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CENTRAL IDEMPOTENT MEASURES ON COMPACT GROUPS

BY

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ABSTRACT. Let G be a compact group with dual object $\Gamma = \Gamma(G)$ and let $M(G)$ be the convolution algebra of regular finite Borel measures on G. The author has characterized the central idempotent measures on certain G, including the unitary groups, in terms of the hypercoset structure of Γ. The characterization also says that, on certain G, a central idempotent measure is a sum of such measures each of which is absolutely continuous with respect to the Haar measure of a closed normal subgroup. The main result of this paper is an extension of this characterization to products of certain groups. The known structure of connected groups and a recent result of Ragozin on connected simple Lie groups will then show that the characterization is valid for connected groups. On the other hand, a simple example will show it is false in general for non-connected groups. This characterization was done by Cohen for abelian groups and the proof borrows extensively from Amemiya and Itô's simplified proof of Cohen's result.

1. Canonical measures. Throughout the paper G will be a compact group. The dual object Γ of G is the set of equivalence classes of irreducible unitary representations of G. For $\alpha \in \Gamma$, χ_α will denote the character of the class and $d(\alpha)$ its degree. For ease of notation we define $\Psi_\alpha = \chi_\alpha / d(\alpha)$. A measure $\mu \in M^Z(G)$, the center of $M(G)$, has a Fourier-Stieltjes transform

$$\hat{\mu}(\alpha) = \int \overline{\Psi}_\alpha d\mu \qquad (\alpha \in \Gamma).$$

μ is idempotent, that is $\mu * \mu = \mu$, provided $\hat{\mu}(\alpha)$ is always 0 or 1. $J(G)$ will denote the class of central idempotent measures on G.

If H is a closed subgroup of G let $\mathfrak{M}_H$ denote the normalized Haar measure of H. $\mathfrak{M}_H$ is idempotent; $\mathfrak{M}_H \in J(G)$ provided H is normal.

It is convenient to consider a larger class

$$F(G) = \{\mu \in M^Z(G) : \hat{\mu}(\alpha) \text{ is an integer}\}.$$

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$F(G)$ consists then of those central measures μ that satisfy $P(\mu) = 0$ for some polynomial P with integral roots (for a noncompact G, $F(G)$ will be defined in this way). For $\mu \in F(G)$ we let $E(\mu) = \{\alpha: \hat{\mu}(\alpha) \neq 0\}$.

The hypercoset structure of Γ is described in [5]. If H is a closed normal subgroup of G then $E(\mathfrak{M}_H) = \{\alpha: \Psi_\alpha|_H \equiv 1\} = H^\perp$ is a normal subhypergroup of Γ. If $\beta \in \Gamma$ the hypercoset $\beta H^\perp$ consists of the $\alpha \in \Gamma$ such that χ_α appears in the decomposition of $\chi_\beta \chi_\gamma$ for some $\gamma \in H^\perp$. Also $\beta H^\perp = \{\alpha: \Psi_\alpha|_H = \Psi_\beta|_H\}$. The hypercoset ring of Γ is the smallest ring of sets containing all hypercosets.

There are two ways to attempt to characterize the measures in $F(G)$.. First, for $\mu \in F(G)$ and n an integer let $E_n(\mu) = \{\alpha: \hat{\mu}(\alpha) = n\}$. If $\mu \in J(G)$ then $E_1(\mu) = E(\mu)$. It is shown in [5] that every set in the hypercoset ring of Γ is $E(\mu)$ for some $\mu \in J(G)$ and that for certain groups the converse is also true. This implies, for such groups, that for $\mu \in F(G)$ each $E_n(\mu)$ is in the hypercoset ring.

Second, some measures in $F(G)$ arise naturally from well-known measures on G.

