum_j y_j \right) \\ &= \sum_j \left[b_p \left(\sum_{b \in H \cap M} b \right) \right] y_j x_0 + \sum_j \left[- \sum_{b \in H \cap M} b \right] y_j. \end{aligned}$$

Since $\{y_j x_i\}$ are coset representatives of G over H and (G, H) has property ρ , we conclude that $b_p \left(\sum_{b \in H \cap M} b \right)$ and $\sum_{b \in H \cap M} b$ are in $\text{Rad } FH$. Now let F be the trivial H -module. We then have $F \cdot \left(\sum_{b \in H \cap M} b \right) = 0$. In particular, $0 = 1 \cdot \left(\sum_{b \in H \cap M} b \right) = \sum_{b \in H \cap M} 1 = |H \cap M|$, which is a contradiction, since $(|H \cap M|, p) = 1$.

Hence (G, H) is a projective pairing and this completes the proof of the theorem.

Remark. It should be observed that for the inclusion $\mathcal{R} \subseteq \mathcal{P}$ we only require G to be p -nilpotent.

5. Extensions by groups of orders prime to p . In the last two sections we showed the equivalence of the classes \mathcal{P} , \mathcal{R} and \mathcal{C} for two classes of groups. We now extend the class of groups for which $\mathcal{P} = \mathcal{R} = \mathcal{C}$. To save space and avoid lengthy repetitions we make

Definition 5. A finite group G is called a *PRC-group* over F if

$$\mathcal{P}(G, F) = \mathcal{R}(G, F) = \mathcal{C}(G, F).$$

Trivially, if $p \nmid |G|$, then G is *PRC* over F ; each class being all subgroups of G .

We start with

Lemma 2. If $H \triangleleft G$, then $\text{Rad } FH \subseteq \text{Rad } FG$.

Proof. Let \mathfrak{M} be an irreducible FG -module. Since $H \triangleleft G$, Clifford's theorem gives $\mathfrak{M}_H = \bigoplus_i \mathfrak{N}_i$, each \mathfrak{N}_i being an irreducible FH -module. We then have $\mathfrak{M} \cdot (\text{Rad } FH) = \mathfrak{M}_H \cdot (\text{Rad } FH) = \bigoplus_i \mathfrak{N}_i (\text{Rad } FH) = 0$. Since \mathfrak{M} is arbitrary, we obtain the desired result.

We next prove

Theorem 5. An extension of a *PRC-group* by a group with order prime to p ($p = \text{characteristic } F$) is a *PRC-group*.

Proof. Let G be an extension of a PRC-group L by a p' -group. We wish to show $\mathcal{P}(G, F) = \mathcal{R}(G, F) = \mathcal{C}(G, F)$.

Since $\mathcal{C} \subseteq \mathcal{R}$ always by Theorem 1, it is sufficient to show $\mathcal{P} \subseteq \mathcal{C}$ and $\mathcal{R} \subseteq \mathcal{P}$.

$\mathcal{P}(G, F) \subseteq \mathcal{C}(G, F)$. Let H be a subgroup of G with (G, H) a projective pairing and let \mathfrak{N} be an irreducible FH -module. Then $H \supseteq P$, a p -Sylow subgroup of G ; which is also a p -Sylow subgroup of L since $[G:L]$ is prime to p . Since $L \triangleleft G$, $H \cap L \triangleleft H$ and so Clifford's theorem yields $\mathfrak{N}_{H \cap L} = \bigoplus \sum_i \mathfrak{N}_i$, where each \mathfrak{N}_i is an irreducible $F(H \cap L)$ -module. By the hypothesis for L , since $H \cap L \supseteq P =$ a p -Sylow subgroup of L , each \mathfrak{N}_i^L is completely reducible over FL . Now $L \triangleleft G$ and (G, L) is a projective pairing. Therefore by Theorem 2, each $\mathfrak{N}_i^G = (\mathfrak{N}_i^L)^G$ is completely reducible over FG . This shows $(\mathfrak{N}_{H \cap L})^G = \bigoplus \sum_i \mathfrak{N}_i^G$ is completely reducible over FG .

Now $(H, H \cap L)$ is a projective pairing. Therefore, by Higman's criteria (2.1) (ii), \mathfrak{N} is a component of $(\mathfrak{N}_{H \cap L})^H$. This in turn implies that \mathfrak{N}^G is a component of the completely reducible FG -module $(\mathfrak{N}_{H \cap L})^G = (\mathfrak{N}_{H \cap L})^H)^G$; thus proving its complete reducibility over FG .

Since \mathfrak{N} was arbitrary, $H \in \mathcal{C}(G, F)$.

