THE MODULE DECOMPOSITION OF $l(\mathcal{A}/A)$

BY

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ABSTRACT. Let $A$ and $B$ be scalar rings with $B$ an $A$-algebra. The $B$-algebra $D^n(B/A) = l(B/A) / I^n(B/A)$ is universal for $n$-truncated $A$-Taylor series on $B$. In this paper, we consider the $A$ module decomposition of $D^n(\mathcal{A}/A)$ with a view to classifying the singularity $A$ which is assumed to be the complete local ring of a point on an algebraic curve at a one-branch singularity. We assume that $A/M = k < A$ and that $k$ is algebraically closed with no assumption on the characteristic.

We show that $D^n(\mathcal{A}/A) = l(\mathcal{A}/A)$ for $n$ large and that the decomposition of $l(\mathcal{A}/A)$ as a module over the P.I.D. $\mathcal{A}$ is completely determined by the multiplicity sequence of $A$. The decomposition is displayed and a length formula for $l(\mathcal{A}/A)$ developed.

1. Preliminaries. Suppose $B$ and $A$ are commutative rings with identity elements. Such rings are known as scalar rings. Suppose also that $B$ is an $A$ algebra.

Definition. An $A$ linear map $T$ from $B$ to a $B$ algebra $S$ is called an $S$-valued $A$-Taylor series if, for each $x, y \in B$, $T(xy) = xT(y) + yT(x) + T(x)T(y)$ and if $T(1) = 0$.

Denote by $\pi$ the map from $B \otimes_A B$ to $B$ given by multiplication: i.e. $\pi(a \otimes_A b) = a \cdot b$. If $l(B/A)$ is defined by the exact sequence

$$0 \to l(B/A) \to B \otimes_A B \xrightarrow{\pi} B \to 0,$$

then the ideal $l(B/A)$ has the structure of a left $B$-algebra and a left $B$ module (for $b \in B$ and $c \otimes_A d \in B \otimes_A B$, $b(c \otimes_A d) = (b \otimes_A 1)(c \otimes_A d)$). The $B$
module \( l(B/A) \) is generated by the set of elements \((1 \otimes_A x - x \otimes_A 1)\) for \( x \in B \). If \( z_i \ (i \in H) \) is a set of \( A \)-algebra generators for \( B \), then the elements
\[(1 \otimes_A z_i - z_i \otimes_A 1)\] are \( B \)-algebra generators for \( l(B/A) \) [6, p. 4].

Let \( \delta_A \) be the map from \( B \) to \( l(B/A) \) given by \( \delta_A(x) = 1 \otimes_A x - x \otimes_A 1 \). This map is easily checked to be an \( l(B/A) \)-valued \( A \)-Taylor series and is called the canonical \( A \)-Taylor series on \( B \). It is known that the pair \((\delta_A, l(B/A))\) is universal for \( S \)-valued \( A \)-Taylor series [6, p. 5]. Hence, if \( r: B \to S \) is an \( S \)-valued \( A \)-Taylor series, then there exists a unique \( B \)-algebra homomorphism \( r^* \) from \( l(B/A) \) to \( S \) so that \( r^* \delta_A = r \).

Definition. Suppose \( A \) and \( B \) are scalar rings with \( B \) an \( A \)-algebra. If \( S \) is a \( B \)-algebra then \( S \) is said to be \( n \)-truncated if for each sequence of \( n + 1 \) elements \( s_0, \ldots, s_n \) in \( S \), \( s_0 \cdots s_1 \cdots s_n = 0 \).

Definition. An \( S \)-valued \( A \)-Taylor series \( r \) is said to be \( n \)-truncated if \( S \) is \( n \)-truncated.

Denote by \( \theta \) the natural map of \( l(B/A) \) to \( l(B/A)/l(B/A)^{n+1} \). If \( \delta_A \) is the canonical \( A \)-Taylor series on \( B \), then \( \delta_A^n = \theta \delta_A \) is an \( n \)-truncated \( A \)-Taylor series and the pair \((\delta_A^n, l(B/A)/l(B/A)^{n+1})\) is universal for \( n \)-truncated \( S \)-valued \( A \)-Taylor series [6, p. 17].

The universal object \( l(B/A)/l(B/A)^{n+1} \) will be denoted by \( D^n(B/A) \) and has again the structure of a left \( B \)-algebra and hence \( B \)-module.

Lemma 1.1. Suppose \( A \) and \( B \) are scalar rings and \( B \) is a finitely generated \( A \)-module. Then \( l(B/A) \) is finitely generated as a \( B \)-module.

Proof. Let \( l(B/A) \) be defined by the sequence
\[0 \to l(B/A) \to B \otimes_A B \xrightarrow{\pi} B \to 0\]
where \( \pi(a \otimes_A b) = ab \). If \( \sum_{i=1}^n (a_i \otimes b_i) \in l(B/A) \), then \( \sum_{i=1}^n a_i b_i = 0 \) and consequently
\[n \sum_{i=1}^n (a_i \otimes b_i) = \sum_{i=1}^n (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1) \].

Hence, \( l(B/A) \) is generated as a \( B \)-module by elements of the form \((1 \otimes_A b - b \otimes_A 1) = \delta_A(b)\) where \( b \in B \). But \( B \) finitely generated as an \( A \)-module implies \( b = \sum x_i y_i \) where \( x_i \in A \) and \( y_i \) are the generators. Since \( \delta_A \) is \( A \)-linear, \( \delta_A(b) = \sum x_i \delta(y_i) \) and hence \( l(B/A) \) is finitely generated as a \( B \)-module. Q.E.D.

Let \( A \) be a Noetherian local domain and \( F \) its field of quotients. If \( \overline{A} \) is the integral closure of \( A \) in \( F \), assume that \( \overline{A} \) is finitely generated as an \( A \)-module. Lemma 1.1 then implies that \( l(\overline{A}/A) \) is finitely generated as an \( \overline{A} \)-module and, consequently, also \( D^n(\overline{A}/A) \).

Theorem 1.2. Assume that \( A \) is a local Noetherian domain of Krull dimension 1. If \( M \) is the maximal ideal of \( A \), assume \( A/M = k \) is algebraically closed and
k < A. Let $\overline{A}$ be the integral closure of $A$ inside its field of quotients and suppose $\overline{A}$ is finitely generated as an $A$-module. If $\overline{A}$ is local then the ideal $I(\overline{A}/A)$ is nilpotent.