Definition 1.1. *A measure μ is canonical if*

$$\mu = \sum_\alpha n_\alpha c_\alpha d(\alpha) \chi_\alpha \mathfrak{M}_{H_\alpha}$$

where the sum is finite, n_α is an integer, H_α is a closed normal subgroup and

$$\frac{1}{c_\alpha} = \int |\chi_\alpha|^2 \, d\mathfrak{M}_{H_\alpha}.$$

The following lemma, which connects the two concepts above, is an immediate consequence of Theorem 1 of [1] and the fact that the intersection of two hypercosets is a finite union of hypercosets.

Lemma 1.2. (a) *Every canonical measure is in $F(G)$.*

(b) *A measure $\mu \in F(G)$ is canonical if and only if each $E_n(\mu)$ belongs to the hypercoset ring of Γ.*

We will use the usual notation $\mu << \nu$ to indicate that μ is absolutely continuous with respect to ν. It is easy to see

Lemma 1.3. *$\mu \in F(G)$ is canonical if and only if there are finitely many closed normal subgroups H_i and $\mu = \Sigma \mu_i$ with $\mu_i << \mathfrak{M}_{H_i}$.*

2. The main result. Let Γ_1 denote those $\alpha \in \Gamma$ with $d(\alpha) = 1$. Γ_1 consists of the complex homomorphisms of G and is the dual group of the abelian group G/G' where G' is the commutator subgroup of G. For $\alpha \in \Gamma_1$ we can identify α and χ_α. Now if $\alpha, \beta \in \Gamma$ it may happen that the tensor product $\alpha \otimes \beta$ is irreducible. If it is we let $\alpha\beta$ denote $\alpha \otimes \beta$ so that $\chi_\alpha \chi_\beta = \chi_{\alpha\beta}$. If $\alpha \in \Gamma_1$

this is always the case. We also let $Z = Z(G)$ denote the center of G.

Definition 2.1. *G is said to satisfy condition* I *provided*

$$\lim_{d(\alpha)\to\infty} \Psi_\alpha(x) = 0$$

for all $x \notin Z$.

G is said to satisfy condition II *provided that for each positive integer t there are finitely many irreducible representations $\beta_1, \cdots, \beta_s$ of degree t such that if $d(\beta) = t$ then $\beta = \alpha\beta_i$ for some i and some $\alpha \in \Gamma_1$.*

It should be noted that groups having open centers, in particular abelian groups and finite groups, satisfy both conditions. In [5] it is shown that unitary groups do also. In §9 it will be shown, using a result of Ragozin [4], that every compact connected simple Lie group satisfies both conditions.

We can now state the main result of the paper.

Theorem 2.2. *Let G_i $(i \in A)$ be compact groups satisfying conditions* I *and* II *and let $G = \Pi_A G_i$. Then every measure in $F(G)$ is canonical.*

Together with Lemma 1.2 this then gives

Corollary 2.3. *If G is as above then $E \subset \Gamma$ is $E(\mu)$ for some $\mu \in J(G)$ if and only if E belongs to the hypercoset ring of Γ.*

It is well known (cf. [3, Theorem 2.1.4]) that an idempotent measure of norm 1 on a locally compact group is of the form $\gamma\mathfrak{M}_H$ for some compact subgroup H and some $\gamma \in \Gamma_1(H)$. It follows that, for any compact group G, the elements of $F(G)$ of norm 1 are canonical.

Definition 2.4. *A measure $\mu \in F(G)$ is irreducible if it cannot be written as the sum of two mutually singular nonzero measures in $F(G)$.*

Definition 2.5. *The support group $L(\mu)$ of a measure $\mu \in M(G)$ is the smallest closed subgroup that carries μ.*

Clearly if $\mu \in M^Z(G)$ then $L(\mu)$ is normal. A rough idea of the proof of Theorem 2.2 is to show that if $\mu \in F(G)$ is irreducible then $\mu << \mathfrak{M}_{L(\mu)}$.