$\mathcal{R}(G, F) \subseteq \mathcal{P}(G, F)$. Let H be a subgroup of G with (G, H) having property ρ . Let P be a p -Sylow subgroup of G such that $H \cap P$ is a p -Sylow subgroup of H . Now $H \cap L$ is a normal subgroup of H and $H \cap L \supseteq H \cap P =$ a p -Sylow of H . Therefore Theorem 2 gives property ρ for the pair $(H, H \cap L)$. Since property ρ is transitive and (G, H) has property ρ by hypothesis, consequently we get property ρ for $(G, H \cap L)$.

Suppose now that $\sum_i p_i x_i \in \text{Rad } FL$ with $p_i \in F(H \cap L)$, where $L = \bigcup_i (H \cap L)x_i$ is a coset decomposition of L over $H \cap L$. We may also take $\{x_i\}$ to be some of the coset representatives of G over $H \cap L$. Since $L \triangleleft G$, $\text{Rad } FL \subseteq \text{Rad } FG$ by Lemma 2 and so $\sum_i p_i x_i \in \text{Rad } FG$, $p_i \in F(H \cap L)$. By above $(G, H \cap L)$ has property ρ , hence each p_i is in $\text{Rad } F(H \cap L)$. This gives property ρ for $(L, H \cap L)$.

By hypothesis for L , $(L, H \cap L)$ is a projective pairing. Hence $H \cap L$, and hence H , contains a p -Sylow subgroup of L , which is also a p -Sylow subgroup of G . Thus $H \in \mathcal{P}(G, F)$.

This is what we wanted to prove.

Corollary 1. *If $P \triangleleft G$, then $\mathcal{P}(G, F) = \mathcal{R}(G, F) = \mathcal{C}(G, F)$.*

Proof. Clear from Theorem 3.

Corollary 2. *If $\text{Rad } FG \subseteq Z(FG)$ and p is odd then G is PRC.*

Proof. By Wallace's result (2.5), $L = G'P$ is a Frobenius group with kernel G' , the derived group of G , P being a p -Sylow subgroup of G . But L is a PRC-

group by Theorem 4 and $([G:L], p) = 1$, so the above theorem is applicable.

Corollary 3. *If G is supersolvable, then $\mathcal{R} \subseteq \mathcal{P}$.*

Proof. If p is the highest prime dividing $|G|$, then $P \triangleleft G$ and Corollary 1 above applies. On the other hand, if p is the smallest prime divisor of G then G is p -nilpotent, since it is supersolvable (Scott [8, p. 158]) and $\mathcal{R} \subseteq \mathcal{P}$ by the remark following Theorem 4. Suppose now that p is an intermediate prime divisor of $|G|$. In this case there exists a normal p -nilpotent subgroup K of G (Sylow Tower Theorem, see Scott [8, p. 158]) and G is an extension of K by a p' -group. By the remark following Theorem 4, $\mathcal{R}(K, F) \subseteq \mathcal{P}(G, F)$ by the above theorem.

We observe that using well-known results in Frobenius groups and complements (see, for example, Scott [8, p. 358]) one can obtain several types of PRC-groups. In particular we have

Corollary 4. *If $|G| = p \cdot q$, p, q being primes, or if $|G| = pq^2$, $p \nmid q - 1$, then G is a PRC-group.*

We omit the proof.

6. Extensions of p -groups. In this section we further extend the class of PRC-groups. In fact, we prove that an extension of a p -group by a PRC-group is a PRC-group.

Let $H < G$. We denote by $\mathcal{O}_G(H)$ the ideal in FG generated by $\{b - 1 \mid b \in H\}$. Similarly if I is an ideal in FG , the set $\{g \in G \mid g - 1 \in I\}$ is denoted by $\mathcal{O}_G^{-1}(I)$. It is easy to check that $\mathcal{O}_G^{-1}(I)$ is a normal subgroup of G . It is well known that if $\phi: FG \rightarrow F(G/H)$ is the map obtained by extending the canonical map $G \rightarrow G/H$ ($H \triangleleft G$) to FG by linearity, then $\text{Ker } \phi = \mathcal{O}_G(H)$ and so $F(G/H) \cong FG/\mathcal{O}_G(H)$.

We start with

Lemma 3. *Let P be a p -Sylow subgroup of G . Then*

$$\mathcal{O}_G^{-1}(\text{Rad } FG) = \text{core}_G(P).$$

Proof. Let $A = \mathcal{O}_G^{-1}(\text{Rad } FG) = \{g \in G \mid g - 1 \in \text{Rad } FG\}$. Then $A \triangleleft G$, since $\text{Rad } FG$ is a two-sided ideal in FG . Since $\text{core } P \triangleleft G$, by Lemma 2, $\text{Rad}(F(\text{core } P)) = \langle g - 1 \mid g \in \text{core } P \rangle \subseteq \text{Rad } FG$. Thus for each $g \in \text{core } P$, $g - 1 \in \text{Rad } FG$ and so $\text{core } P \subseteq A$.