Proof. Since $\overline{A}$ is finitely generated over $A$, the conductor $C = \text{Ann}_A(\overline{A}/A) \neq 0$. (Take the product of the denominators of the generators of $A$.) We may assume that $C \neq (1)$, since otherwise $A = \overline{A}$.

Let $x_1, \ldots, x_s$ be the generators for $\overline{A}$ over $A$. Then $\delta_A(x_1), \ldots, \delta_A(x_s)$ are $\overline{A}$ algebra generators for $I(\overline{A}/A)$. For the maximal ideal $M'$ of $\overline{A}$, $\overline{A}/M' \cong k$ since $k$ is algebraically closed. If $a + y \in \overline{A}$ where $a \in k$, then $\delta_A(a + y) = \delta_A(y)$.

Hence, we may assume that the $x_i$'s lie in $M'$. Because $A$ is one dimensional, we may assume that $x_i^d \in C < A$ for some $d$ and all $i$. But

$$\delta_A(x_i^d) = \left(\begin{array}{c} d \\ 1 \end{array}\right)x_i^{d-1}\delta_A(x_i) + \left(\begin{array}{c} d \\ 2 \end{array}\right)x_i^{d-2}[\delta_A(x_i)]^2 + \cdots + [\delta_A(x_i)]^d$$

[6, p. 16]. Since $x_i^d \in A$, $\delta_A(x_i^d) = 0$ and hence

$$[\delta_A(x_i)]^d = (-1)^{d-1}d! x_i^{d-2}\delta_A(x_i) + \cdots + \left(\begin{array}{c} d \\ d-1 \end{array}\right)[\delta_A(x_i)]^{d-1}.$$ 

Since $(x_i)^d I(\overline{A}/A) = 0$, we conclude that $\delta_A(x_i)$ is nilpotent for $i = 1, \ldots, s$. But these are algebra generators for $I(\overline{A}/A)$ over $\overline{A}$. Therefore, $I(\overline{A}/A)$ is nilpotent. Q.E.D.

Theorem states that with the assumptions on $A$, $D^n(\overline{A}/A) \cong I(\overline{A}/A)$ as $\overline{A}$-algebras for $n > 0$. Specifically, if $A$ is the complete local ring of a point on algebraic curve at a "one-branch singularity", then $D^n(\overline{A}/A) \cong I(\overline{A}/A)$ for $n$ large and it is this observation which leads to the study of $I(\overline{A}/A)$ in the following sections.

Definition. Suppose $A$ and $B$ are scalar rings with $B$ an $A$-algebra. Suppose $M$ is a $B$-module. An $A$-linear map $L$ from $B$ to $M$ is called a $q$th order derivation of $B/A$ into $M$ if it satisfies the following conditions:

1. $L(x_0 x_1 \ldots x_q) = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1 < \ldots < i_s} x_{i_1} \ldots x_{i_s} L(x_0 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_s} \ldots x_q)$

for any set $x_0, \ldots, x_q$ of $(q + 1)$ elements in $B$.

2. $L(1) = 0$.

Note that a derivation of order 1 is a standard derivation from $B$ to $M$ over $A$.

The map $\delta_A^q$ is itself a $q$th order derivation of $B/A$ into $D^q(B/A)$. If $L$ is a $q$th order derivation of $B/A$ into $M$, then there exists a unique $B$-homomorphism $b$ from $D^q(B/A)$ to $M$ so that $b \cdot \delta_A^q = L$ [6, p. 35]. Therefore, when considered as a $B$-module, $D^q(B/A)$ together with $\delta_A^q$ is universal for $q$th order derivations of $B/A$ into $M$ (cf. [8]).
2. The blow-up and strict closure of $A$. Throughout this section we will assume that $A$ is the complete local ring of a point on an algebraic curve at a "one-branch singularity" whose residue field $k$ is algebraically closed and contained in $A$. Hence, $A$ is a complete local Noetherian domain whose integral closure $\overline{A}$ inside its quotient field $F$ is a power series ring in one variable with coefficients in $k$. If $k[[t]]$ is the power series ring where $t$ is a uniformizing parameter, then $A \subset k[[t]]$ and the inclusion is proper since we assume that $A$ is the local ring of a singular point.

Since $A$ is a complete local integral domain, $\overline{A}$ is a finitely generated $A$ module [7, p. 112]. Hence, $l(\overline{A}/A)$ is nilpotent according to Theorem 1.1.

The natural valuation on $F$ will be denoted by $v$ and for any $x \in A$, $v(x)$ is called the order of $x$. The valuation ring of $v$ is, of course, $k[[t]]$.

Let $E$ be a set contained in $A$. Then $E$ is said to be open in the $M$-adic topology of $A$ if $M^n < E$ for some $n$. Note that for any $x \in A$, $x$ is not a zero divisor. Hence, since $A$ is one-dimensional, $xA$ is an open ideal. Let $E$ be any $A$ module and denote $\lambda_A(E)$ as the length of this module. For any open ideal $J < A$, $\lambda_A(A/J^n)$ becomes a polynomial of degree 1 in $n$ for $n$ large [11, Volume 1, p. 284]. This polynomial, say $en + b$, is known as the characteristic polynomial of $J$ and $e$ is called the multiplicity of the ideal $J$. We shall denote the multiplicity of an ideal $J < A$ as $e(J)$. The multiplicity of the maximal ideal $M$ is by definition the multiplicity of $A$. We shall write $e(A)$ for $e(M)$. It is easy to see that $\lambda_A(M^n/M^{n+1}) = e(A)$ for $n$ large. If $v(M)$ is the least integer in the set $\{v(x): x \in M\}$, then $v$ is the valuation, then $v(M) = e(M)$.

Definition. Let $M$ be the maximal ideal of $A$. Then $x \in M$ is said to be transversal to $M$ if $v(x) = e(M)$.

Let $A < A = k[[t]]$ and let $e(A) = e > 0$. For any $x \in A$ where $v(x) = e$, $x = t^e u$ where $u$ is a unit in $k[[t]]$. Hensel's lemma [11, Volume 2, p. 279] assures that there exists a unit $u' \in k[[t]]$ so that $(u')^e = u$. Setting $t' = tu'$, $t'$ is necessarily of order one and $(t')^e = x$. Since $v(t') = 1$, $k[[t]] = k[[t']]) = \overline{A}$ and consequently, $F = k((t'))$. Hence, we may always assume that if $x$ is transversal to $M < A$, then $x = t^e$ where $\overline{A} = k[[t]]$ and $e = e(A)$.