The proof of Theorem 2.2 is in §8. §3 deals with projections of $M(G)$ onto the measures carried by the cosets of a normal Borel subgroup. §4 contains results concerning $F(G)$ for an arbitrary compact group G. In §5 it is shown that if μ is canonical and $\|\mu\| > 1$ then $\|\mu\| > 1 + 1/700$. This generalizes a well-known result on abelian groups and is perhaps of independent interest. §§6 and 7 contain results about $F(G)$ for G as in the hypotheses of Theorem 2.2. They are an attempt to use the methods of Amemiya and Itô [1] for abelian groups in this more general setting. In §9 a result of Ragozin [4] is used to show that the conclusions of Theorem 2.2 and Corollary 2.3 are valid for connected groups. An example of where Theorem 2.2 fails is given in §10.

3. **Projections.** Let H be a normal Borel subgroup of G. For $\mu \in M(G)$ define

$$\Pi_H \mu(E) = \sum_x \mu(E \cap Hx),$$

the sum being over distinct coset representatives of H.

Lemma 3.1. (a) Π_H *is a homomorphism of the algebra* $M(G)$ *into itself.*
(b) Π_H *maps* $F(G)$ *into itself.*

Proof. (a) is proved in [7, Theorem 3.4.1] for H a closed subgroup and G abelian. The proof works equally well for this more general situation. Since H is normal it is easily seen that Π_H maps $M^Z(G)$ into itself and (b) then follows since Π_H is a homomorphism.

The following theorem gives the first indication that an irreducible measure in $F(G)$ is absolutely continuous with respect to the Haar measure of its support group.

Theorem 3.2. *Let* $\mu \in F(G)$ *be irreducible and have support group* L. *If* H *is a closed normal subgroup of* G *and* $\Pi_H\mu \neq 0$ *then* $H \cap L$ *is open in* L.

Proof. Write $\mu = \Pi_H\mu + (\mu - \Pi_H\mu)$. By Lemma 3.1 these last two measures are in $F(G)$. Since they are singular and μ is irreducible we must have $\mu = \Pi_H\mu$. Also $\mu = \Pi_L\mu$ and it is then easily seen that $\mu = \Pi_{H \cap L}\mu$.

Retopologize G so that the closed subgroup $H \cap L$ is open; let G with this new topology be denoted by G_0. Since μ is supported on countably many cosets of $H \cap L$ we have then that $\mu \in F(G_0)$. Now G_0 is a locally compact group and, since $H \cap L$ is a compact open normal subgroup, G_0 has small invariant neighborhoods. It follows from [6, Theorem 1] that μ is supported on a compact subgroup of G_0. Thus, as an element of $F(G_0)$, μ is supported on a finite extension P of $H \cap L$. P is then a closed subgroup of G that carries μ. Hence $L \subset P$ and so L is a finite extension of $H \cap L$; that is $H \cap L$ is open in L.

4. **$F(G)$ for arbitrary G.** This section contains some lemmas concerning $F(G)$ for an arbitrary compact group G.

Lemma 4.1. *If* $\mu \in F(G)$ *has support group* L *and* $T = \{\Psi_\alpha|_L : \alpha \in E(\mu)\}$ *is finite then* μ *is canonical.*

Proof. Let $\Psi_{\alpha_1}|_L, \cdots, \Psi_{\alpha_t}|_L$ be the distinct elements of T. Then $E(\mu) = \bigcup_1^t \alpha_i L^\perp$. These hypercosets are disjoint and, since L carries μ, $\hat{\mu}$ is constant on each $\alpha_i L^\perp$. Thus $\mu = \Sigma_1^t a_i \chi_{\alpha_i} \mathfrak{M}_L$ for some constants a_i so that μ is canonical by Lemma 1.3.