Now let $g \in A$. Then $g - 1$ is nilpotent, so if $H = \langle g \rangle$ then H is abelian and $\text{Rad } FH$ is not 0. In such a case, it is well known that H has an element in order p . Since $|H| = |g|$, so by Lagrange's theorem, p divides $|g|$. Thus for each $g \in A$, p divides $|g|$. Suppose $|A| = p^b m$, $(p, m) = 1$. If $m > 1$ then A will have elements with orders prime to p , contrary to what we have proved above. Therefore

$m = 1$ and A is a p -group. Since A is a normal subgroup of G and $\text{core } P \subseteq A$, we have $A = \text{core } P$. This proves the lemma.

Theorem 6. *Let $H < G$ such that (G, H) has property ρ . Then $H \supseteq \text{core } P$.*

Proof. If the assertion of the theorem is false then there is a $g \in \text{core } P$ and $g \notin H$. Write $G = \bigcup_i Hx_i$: coset decomposition of G over H . Let $g = bx_j$ for some $b \in H$ and some $x_j \neq 1$. By Lemma 3, $g - 1 = bx_j - 1 \cdot 1 \in \text{Rad } FG$. Since (G, H) has property ρ , we must have $b, 1 \in \text{Rad } FH$, which is impossible and we are done.

Before we come to the main theorem, we need

Lemma 4. *Let I be a nilpotent ideal in a ring R and $\phi: R \rightarrow R/I$ the natural map. If $\text{Rad } R$ denotes the nilpotent radical of R , then $\phi(\text{Rad } R) = \text{Rad}(R/I)$.*

Proof. We always have $\phi(\text{Rad } R) \subseteq \text{Rad}(R/I)$. For the reverse inclusion, suppose $a + I \in \text{Rad}(R/I)$, $a \in R$. Then for each $b + I \in R/I$, $(a + I)(b + I) = ab + I$ is nilpotent. Thus there is an integer m such that $(ab + I)^m = (ab)^m + I = I$. Therefore $(ab)^m \in I$. Since I is nilpotent by hypothesis, this yields nilpotency of ab . Since b in R was arbitrary, $a \in \text{Rad } R$ and $\phi(a) = a + I \in \text{Rad}(R/I)$. This proves the lemma.

Remark. In fact, we proved a little more than what is asserted in the statement of the lemma. We showed that $a + I$ is in $\text{Rad}(R/I)$ if and only if a is in $\text{Rad } R$.

We are now in a position to prove the main theorem of this section.

Theorem 7. *Any extension G of a p -group A by a PRC-group is PRC.*

Proof. Once again we show $\mathcal{P} \subseteq \mathcal{C}$ and $\mathcal{R} \subseteq \mathcal{P}$. Let G be an extension of a p -group A by a PRC-group. Hence $A \trianglelefteq G$ and G/A is PRC.

$\mathcal{P}(G, F) \subseteq \mathcal{C}(G, F)$. Let (G, H) be a projective pairing. Then H contains a p -Sylow subgroup of G and hence contains A . Since $A \triangleleft H$, $\bigoplus_A(A) \subseteq \text{Rad } FH$ by Lemma 2 and so if \mathfrak{N} is an irreducible FH -module, each $a \in A$ acts trivially on \mathfrak{N} and \mathfrak{N} can be regarded as an irreducible $F(H/A)$ -module. Now clearly $(G/A, H/A)$ is a projective pairing, hence, since G/A is PRC, the induced $F(G/A)$ -module $\mathfrak{N}^{G/A} = \mathfrak{N} \otimes_{F(H/A)} F(G/A)$ is completely reducible, over $F(G/A)$. Since $a \in A$ acts trivially on \mathfrak{N} , we can view $\mathfrak{N}^{G/A}$ as an FG -module in a natural way by defining $(n \otimes Ag) \cdot b = (n \otimes Ag) \cdot Ab$, $n \in \mathfrak{N}$, $g, b \in G$.

Define $\phi: \mathfrak{N}^{G/A} \rightarrow \mathfrak{N}^G$ by $\phi(n \otimes Ag) = n \otimes g$ and extend by linearity. Since

$$\begin{aligned} \phi((n \otimes Ag)b) &= \phi((n \otimes Ag)Ab) = \phi(n \otimes Agb) = n \otimes gb \\ &= (n \otimes g)b = \phi(n \otimes Ag) \cdot b \end{aligned}$$

and since (obviously) ϕ is 1-1 and onto, ϕ is an FG -isomorphism.

Thus \mathfrak{R}^G is completely reducible over FG implying $H \in \mathcal{C}(G, F)$.

$\mathfrak{R}(G, F) \subseteq \mathcal{P}(G, F)$. Suppose that (G, H) has property ρ . Then $H \supseteq \text{core } P \supseteq A$, by Theorem 6. Let $\phi: G \rightarrow G/A$ be the natural map. We extend ϕ to FG by linearity.