Let $n$ be any positive integer. The power series ring $k[[t^n]]$ is contained in $k[[t]]$ and $k[[t]]$ is freely generated over $k[[t^n]]$ by the elements $1, t, \ldots, t^{n-1}$. Denote by $R$ the power series ring $k[[t^n]]$ and let $\overline{A} = k[[t]]$.

Lemma 2.1. The $\overline{A}$ module $l(\overline{A}/R)$ is a free $\overline{A}$-module on the generators $\delta_R(t), \ldots, \delta_R(t^{n-1})$.

Proof. The sequence

$$0 \to l(\overline{A}/R) \to \overline{A} \otimes_R \overline{A} \twoheadrightarrow \overline{A} \to 0$$

is split exact and hence $\overline{A} \otimes_R \overline{A} \cong \overline{A} \otimes l(\overline{A}/R)$. But $\overline{A} \otimes_R \overline{A}$ is a free $\overline{A}$-module.
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of rank $n$ and hence, $l(\mathcal{A} / R)$ is a free $\mathcal{A}$-module of rank $n - 1$. Since by Lemma 1.1, $\delta_R(t), \ldots, \delta_R(t^{n-1})$ generate $l(\mathcal{A} / R)$, they must form a free basis. \hfill Q.E.D.

Definition. Let $x$ be transversal to $M$, the maximal ideal of $A$ and let $\{x_1, \ldots, x_n\}$ generate $M$. The ring $A^M = A[x_1/x, \ldots, x_n/x]$ is called the blow-up of $A$ along $M$ [5, p. 651], (cf. [9]).

Note that $A^M$ is finitely generated over $A$ and that $\overline{A^M} = \mathbb{k}[[t]]$. Hence, $\overline{A^M}$ is itself the complete local ring of an algebraic curve at a one-branch singularity. Since $M$ is contained in the maximal ideal of $A^M$, $e(A^M) \leq e(A)$.

Set $A^M = A_1$ and let $M_1$ be the maximal ideal of this ring. We may form the blow up of $A_1$ along the maximal ideal $M_1$ and set $A_1^M = A_2$ with maximal ideal $M_2$. Continuing in this way, we have a sequence of increasing rings

$A < A_1 < \ldots < A_N < \ldots < \mathcal{A}$.

Since $A$ is Noetherian and for each $i$, $A_i$ is finitely generated, $A_N = A_{N+1}$ for some $N$. Hence, by [9, p. 372], $A_N$ is a regular local ring and therefore, $A_N = \mathcal{A} = \mathbb{k}[[t]]$. That is, the singularity can be resolved after applying a finite number of blow-ups. The sequence $A = A_0 < A_1 < \ldots < A_N < \ldots < \mathcal{A}$ is known as the branch sequence of $A$ along the maximal ideal $M$ of $\mathcal{A}$ and $\{e(A), e(A_1), \ldots\}$ is called the multiplicity sequence of $A$.

Definition. Suppose $\mathcal{A}$ is the integral closure of $A$ inside its quotient field. Then the ring $A' = \{x \in \mathcal{A} : 1 \otimes_A x = x \otimes_A 1 = 1 \}$ is called the strict closure of $A$ inside $\mathcal{A}$. Clearly, $A < A'$ and $\mathcal{A} = \mathcal{A}$. \hfill \textit{Lemma 2.2.} Let $A < A$ and let $A'$ be the strict closure of $A$ in $\mathcal{A}$. Then $\mathcal{A} \otimes_A \mathcal{A} \simeq \mathcal{A} \otimes_{A'} \mathcal{A}$, $l(\mathcal{A} / A) \simeq l(\mathcal{A} / A')$ and $D^n(\mathcal{A} / A) \simeq D^n(\mathcal{A} / A')$.

\textbf{Proof.} Consider the diagram

\[ 0 \rightarrow l(\mathcal{A} / A') \overset{i'}{\rightarrow} \mathcal{A} \otimes_{A'} \mathcal{A} \rightarrow 0 \]

\[ 0 \rightarrow l(\mathcal{A} / A) \overset{j}{\rightarrow} \mathcal{A} \otimes_A \mathcal{A} \rightarrow 0 \]

\[ \psi/l(\mathcal{A} / A) \]

\[ \psi/l(\mathcal{A} / A') \]

where $\psi$ is the canonical map.

Note that for any three scalar rings $A < B < C$, the kernel of the canonical map from $C \otimes_A C$ to $C \otimes_B C$ is generated as an ideal by the elements $1 \otimes_A x - x \otimes_A 1 : x \in B$ [6, p. 12]. Hence, if $K$ is the kernel of $\psi$, then $K$ is generated as an ideal by the elements $1 \otimes_A x - x \otimes_A 1$ where $x \in A'$. Therefore $K = 0$. The map $\psi$ is clearly onto and the restriction of $\psi$ to $l(B / A)$ gives the required $B$-algebra isomorphism. Hence, $l(B / A) \simeq l(B / A')$ and $D^n(B / A) \simeq D^n(B / A')$. \hfill Q.E.D.

Remark 2.3. The proof of the lemma shows that $(A')' = A'$. If $A = A'$, then $A$ is said to be strictly closed in $\mathcal{A}$. Under our standing assumption that $k < A$ (k
a field), [5, p. 677] shows that $A'$ coincides with the Arf closure of $A$. In particular, $e(A) = e(A')$ [5, p. 668] and if $M'$ is the maximal ideal of $A'$, then $(A')^M = (A^M)'$ [5, p. 668]. The geometric properties of the strict closure of $A$ (= Arf closure of $A$) will be used in § 4.

Remark 2.4. The blow-up of $A'$ along its maximal ideal has an easier form. In fact, if $A$ is strictly closed (i.e. $A = A'$) with maximal ideal $M$ and if $x$ is transversal to $M$, then $A^M = \{m/x : m \in M\}$. To show this we need only prove that $Mx^{-1}$ forms a ring. But for any $y/x, z/x \in Mx^{-1}$,

$$y \cdot z/x \otimes_A 1 = y/x \otimes_A z = y/x \otimes_A zx/x = y \otimes_A z/x = 1 \otimes_A yz/x.$$  

Hence, $y \cdot z/x \in A$ since $A$ is strictly closed. Equivalently, $y \cdot z = x \cdot a, a \in M < A$. Therefore, $y/x \cdot z/x = a/x \in Mx^{-1}$.