Lemma 4.2. *Let H be a closed normal subgroup of G. If $\mu \in F(G)$ and $|\mu|(H^c) < \frac{1}{2}$ then $|\mu|(H^c) = 0$.*

Proof. Let α belong to the hypercoset $\beta H^{\perp}$. This implies that $\Psi_\alpha|_H = \Psi_\beta|_H$ so that

$$|\hat{\mu}(\alpha) - \hat{\mu}(\beta)| \leq \int |\overline{\Psi}_\alpha - \overline{\Psi}_\beta|\, d|\mu| \leq 2|\mu|(H^c) < 1.$$

Since $\hat{\mu}$ is integer valued this gives that $\hat{\mu}$ is constant on $\beta H^{\perp}$. It follows easily that μ is carried by H; that is $|\mu|(H^c) = 0$.

If $\alpha \in \Gamma$ and $\mu \in F(G)$ then it does not follow that $\Psi_\alpha \mu \in F(G)$. The problem of course is that the product of two irreducible characters decomposes, in general, into a sum of several irreducible characters. However a sequence in Γ may have the following property.

Definition 4.3. *A sequence $\{\alpha\} \subset \Gamma$ is an irreducible sequence if for each $\beta \in \Gamma$ there is $\alpha(\beta)$ such that $\alpha \otimes \beta$ is irreducible whenever $\alpha > \alpha(\beta)$.*

For example if $G = \Pi_1^\infty G_i$ and $\alpha_i \in \Gamma(G)$ is given by some $\alpha_i \in \Gamma(G_i)$ then $\{\alpha_i\}$ is an irreducible sequence. This example will be useful in the proof of Theorem 2.2 because if $\{\alpha\}$ is an irreducible sequence and $\mu \in F(G)$ then every weak limit point of $\{\Psi_\alpha \mu\}$ also belongs to $F(G)$.

The following lemma is a generalization of Helson's translation lemma [7, Lemma 3.5.1].

Lemma 4.4. *Let L be a closed normal subgroup of G. Suppose $\{\alpha\}$ is a sequence in Γ with $\Psi_\alpha|_L$ being distinct. Let $\mu \in M^Z(G)$ be carried by L. If λ is a weak limit point of $\{\Psi_\alpha \mu\}$ then λ and $\mathfrak{M}_L$ are mutually singular.*

Proof. $\chi_\alpha|_L$ decomposes into a sum of irreducible characters on L. Since the $\Psi_\alpha|_L$ are distinct any character of $\Gamma(L)$ which appears in $\chi_\alpha|_L$ does not appear in $\chi_\beta|_L$ (for $\alpha \neq \beta$). The remainder of the proof follows that of [7, Lemma 3.5.1] exactly.

Lemma 4.5. *Let $\mu \in F(G)$ be irreducible. If $\{\alpha\}$ is an irreducible sequence and $\Psi_\alpha \mu$ converges weakly to a nonzero canonical measure λ then μ is also canonical.*

Proof. Let L be the support group of μ. λ is then also carried by L. Now if H is a closed normal subgroup with $\Pi_H \lambda \neq 0$ then $\Pi_H \mu \neq 0$. By Theorem 3.2 this implies that $H \cap L$ is open in L. Thus the Haar measures that appear in the canonical measure λ are all absolutely continuous with respect to $\mathfrak{M}_L$. By Lemma 4.4 we then have that $\Psi_\alpha|_L$ are the same for $\alpha \geq \alpha_0$ so that

$$\Psi_{\alpha_0} \mu = \lambda, \tag{1}$$

and

$$|\Psi_{\alpha_0}|_L^2 = \overline{\Psi}_{\alpha_0}\Psi_\alpha|_L \quad \text{for } \alpha \geq \alpha_0. \tag{2}$$

We will now show that

$$|\Psi_{\alpha_0}|_L \equiv 1. \tag{3}$$

This gives, by (1), that $\mu << \lambda$ and so $\mu << \mathfrak{M}_L$ which implies, by Lemma 1.3, that μ is canonical.

Since $\{\alpha\}$ is an irreducible sequence there is $\alpha \geq \alpha_0$ so that $\beta = \overline{\alpha}_0 \alpha$ is irreducible. Because of (2), $\int_L \Psi_\beta d\mathfrak{M}_L \neq 0$ so that $\beta \in L^\perp$; that is $\Psi_\beta|_L \equiv 1$. (3) then holds because of (2).