Let $G = \bigcup_i Hx_i$ be a coset decomposition of G over H . Then $G/A = \bigcup_i (H/A)Hx_i$ is a coset decomposition of G/A over H/A . We show that $(G/A, H/A)$ has property ρ .

Suppose that $\sum_i p'_i Ax_i$ is in $\text{Rad } F(G/A)$, where $p'_i \in F(H/A)$. There exist $p_i \in FH$ such that $\phi(p_i) = p'_i$ for all i . Thus $\phi(\sum_i p'_i Ax_i) = \sum_i p'_i Ax_i \in \text{Rad } F(G/A) = \text{Rad}(FG/\mathfrak{O}_G(A))$. Since $\mathfrak{O}_G(A) \subseteq \text{Rad } FG$, Lemma 4 is applicable and we conclude that $\sum_i p_i x_i \in \text{Rad } FG$, with $p_i \in FH$. By hypothesis, (G, H) has property ρ , so each $p_i \in \text{Rad } FH$. Thus each $p'_i = \phi(p_i) \in \phi(\text{Rad } FH) = \text{Rad } F(H/A)$. This yields property ρ for the pair $(G/A, H/A)$.

Now by hypothesis G/A is a PRC -group; hence $(G/A, H/A)$ is a projective pairing and therefore H/A contains a p -Sylow subgroup P/A and $G/A, P$ being a p -Sylow subgroup of G . Thus $H \supseteq P$ and (G, H) is a projective pairing.

This completes the proof of the theorem.

This gives another proof of Corollary 1 to Theorem 5.

Corollary. *If $P \triangleleft G$, then G is PRC .*

We observe that using the above theorem and easy known results on group theory one can prove that all groups of order $p^2 \cdot q$ are PRC without any extra conditions (contrast Corollary 4 to Theorem 5).

Remark. We do not know whether an extension of PRC by PRC is PRC or even whether an extension of a p' -group by a p -group is PRC . However, it can easily be seen that the direct product of two PRC -groups is a PRC -group.

7. **Example.** In the previous sections we gave some sufficient conditions for the equivalence of the three classes \mathcal{P} , \mathfrak{R} and \mathcal{C} . That, in general, they are not equivalent is shown by the following example, which, in fact, shows that projective pairing does not imply property ρ . Since \mathcal{C} is a subclass of \mathfrak{R} , this would mean that projective pairing also does not imply the complete reducibility of induced modules.

Let $G = A_5$, $H = A_4$ and $p = \text{characteristic of } F = 3$, where A_n denotes the alternating group on n symbols. Since H contains a 3-Sylow subgroup of G , (G, H) is a projective pairing over F . We show that (G, H) cannot have property ρ . We may take F to be a splitting field for G .

It is well known that over the complex field, A_5 has five irreducible representations, say, T_1, T_2, T_3, T_4, T_5 of degrees 1, 2, 3, 4, 5 respectively. Taking

mod 3, it follows from the results of Brauer and Nesbitt (Theorem 1 of [2] and a remark on p. 936 together with Theorem 5 of [1]) that $\bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{T}_4$ remain irreducible while $\bar{T}_5 \approx \bar{T}_1 + \bar{T}_4$. Since A_5 has four 3-regular classes, these are all the irreducible representations of A_5 over the field F of characteristic 3.

Now by Brauer and Nesbitt's result (2.6) $\dim_F(\text{Rad } FA_5) = 60 - (1^2 + 3^2 + 3^2 + 4^2) = 25$.

Also since $H = A_4$ is a Frobenius group with complement a 3-Sylow subgroup of H , Wallace's result (2.4) gives $\dim_F(\text{Rad } FA_4) = 3 - 1 = 2$. Now if (G, H) has property ρ , then we must have $\text{Rad } FG \subseteq (\text{Rad } FH)FG$. This yields

$$\dim_F(\text{Rad } FG) \leq \dim_F(\text{Rad } FH) \cdot [G:H].$$

That is, we must have $25 \leq 2 \cdot 5 = 10$, which is impossible. Thus (G, H) cannot have property ρ .

Incidentally, in this case property ρ does imply projective pairing; for if (G, H) has property ρ then 3 divides $|H|$ and since 3 occurs to the first power in $|G| = 2^2 \cdot 3 \cdot 5$, H would contain a 3-Sylow subgroup of G . This example also shows that in Theorem 5 we cannot drop the normality of L : A_5 has a subgroup A_4 with $\mathcal{P}(A_4, F) = \mathcal{R}(A_4, F) = \mathcal{C}(A_4, F)$, since A_4 is Frobenius group with 3-Sylow subgroup as complement, and $[A_5 : A_4] = 5$ with $(3, 5) = 1$.

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