3. The relation of $I(\overline{A}/A)$ to $I(\overline{A}/A^M)$. Suppose $A$ is as in § 2 with maximal ideal $M$. Choose a transversal element $x$ so that $x = t^e$ where $e = e(A)$ and $\overline{A} = k[[t]]$. Hence, $A$ is properly contained in $\overline{A} = k[[t]]$ and if $A^M$ is the blow-up of $A$ along $M$, then also $\overline{A}^M = k[[t]]$. Let $\theta$ be the canonical homomorphism from $I(\overline{A}/A)$ to $I(\overline{A}/A^M)$ given by

$$\theta \left( \sum_{i=1}^{n} (a_i \otimes_A b_i) \right) = \sum_{i=1}^{n} (a_i \otimes_{A^M} b_i).$$

Note that $\theta$ is onto.

Let $R = k[[t^e]]$; it is clear that $R$ is a complete subring of both $A$ and $A^M$.

Lemma 2.1 asserts that $I(\overline{A}/R)$ is a free $\overline{A} = k[[t]]$ module with $\delta_R(t), \ldots, \delta_R(t^{e-1})$ as generators. Let $\phi_1$ and $\phi_2$ be the canonical maps from $I(\overline{A}/R)$ to $I(\overline{A}/A)$ and $I(\overline{A}/A^M)$, respectively. Since $I(\overline{A}/A)$ is generated by $\delta_A(t), \ldots, \delta_A(t^{e-1})$ (Lemma 1.1) and since $e(A^M) \leq e(A)$, it follows that $\phi_1$ and $\phi_2$ are both onto maps. Let $N(A)$ and $N(A^M)$ be the kernel of $\phi_1$ and $\phi_2$ respectively. We have then the diagram:

\[
\begin{array}{ccc}
N(A) & \xrightarrow{i_1} & I(\overline{A}/A) \\
\phi_2 & & \theta \\
N(A^M) & \xrightarrow{i_2} & I(\overline{A}/A^M)
\end{array}
\]

\[
(\ast)
\]

where clearly, $\phi_1 \theta = \phi_2$.

Both $N(A)$ and $N(A^M)$ are submodules of the free $\overline{A}$ module $I(\overline{A}/R)$ and are finitely generated.
Lemma 3.1. With the situation as above, suppose $\eta \in N(A^M)$, then $t^e\eta \in N(A)$.

Proof. If $A'$ is the strict closure of $A$ in $\overline{A}$, then $l(\overline{A}/A) \cong l(\overline{A}/A')$ where the $\overline{A}$-algebra isomorphism is given by the canonical morphism $\psi_1$ from $\overline{A} \otimes_A \overline{A}$ to $\overline{A} \otimes_A \overline{A}$.

Let $M'$ be the maximal ideal of $A'$. Since the blow-up of $A$ commutes with taking the strict closure (Remark 2.3), we have $(A')^{M'} = (A^M)'$ and hence the commutative diagram:

$$
\begin{array}{ccc}
I(\overline{A}/AM^M) & \xrightarrow{\psi_2} & I(\overline{A}/(AM')^M) \\
\downarrow{\theta} & & \downarrow{\theta'} \\
I(\overline{A}/A) & \xrightarrow{\psi_1} & I(\overline{A}/A')
\end{array}
$$

where $\psi_2$ and $\theta'$ are the obvious canonical maps.

If $\eta = \sum_{j=1}^{e-1} a_j \delta_R(t^j) \in N(A^M)$, then

$$
\psi_2 \phi_2(\eta) = \sum_{j=1}^{e-1} a_j \delta_{A^M}(t^j) = 0.
$$

Since $\epsilon(A) = \epsilon(A')$ (Remark 2.3), $t^e$ is also transversal to $M'$ and since $(A')^{M'} = (A^M)'$, $A'^{M'}$ is strictly closed and $(A')^{M'} = \{m'/t^e : m' \in M'\}$ (Remark 2.4). But

$$
\sum_{j=1}^{e-1} a_j \delta_{A^M}(t^j) = \sum_{j=1}^{e-1} a_j \delta_{(A')^{M'}}(t^j) = 0
$$

and hence

$$
\sum_{j=1}^{e-1} c_i \otimes_A d_i \left(1 \otimes_A \frac{m_i'}{t^e} - \frac{m_i'}{t^e} \otimes_A 1\right)
$$

where $m_i' \in M'$ for all $i$ and $c_i, d_i \in \overline{A}$. Then clearly

$$
(t^e) \left(\sum_{j=1}^{e-1} a_j \delta_{A^M}(t^j)\right) = 0.
$$

Since $\psi_1$ is an isomorphism it follows that $t^e \sum_{j=1}^{e-1} a_j \delta_A(t^j) = 0$ and consequently, $\phi_1[t^e(\sum_{j=1}^{e-1} a_j \delta_R(t^j))] = 0$. Hence, $t^e\eta \in N(A)$. Q.E.D.

Let $\eta_1, \ldots, \eta_s \in N(A^M)$ be a set of generators for $N(A^M)$. (This generating set is finite since the rank of $l(\overline{A}/R)$ is finite.) The next theorem asserts that $t^e\eta_1, \ldots, t^e\eta_s$ is then a generating set for $N(A)$. In fact, let $\eta_1, \ldots, \eta_s$ form a basis for $N(A^M)$. That is, if $\sum_{i=1}^{s} a_i \eta_i = 0$ where $a_i \in \overline{A}$, then necessarily $a_i \eta_i = 0$ for all $i$. We may then claim that $t^e\eta_1, \ldots, t^e\eta_s$ is a basis for $N(A^M)$.

Theorem 3.2. Suppose $\eta_1, \ldots, \eta_s \in N(A^M)$ form a basis for $N(A^M)$. Then the
elements $t^e \eta_1, \ldots, t^e \eta_s$ form a basis for $N(A)$.

Proof. Lemma 3.1 assures that $t^e \eta_1, \ldots, t^e \eta_s \in N(A)$. In order to show that these elements generate $N(A)$, we refer to diagram (*).