The example of §10 will show that we can have $\mu \in F(G)$ being carried by a closed normal subgroup H with μ canonical when considered as an element of $F(H)$ but not canonical as an element of $F(G)$. However we do have the following.

Lemma 4.6. *Let $\mu \in F(G_1 \times G_2)$ have support group L. Suppose that $L \subset G_1 \times K$ where K is a finite normal extension of $Z = Z(G_2)$. Then if μ is canonical with respect to $G_1 \times K$ it is also canonical with respect to $G_1 \times G_2$.*

Proof. Without loss of generality we can assume that μ is irreducible as an element of $F(G_1 \times G_2)$. Now μ is the sum of finitely many nonzero measures μ_i each of which is absolutely continuous with respect to $\mathfrak{M}_{H_i}$, where H_i is a closed normal subgroup of $G_1 \times K$. Now each $H_i \subset L$ so that $K_i = H_i \cap (G_1 \times Z)$ is open in H_i. Thus $\Pi_{K_i}\mu_i = \mu_i$. But also K_i is normal in $G_1 \times G_2$ and $\Pi_{K_i}\mu \neq 0$ so that, by Theorem 3.2, $K_i = K_i \cap L$ is open in L. Thus H_i is also open in L_i so that $\mu_i << \mathfrak{M}_{H_i} << \mathfrak{M}_L$. This implies $\mu << \mathfrak{M}_L$ which, by Lemma 1.3, makes μ canonical on $G_1 \times G_2$.

The previous lemma can be generalized to infinite products.

Lemma 4.7. *Let $G = \Pi_1^\infty G_i$ and let $\mu \in F(G)$ have support group L. Suppose that $L \subset G_1 \times \Pi_2^\infty K_i$ where each K_i is a finite normal extension of $Z_i = Z(G_i)$. Then if μ is canonical with respect to $G_1 \times \Pi_2^\infty K_i$ it is also canonical with respect to G.*

Proof. Without loss of generality we can assume μ is irreducible as an element of $F(G)$. By Lemma 4.6 we have that, for each n, μ is canonical with respect to $A_n = \Pi_1^n G_i \times \Pi_{n+1}^\infty K_i$. Thus for each n we can write

$$\mu = \sum_{j=1}^{a(n)} \mu_{j,n} \tag{4}$$

where each $\mu_{j,n}$ is absolutely continuous with respect to $\mathfrak{M}_{j,n}$, the Haar measure of $H_{j,n}$, a closed normal subgroup of A_n. Also $H_{j,n} \subset L$. We can also assume, for each fixed n, that the $\mathfrak{M}_{j,n}$ and hence the $\mu_{j,n}$ are mutually singular. Now $\|\mu_{j,n}\| \geq 1$ so that $a(n) \leq \|\mu\|$. Thus we can assume that $a(n) = a$ for $n \geq n_0$. For each n let the $\mathfrak{M}_{j,n}$ be ordered so that

(5) $$\Pi_{H_{j,n}} \mathfrak{M}_{i,n} = 0 \quad \text{for } j < i.$$

Then

(6) $$\Pi_{H_{1,n}} \mu = \mu_{1,n} = \sum_j \Pi_{H_{1,n}} \mu_{j,n+1}.$$

Now $\Pi_{H_{1,n}} \mu_{j,n+1} = \mu_{j,n+1}$ or 0. Pick the smallest j so that it is not 0 and apply $\Pi_{H_{j,n+1}}$ to (6). Using the fact that $H_{j,n+1}$ is normal in A_n as well as A_{n+1} (5) gives $\mu_{1,n} = \mu_{j,n+1}$. By repeating this process, and also reordering the $\mathfrak{M}_{j,n}$, we obtain that

(7) $$\mu_{j,n} = \mu_{j,n+1} \qquad (1 \leq j \leq a;\ n \geq n_0).$$

Now let $\mu_j = \mu_{j,n}$. Then, for $n \geq n_0$, $\mu_j \in F(A_n)$. Since $\bigcup_{n_0}^{\infty} A_n$ is dense in G we must then have that $\mu_j \in M^Z(G)$ and so $\mu_j \in F(G)$. But the μ_j are mutually singular and μ is irreducible so that $a = 1$. Hence μ itself can be written, for $n \geq n_0$, as $\mu = \mu_{1,n}$. Thus $\mu << \mathfrak{M}_{1,n}$ and so μ is carried by $H_{1,n} \subset L$. But since L is the support group of μ and $H_{1,n}$ is closed we must have $H_{1,n} = L$. Thus μ, being absolutely continuous with respect to the Haar measure of L, a closed normal subgroup of G, is canonical with respect to G

5. **Norms of canonical measures.** It is known [7, Theorem 3.7.2] that if μ is an idempotent measure on an abelian group and $\|\mu\| > 1$ then $\|\mu\| > \sqrt{5}/2$. This section contains a generalization of this to canonical measures on compact groups.

Lemma 5.1. *Let $P = \Sigma a_\alpha \chi_\alpha$ be a polynomial on G. Suppose $a_\alpha \geq 0$ and $\|P\|_\infty = P(e) = 1$. If*

(1) $$|P(g_i) - z_i| \leq \delta_i \quad (1 \leq i \leq p)$$

where $|z_i| = 1$ then

(2) $$\left| P(g_1 \cdots g_p) - \prod_1^p z_i \right| \leq \left(\sum \delta_i^{1/2}\right)^2.$$

Proof. The lemma needs only to be proved for $p = 2$ as the general case then easily follows by induction. If T_α is a representation affording χ_α we can assume $T_\alpha(g_1)$ is a diagonal unitary matrix with diagonal entries $b_{\alpha,i}$ $(1 \le i \le d_\alpha)$ and $T_\alpha(g_2)$ is a unitary matrix with diagonal entries $c_{\alpha,i}$. Then

$$\chi_\alpha(g_1) = \sum_i b_{\alpha,i}, \qquad \chi_\alpha(g_2) = \sum_i c_{\alpha,i} \quad \text{and}$$

$$\chi_\alpha(g_1 g_2) = \sum_i b_{\alpha,i} c_{\alpha,i}.$$

Since $\Sigma a_\alpha d(\alpha) = 1$, (2) follows directly from (1).

Lemma 5.2. *Let $Q = \Sigma_E d(\alpha)\chi_\alpha$ be a central idempotent polynomial on G. If $\|Q\|_1 > 1$ then*

(3) $$\|Q\|_1 > 1 + 1/300.$$

Proof. Suppose

(4) $$\|Q\|_1 \le 1 + 1/300;$$

we will show $\|Q\|_1 = 1$. It can be assumed that G is the support group of Q. Write $|Q|^2 = \Sigma a_\alpha \chi_\alpha$ and $|Q|^4 = \Sigma b_\alpha \chi_\alpha$. The a_α and b_α are nonnegative integers and

(5) $$a_\alpha = \int |Q|^2 \overline{\chi}_\alpha \le M d(\alpha), \qquad b_\alpha = \int |Q|^4 \overline{\chi}_\alpha \le M^3 d(\alpha)$$

where

(6) $$M = \sum_E d^2(\alpha) = \|Q\|_\infty = Q(e) = \|Q\|_2^2.$$

Also, by Hölder's inequality,

(7) $$\sum a_\alpha b_\alpha = \int |Q|^6 \ge M^5 \|Q\|_1^{-4}.$$

Define $A_1(g) = M^{-4} \Sigma b_\alpha (1 - a_\alpha (M d(\alpha))^{-1}) \chi_\alpha(g)$ and

$$A_2(g) = M^{-2} \sum a_\alpha (1 - b_\alpha (M^3 d(\alpha))^{-1}) \chi_\alpha(g).$$