If $\xi = \sum_{j=1}^{e-1} a_j \delta_R(t^j) \in N(A)$, then

$$\phi_1(\xi) = \sum_{j=1}^{e-1} a_j \delta_A(t^j) = 0 \quad \text{in } k[[t]] \otimes_A k[[t]].$$

Hence,

$$\xi = \sum_{i=1}^n (a_i \otimes_R b_i) (1 \otimes_R t^i - t^i \otimes_R 1)$$

where $a_i, b_i \in \overline{A}$ and $x_i \in M < A$. Since $t^e \in R = k[[t]]$,

$$\xi = t^e \left( \sum_{i=1}^n (a_i \otimes_R b_i) \left( 1 \otimes_R \frac{x_i}{t^e} - \frac{x_i}{t^e} \otimes_R 1 \right) \right).$$

Now,

$$\phi_2 \left( \sum_{i=1}^n (a_i \otimes_R b_i) \left( 1 \otimes_R \frac{x_i}{t^e} - \frac{x_i}{t^e} \otimes_R 1 \right) \right) = \theta \cdot \phi_1 \left( \sum_{i=1}^n (a_i \otimes_R b_i) \left( 1 \otimes_R \frac{x_i}{t^e} - \frac{x_i}{t^e} \otimes_R 1 \right) \right) = 0$$

since $x_i / t^e \in A^M$. Therefore,

$$\sum_{i=1}^n (a_i \otimes_R b_i) \left( 1 \otimes_R \frac{x_i}{t^e} - \frac{x_i}{t^e} \otimes_R 1 \right) = \sum_{i=1}^s \beta_i \eta_i$$

where $\beta_i \in k[[t]]$ for $i = 1, \ldots, s$. Hence,

$$\xi = t^e \left( \sum_{i=1}^s \beta_i \eta_i \right) = \sum_{i=1}^s \beta_i t^e \eta_i$$

and, consequently, $t^e \eta_1, \ldots, t^e \eta_s$ generate $N(A)$.

To show that $t^e \eta_1, \ldots, t^e \eta_s$ form a basis we need only point out that the $\overline{A}$-module homomorphism $p$ from the free $\overline{A}$ module $l(\overline{A}/R)$ to $l(\overline{A}/R)$ given by

$$p \left( \sum_{j=1}^{e-1} a_j \delta_R(t^j) \right) = \sum_{j=1}^{e-1} t^e a_j \delta_R(t^j)$$

is injective. Q.E.D.

It should be noted that in the case where $A^M$ is the full power series ring $k[[t]]$, then a set of generators for $N(A)$ is $\{t^e \delta_A(t), \ldots, t^e \delta_A(t^{e-1})\}$ where $e = \ldots$
When $A$ is itself equal to $k[[t]]$, then $l(\bar{A}/A) = l(\bar{A}/AM) = 0$ and the theorem is trivial.

**Corollary 3.3.** If $z = \sum_{j=1}^{e-1} a_j \delta_A(t^j) = 0$ in $\bar{A} \otimes_A \bar{A}$ where $e = e(A) > 1$, then $v(a_j) \geq e$ for all $j$. Consequently, $a_j / t^e = a_j \in k[[t]]$. Furthermore,

$$e-1 \sum_{j=1} \alpha_j \delta_{AM}(t^j) = 0 \quad \text{in} \quad \bar{A} \otimes_{AM} \bar{A}.$$

**Proof.** Let $\sum_{j=1}^{e-1} a_j \delta_R(t^j)$ be the preimage of $z$ under $\phi_1$. The theorem asserts that

$$e-1 \sum_{j=1} a_j \delta_R(t^j) = t^e \left( \sum_{j=1}^{e-1} a_j \delta_R(t^j) \right)$$

for some $a_j \in k[[t]]$. But $l(\bar{A}/R)$ is a free module and hence, $a_j = t^e a_j$ for all $j$. Consequently, $v(a_j) = e + v(a_j) \geq e$ since $a_j \in k[[t]]$.

Since $z = 0$ in $\bar{A} \otimes_A \bar{A}$ and because $e \in R = k[[t^e]]$ we have

$$e-1 \sum_{j=1} a_j \delta_R(t^j) = \sum_{i=1}^n (c_i \otimes_R d_i)(1 \otimes_R y_i - y_i \otimes_R 1)$$

$$= t^e \left[ \sum_{i=1}^n (c_i \otimes_R d_i) \left( 1 \otimes_R \frac{y_i}{t^e} - \frac{y_i}{t^e} \otimes_R 1 \right) \right],$$

where $c_i, d_i \in \bar{A}$ and $y_i \in A$.

But $a_j = t^e a_j$, and hence

$$(t^e) \left[ \sum_{j=1}^{e-1} a_j \delta_R(t^j) \right] = (t^e) \left[ \sum_{i=1}^n (c_i \otimes_R d_i) \left( 1 \otimes_R \frac{y_i}{t^e} - \frac{y_i}{t^e} \otimes_R 1 \right) \right].$$

But $l(\bar{A}/R)$ is a free module (Lemma 2.1) and therefore,

$$e-1 \sum_{j=1} a_j \delta_R(t^j) = \sum_{i=1}^n (c_i \otimes_R d_i) \left( 1 \otimes_R \frac{y_i}{t^e} - \frac{y_i}{t^e} \otimes_R 1 \right).$$

Under the mapping $\phi_2$, the right side of this equation goes to zero since $y_i / t^e \in A^M$ and hence,

$$e-1 \sum_{j=1} a_j \delta_{AM}(t^j) = 0 \quad \text{in} \quad \bar{A} \otimes_{AM} \bar{A}. \quad \text{Q.E.D.}$$

We refer once more to diagram (*). Since $l(\bar{A}/R)$ is a free module over the principal ideal domain $k[[t]] = \bar{A}$, $N(A) = \ker \phi_1$ is itself free and finitely generated.

Let $\eta_i \in N(A)$, $i = 1, \ldots, s$ be a basis for $N(A)$. If $\eta_i = \sum_{j=1}^{e-1} \alpha_{ij} \delta_R(t^j)$ for each $i$, let $(\alpha_{ij})$ be the matrix of the coefficients. It is known that a set of in-
variant factors of \((a_{ij})\) are found by considering the highest common factor of all \((k \times k)\) subdeterminants of \((a_{ij})\) [4, p. 92]. These invariant factors completely determine the structure of the \(k[[t]]\) module \(l(\overline{A}/A)\) and are unique up to units from \(k[[t]]\).

Likewise, let \((\beta_{ij})\) be the matrix of the coefficients of a basis for \(N(A^M) = \ker \phi_2\). We shall relate the invariant factors of \((a_{ij})\) to those of \((\beta_{ij})\).

Lemma 3.4. Suppose \(e\) is the multiplicity of \(A\) and \(A^M\) is the blow-up of \(A\). Let \(\{E_1, \ldots, E_{e-1}\}, E_i \in k[[t]]\) be a set of invariant factors of \((\beta_{ij})\). Then \(\{t^eE_1, \ldots, t^eE_{e-1}\}\) constitutes a set of invariant factors of \((a_{ij})\).

Conversely, suppose \(\{F_1, \ldots, F_{e-1}\}\) is a set of invariant factors of \((a_{ij})\) then \(F_i/t^e \in k[[t]]\) for all \(i = 1, \ldots, e-1\) and \(\{F_1/t^e, \ldots, F_{e-1}/t^e\}\) is a set of invariant factors of \((\beta_{ij})\).

Proof. Let \((\beta_{ij})\) be the matrix of the relations for \(N(A^M)\). Theorem 3.2 implies that \((t^e\beta_{ij})\) is the matrix of the relations of \(N(A)\). Let \(\sigma_1, \ldots, \sigma_e\) be the highest common factor of the \((k \times k)\) subdeterminants of \((\beta_{ij})\). We have [4, p. 92]

\[
\sigma_1 = E_1, \quad \sigma_2/\sigma_1 = E_2, \quad \ldots, \quad \sigma_{e-1}/\sigma_{e-2} = E_{e-1}.
\]

Hence,

\[
t^e\sigma_1 = t^eE_1, \quad \frac{(t^e)^2\sigma_2}{(t^e)\sigma_1} = t^eE_2, \quad \ldots, \quad \frac{(t^e)^{e-1}\sigma_{e-1}}{(t^e)^{e-2}\sigma_{e-2}} = t^eE_{e-1}
\]

is a set of invariant factors of \((a_{ij})\).

Conversely, Theorem 3.2 asserts that \((t^e\beta_{ij})\) is the matrix of relations of \(N(A)\). Hence, \(t^eE_i\) and \(F_i\) are associates for each \(i\). The conclusion follows easily. Q.E.D.

Suppose \(A = A_0 < A_1 < \ldots < A_N < \ldots < \overline{A}\) is the branch sequence of \(A\) along \(M < A\), the maximal ideal of \(\overline{A}\). Let \(e_i = e(A_i)\) for each \(i\). Assume that \(A_{N+1} = k[[t]]\) and \(e(A_N) = e_N > 1\). Hence, the multiplicity sequence of \(A\) has the form \(\{e_0, e_1, \ldots, e_N, 1, \ldots\}\).

Theorem 3.5. The decomposition of \(l(\overline{A}/A)\) as a module over the P.I.D. \(\overline{A}\) depends only on the multiplicity sequence of \(A\).

Proof. If \((a_{ij})\) is the matrix of relations for \(N(A_i)\) and \(\{F_1, \ldots, F_{e_i-1}\}\) constitutes a set of invariant factors of \((a_{ij})\) then \(l(\overline{A}/A_i) \cong \overline{A}/F_1 \oplus \ldots \oplus \overline{A}/F_{e_i-1}\) \[4, p. 86\], as \(\overline{A}\) modules. We shall write \(l(\overline{A}/A_i) \sim \{F_1, \ldots, F_{e_i-1}\}\) to mean that \(\{F_1, \ldots, F_{e_i-1}\}\) is a set of invariant factors of \((a_{ij})\). Since \(A_{N+1} = k[[t]]\), we may proceed backwards to \(A\) by using Lemma 3.4. In fact, if \(\{e_0, \ldots, e_N, 1, \ldots\}\) is the multiplicity sequence of \(A\), write \(E_i = t^{e_i}\) for \(i = 0, \ldots, N\). Then using Lemma 3.4 repeatedly:
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\[ l(\overline{A}/A_N) \sim \{E_N, \ldots, E_N\} \]
\[ e_N - 1 \]

\[ l(\overline{A}/A_{N-1}) \sim \{E_{N-1}, \ldots, E_{N-1}\} \]
\[ E_N \cdot E_{N-1}, \ldots, E_N \cdot E_{N-1} \]
\[ e_{N-1} - e_N \geq 0 \]
\[ e_N - 1 \]

\[ l(\overline{A}/A_{N-2}) \sim \{E_{N-2}, \ldots, E_{N-2}\} \]
\[ E_{N-1} \cdot E_{N-2}, \ldots, E_{N-1} \cdot E_{N-2} \]
\[ e_{N-1} - e_{N-1} \]
\[ e_N - 1 \]

\[ \frac{E_N \cdot E_{N-1} \cdot E_{N-2}, \ldots, E_N \cdot E_{N-1} \cdot E_{N-2}}{e_N - 1} \]

Note that it may happen that $e_{i+1} - e_i = 0$. Q.E.D.

Theorem 3.6. Let \{e_0, e_1, \ldots, e_N, 1, \ldots\} be the multiplicity sequence of $A$.

Then

\[ \lambda^{-1}_A(l(\overline{A}/A)) = \sum_{i=0}^{\infty} e_i(e_i - 1). \]

Proof. If $l(\overline{A}/A) \sim \{F_1, \ldots, F_r\}$, then it is clear that $\lambda^{-1}_A(l(\overline{A}/A)) = v(F_1) + \ldots + v(F_r)$ where $v$ is the valuation. Hence, by Theorem 3.5

\[ \lambda^{-1}_A(l(\overline{A}/A)) = (e_0 - e_1)v(E_0) + (e_1 - e_2)[v(E_0) + v(E_1)] \]
\[ + \ldots + (e_{N-1})[v(E_0) + \ldots + v(E_N)] \]
\[ = (e_0 - 1)v(E_0) + (e_1 - 1)v(E_1) + \ldots + (e_{N-1})v(E_N) \]
\[ = \sum_{i=0}^{N} e_i(e_i - 1) \quad \text{since} \quad v(E_i) = e_i \text{ for all } i. \]
Since $e_{N+k} = 1$ for $k \geq 1$, the formula holds when taking the infinite sum. Q.E.D.

We mention that if $A$ is the complete local ring of a plane curve with only one characteristic pair, i.e. if $A = k[[t^p, t^q + a_1 t^{q+1} + \cdots]]$ where $(p, q) = 1$, then the length formula reduces to the following: $\lambda_A(l(A/A)) = (p-1)(q-1)$.