It follows from (4)–(7) that

(8) $$\|A_i\|_\infty = A_i(e) \le 1 - \|Q\|_1^{-4} < 1/60 \qquad (i = 1, 2).$$

Thus

(9) $$0 \le |Q|^2/M^2 - |Q|^4/M^4 = A_2 - A_1 < 1/30.$$

Hence, for all g, either

$$(10) \qquad |Q(g)| < M/5 \quad \text{or}$$

$$(11) \qquad |Q(g)| > 4M/5.$$

If g_1 and g_2 satisfy (11) then applying Lemma 5.1 to $P = Q/M$ shows $|Q(g_1g_2)| \geq M/5$ so that g_1g_2 must also satisfy (11). Hence the g for which (11) holds form a closed normal subgroup H. Since (10) holds on H^c we have from (4) and (6) that

$$(12) \qquad \frac{4}{5}\int_{H^c}|Q| \leq \int_{H^c}|Q|\left(1-\frac{|Q|}{M}\right) \leq \int |Q|\left(1-\frac{|Q|}{M}\right) \leq \frac{1}{300}.$$

It follows from (12) and Lemma 4.2 that the idempotent measure $Q\mathfrak{M}_G$ is carried on H; that is Q vanishes on H^c. It was assumed however that G is the support group of Q. Hence (11) holds for all g. But then $4M/5 \leq \|Q\|_1 \leq 1 + 1/300$ so that $M = 1$. This then implies that Q is just a character of degree 1 and the proof is complete.

Theorem 5.3. *If μ is a canonical measure and $\|\mu\| > 1$ then $\|\mu\| > 1 + 1/700$.*

Proof. If μ is reducible then $\|\mu\| \geq 2$. An irreducible canonical measure is absolutely continuous with respect to $\mathfrak{M}_H$ for some closed normal subgroup H. Also $\|\mu\| \geq |\hat{\mu}(\alpha)|$. Thus we can assume that μ is given by a polynomial $Q = \Sigma(\pm 1)d(\alpha)\chi_\alpha$. Suppose $\|Q\|_1 \leq 1 + 1/700$. If either $\pm Q$ is idempotent then $\|Q\|_1 = 1$ by Lemma 5.2. Otherwise $Q * Q$ and $(Q * Q \pm Q)/2$ are nonzero idempotents with norms less than $1 + 1/300$ and so, by Lemma 5.2, they all have norm 1. It is easily seen from this that $Q = \gamma_1 - \gamma_2$ where γ_1 and γ_2 are distinct characters of degree 1 with $\gamma_1^2 = \gamma_2^2$. It follows that $|Q| = 0$ on a subgroup of index 2 and $|Q| = 2$ on the complement of this subgroup. Thus $\|\mu\| = \|Q\|_1 = 1$.

The estimates in 5.2 and 5.3 can easily be improved. It would be interesting to know the best ones. A special case of 5.2 is that if $\|d(\alpha)\chi_\alpha\|_1 > 1$ then $\|d(\alpha)\chi_\alpha\|_1 > 1 + 1/300$, whenever χ_α is an irreducible character. It should be noted that a group can be constructed having a sequence $\{\alpha\} \subset \Gamma$ with $d(\alpha) \to \infty$ and $\|d(\alpha)\chi_\alpha\|_1 = 1$ and having another sequence $\{\beta\}$ with $d(\beta) \to \infty$ and $1 < \|d(\beta)\chi_\beta\|_1 \leq 2$.

6. $F(G)$ **for special** G. This section contains some rather technical results which are necessary for the proof of Theorem 2.2. We will first prove a special case of that theorem.