4. Comparison of $l(A/A)$ to $l(B/B)$. The purpose of this section is to study the relationship of the $k[[t]]$ module $l(A/A)$ to that of $l(B/B)$ where $A$ and $B$ are two arbitrary rings which satisfy the previous assumption. Namely, $A$ and $B$ are the complete local rings of an algebraic curve at a "one-branch singularity." We shall assume that $\overline{A} = \overline{B} = k[[t]]$ for some uniformizing parameter $t$ (a field algebraically closed). If $C$ is any ring satisfying the above, we write $l_C$ to mean $l(C/C)$ since $C = k[[t]]$ for all of these rings. We shall, for the most part, be interested in the structure of $l_A$ as a module over $k[[t]]$ even though $l_A$ is also an algebra over $k[[t]]$. We will mention explicitly which structure is intended.

Recall that every $k$ automorphism $\sigma$ on $k[[t]]$ is of the form $\sigma: t \rightarrow ut$ where $u$ is a unit in $k[[t]]$. Conversely, every mapping of the form $\sigma$ is a $k$-automorphism on $k[[t]]$.

Definition. If $A, B \leq k[[t]]$, both complete, $A$ and $B$ are said to be analytically equivalent if there exists a $k$-automorphism $\sigma$ on $k[[t]]$ so that $\sigma(A) = B$.

Let $A = A_0 < A_1 < A_2 < \cdots < k[[t]]$ and $B = B_0 < B_1 < B_2 < \cdots < k[[t]]$ be the branch sequence of $A$ and $B$ respectively. Let $\{e(A_0) = e(A), e(A_1), \ldots\}$ and $\{e(B_0) = e(B), e(B_1), \ldots\}$ be the multiplicity sequence of $A$ and $B$ respectively.

Definition. $A$ and $B$ are said to have the same multiplicity sequence if $e(A_i) = e(B_i)$ for every $i = 0, 1, \ldots$.

Lemma 4.1. If $A$ and $B$ are analytically equivalent, then $l_A \cong l_B$ as $k[[t]]$ modules.

Proof. If $M$ and $M'$ are the maximal ideals of $A$ and $B$ respectively, then the $k$-automorphism $\sigma$ of $k[[t]]$ between $A$ and $B$ extends to one between $A^M$ and $B^{M'}$. Hence the multiplicity sequences of $A$ and $B$ are the same, since $\sigma(A) = B$ implies $e(A) = e(B)$. By Theorem 3-5, the decomposition of $l_A$ as $k[[t]]$ modules is dependent only on the multiplicity sequence of $A$. Hence, the decomposition of $l_A$ and $l_B$ are equivalent. Q.E.D.

Lemma 4.2. Let $\overline{M}$ be the maximal ideal of $A$, then

$$\dim_{A/\overline{M}}(l_A/\overline{M}l_A) = \dim_k(l_A/\overline{M}l_A) = e(A) - 1.$$

Proof. We may assume that a transversal element has been so chosen that
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Let $x = t^e$ where $e = e(A)$ and $\mathcal{A} = k[[t]]$. Hence, $M = (t)$ and we need to show

$$\dim_k(S_A/(t)S_A) = e(A) - 1.$$  

Lemma 2.1 says that $\delta_A(t), \ldots, \delta_A(t^{e-1})$ generate $S_A$ as a $k[[t]]$ module. Hence, $\delta_A(t), \ldots, \delta_A(t^{e-1})$ span the $k$ vector space $S_A/(t)S_A$. We need to show these are independent. But if $\sum_{j=1}^{e-1} \alpha_j \delta_A(t^j) = 0$ in $S_A/(t)S_A$ then $\sum_{j=1}^{e-1} \alpha_j \delta_A(t^j) \in (t)S_A$ or, equivalently, $\sum_{j=1}^{e-1} \alpha_j \delta_A(t^j) \in (t)S_A$ where $\alpha_j \in k$ is the constant term of the power series $a_j$. Hence $\sum_{j=1}^{e-1} (a_j - b_j) \delta_A(t^j) = 0$ where $b_j \in (t)$. But Corollary 2.3 implies $v(a_j - b_j) > e(A)$ for all $j$. Hence, $a_j = 0$ and $a_j = 0$ for all $j$. Q.E.D.

Lemma 4.3. If $M$ and $M^*$ are the maximal ideals of $A$ and $B$ respectively, then $S_A \otimes I_A \cong I_B$ implies $I_A = I_B$ (both isomorphisms as $k[[t]]$ isomorphisms.)

Proof. By Lemma 4.2, $S_A \otimes I_B$ implies $e(A) - 1 = e(B) - 1$ and hence $e(A) = e(B) = e$. Therefore, if $t^e$ and $(t')^e$ are transversal parameters to $M$ and $M^*$ respectively, then $t^e = u(t')^e$ where $u$ is a unit in $k[[t]]$.

Let $E_1, \ldots, E_{e-1}$ and $F_1, \ldots, F_{e-1}$ be a set of invariant factors of $S_A$ and $S_B$ respectively (cf. Lemma 3.4). We may assume $E_i$ and $F_i$ are associates and $E_i \neq 1$ for each $i$.

$\sum_{j=1}^{e-1} \alpha_j \delta_A(t^j) \in (t)S_A$ or, equivalently, $\sum_{j=1}^{e-1} \alpha_j \delta_A(t^j) \in (t)S_A$ where $\alpha_j \in k$ is the constant term of the power series $a_j$. Hence $\sum_{j=1}^{e-1} (a_j - b_j) \delta_A(t^j) = 0$ where $b_j \in (t)$. But Corollary 2.3 implies $v(a_j - b_j) > e(A)$ for all $j$. Hence, $a_j = 0$ and $a_j = 0$ for all $j$. Q.E.D.

The converse of this is clearly false. Let $A = k[[t^2, t^3]]$ and $B = k[[t^3, t^4]]$ with maximal ideal $M = (t^2, t^3)$ and $M^* = (t^3, t^4)$ respectively. Then $A^M = B^{M^*} = k[[t]]$ and hence, $I_A = I_B = 0$. But Theorem 3.5 gives

$$I_A \cong \mathcal{A}/(t^2) \text{ and } I_B \cong \mathcal{A}/t^3 \oplus \mathcal{A}/t^3$$

as $k[[t]]$ modules.

Theorem 4.4. $I_A \cong I_B$ as $k[[t]]$ modules if and only if $A$ and $B$ have the same multiplicity sequence.

Proof. Let $A = A_0 < A_1 < \ldots < A_N = k[[t]]$ and $B = B_0 < B_1 < \ldots < B_N = k[[t]]$ be the branch sequence of $A$ and $B$ respectively. Lemma 4.2 implies that if $I_A \cong I_B$, then $e(A) = e(B)$. Lemma 4.3 asserts that $I_A \cong I_B$ and hence $e(A_j) = e(B_j)$. Continuing, the result follows.

Theorem 3.5 asserts the converse. Q.E.D.

Before continuing, we need to indicate some of the geometric properties of the strict closure $A'$ of $A$ in $\mathcal{A}$. Let $C$ be a ring so that $A < C < \mathcal{A} = k[[t]]$. (Our assumptions on $A$ imply that $C$ is necessarily local and complete.) Let $C = C_0
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$C_1 < \cdots < C_N < \overline{A}$ be the branch sequence of $C$ and let $e(C_i) = e_i$. The ring $C$ is said to be an Arf ring (cf. [1]) if it satisfies any one of the following conditions.

1. The embedding dimension of $C_i$ is equal to the multiplicity of $C_i$ for every $i$.
2. $\lambda_C(C/C) = \sum_{i=0}^{\infty} (e_i - 1)$. (Since $e_n = 1$ for $n$ large, the formula makes sense.)
3. The semigroup $G(C) = \{\sigma(x) : x \in C\}$ has the form $G(C) = \{0, e_0, e_0 + e_1, e_0 + e_1 + e_2, \ldots\}$.

J. Lipman in [5] shows the equivalence of the above conditions. In the same paper, he shows that if $A$ is any ring among the collection of all Arf rings between $A$ and $\overline{A}$, there exists one, say $A^*$, contained in all the others [5, p. 666]. The ring $A^*$ is called the Arf closure of $A$ and coincides with the strict closure $A'$ since we assume that $A$ contains a field $k$ [5, p. 677]. Hence, we shall continue to denote the Arf closure $A^*$ of $A$ as $A'$.

Remark 4.5. Note that if $A < C < \overline{A}$, then $A' < C'$. Using (2), the ring $A'$ (= strict closure of $A$ = Arf closure of $A$) can be characterized as the largest ring between $A$ and $\overline{A}$ whose multiplicity sequence is equal to $A$ [5, p. 671]. This implies that if $A < C < \overline{A}$, then $A' = C'$ if and only if the multiplicity sequence of $A$ is equal to the multiplicity sequence of $C$. Similarly, one shows by using (3) that if $A < C < \overline{A}$, then $A' = C'$ if and only if $G(C') = G(A')$.

Definition. Let $d \in G(A)$ be the least integer in $G(A)$ so that $d + j \in G(A)$ for any integer $j \geq 0$. Then $d$ is called the degree of the conductor of $A$.

Theorem 4.6. The annihilator ideal of $I_A$ in $\overline{A} = k[[t]]$ is $(t^d)$ where $d$ is the degree of the conductor of $A'$.

Proof. By induction on the number of blow-ups needed to "resolve the singularity". Note that if $A = k[[t]]$, then $A' = k[[t]]$ and since $d = 0$ and $I_A = 0$, the theorem holds true in this case.

Next note that for an ideal $Q \leq k[[t]]$, $Q|_A = 0$ if and only if $Q\delta_A(t) = 0$. For if $x = \sum_{i=0}^{\infty} a_i t^i \in k[[t]]$, $\delta_A(x) = \sum_{i=1}^{\infty} a_i \delta(t^i)$. But

$$\delta_A(t^n) = \binom{n}{1} t^{n-1} \delta_A(t) + \binom{n}{2} t^{n-2} \delta_A^2(t) + \cdots + \delta^n(t)$$

and hence, $Q|_A = 0$ if $Q\delta_A(t) = 0$. The converse is clear. Therefore, the theorem asserts that the order ideal of $\delta_A(t)$ is $(t^d)$ where $d$ is the degree of the conductor of $A'$.

Let $A$ have the multiplicity sequence $\{e(A), e(A_1), e(A_2), \ldots, e(A_N), 1, \ldots\}$ where $N$ is the largest integer so that $e(A_N) > 1$. By Remark 4.5, $G(A') = \{0, e_0, e_0 + e_1, \ldots\}$ where $e_1 = e(A_1)$, and hence, the degree of the conductor of $A'$ is $d = e_0 + e_1 + \cdots + e_N$. Note that $A_1$ has the multiplicity sequence $\{e_1,$
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Let $A$ and $B$ be complete local rings of points on an algebraic curve at one-branch singularities and assume that $A < B < \overline{A}$. Then the following are equivalent.

1. $I_A \cong I_B$ as $k[[t]] = \overline{A} = B$ modules.
2. The multiplicity sequence of $A$ is equal to the multiplicity sequence of $B$.
3. $A' = B'$.
4. $A \cong B$ as $\overline{A}$-algebras.
5. $D^n(\overline{A}/A) \cong D^n(\overline{B}/B)$ as $\overline{A}$-algebras for all $n$.
6. $G(A') = G(B')$.

Proof. (1) implies (2) by Theorem 4.4, (2) implies (3) by Remark 4.5. To show that (3) implies (4), we consider the canonical isomorphism $\psi$ from $\overline{A} \otimes_A \overline{A}$ to $\overline{A} \otimes_B \overline{A}$. (By assumption, $\overline{A} \otimes_A \overline{A} = \overline{A} \otimes_B \overline{A}$.) From the diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & I_B \\
& \searrow & \downarrow \psi/I_A \\
& & \psi \\
0 & \rightarrow & I_A \\
\end{array}
\quad
\begin{array}{ccc}
\overline{A} & \otimes_B & \overline{A} \\
& \rightarrow & \overline{A} \\
\overline{A} & \otimes_A & \overline{A} \\
& \rightarrow & \overline{A} \\
& \rightarrow & 0 \\
\end{array}
$$

it is clear that the restriction of $\psi$ gives the desired algebra isomorphism.

Clearly, (4) implies (1).

Since by Theorem 1.1, $D^n(\overline{A}/A) = I(\overline{A}/A)$ and $D^n(\overline{B}/B) = I(\overline{B}/B)$ for $n > 0$, (4) is equivalent to (5).

(6) is equivalent to (3) by Remark 4.5. Q.E.D.

REFERENCES


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