Theorem 6.1. *Let* G_i $(1 \leq i \leq n)$ *be compact groups satisfying conditions* I *and* II *and let* $G = \Pi G_i$. *Then every measure in* $F(G)$ *is canonical.*

Proof. The proof will be by induction on $\|\mu\|$. If $\|\mu\| = 1$ then it is known [3, Theorem 2.1.4] that μ is canonical. Suppose that for such G every measure in $F(G)$ with norm less than A is canonical and let $\|\mu\| < A + 1$. We can assume μ is irreducible.

Say that μ is of *bounded representation type* (b.r.t.) if there is $M < \infty$ such that $\hat{\mu}(\alpha) = 0$ whenever $d(\alpha) > M$. If μ is not of b.r.t. then, for some $j \neq 0$, $E_j(\mu)$ contains a sequence $\{\alpha\}$ with $d(\alpha) \to \infty$. Now α can be written as $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ where $\alpha_i \in \Gamma(G_i)$ and $d(\alpha) = d(\alpha_1) \cdots d(\alpha_n)$. For some i, say $i = 1$, $d(\alpha_1) \to \infty$. Since G_1 satisfies condition I, $\Psi_{\alpha_1} \to 0$ off $Z_1 = Z(G_1)$. Thus since

$$j = \hat{\mu}(\alpha) = \int \overline{\Psi}_\alpha d\mu \tag{1}$$

we having, using the Lebesgue dominated convergence theorem, that $|\mu|(Z_1 \times \Pi_2^n G_i) \neq 0$. Since μ is irreducible it follows from Theorem 3.2 that $L \cap (Z_1 \times \Pi_2^n G_i)$ is open in L (L is the support group of μ). Thus $L \subset K_1 \times \Pi_2^n G_i = G_i^*$ where K_1 is a finite normal extension of Z_1.

There are now three possibilities:

(a) μ is irreducible but not of b.r.t. on G^*.

(b) μ is irreducible and of b.r.t. on G^*.

(c) μ is reducible on G^*.

For $\beta \in \Gamma(K_1)$, $d(\beta) \leq [K_1 : Z_1]^{1/2} < \infty$. Thus in case (a) we can repeat the above process and obtain that $L \subset K_1 \times K_2 \times \Pi_3^n G_i$ where K_2 is a finite normal extension of $Z(G_2)$. With respect to this new group, one of the above cases holds.

After a finite number of such steps we obtain that, for some rearrangement of the G_i and some m, $L \subset H_1 \times H_2 = G^*$ where H_1 is a finite normal extension of $Z(\Pi_1^m G_i)$ and $H_2 = \Pi_{m+1}^n G_i$ and either (b) or (c) holds for this G^*. Since H_1 has an open center it satisfies both conditions I and II. G^* also satisfies condition II because it is the product of groups that do.

If case (b) holds then by Theorem 6 of [5] and Lemma 1.1 we have that μ is canonical on G^*. (Theorem 6 of [5] is stated for J but it applies equally well to F.) If case (c) holds then μ is a sum of measures in $F(G^*)$ each of norm less then A and so by the induction hypothesis μ is canonical on G^*. Lemma 4.6 then gives that μ is canonical on G.

Lemma 6.2. *Let* G_i $(1 \leq i < \infty)$ *be compact groups satisfying conditions* I *and* II *and let* $G = \Pi G_i$. *Suppose* $\mu \in J(G)$ *satisfies*

$$\alpha \in E(\mu) \text{ if and only if } \|(\Psi_\alpha - 1)\mu\| < 1/300. \tag{2}$$

Then μ is canonical; in fact $\mu = \mathfrak{M}_H$ for some closed normal subgroup H.

Remark. For abelian groups this is obvious. The problem in the nonabelian case is that $\Psi_\alpha \mu$ need not belong to $F(G)$. It seems likely that the restriction on G can be lifted but I have been unable to do so. It does hold, however, if G is totally disconnected.

Proof. Let $\mathfrak{M}_r$ be the Haar measure of $\Pi_{r+1}^{\infty} G_i \subset G$ and let $\mu_r = \mu * \mathfrak{M}_r$. Then $\mu_r \to \mu$ weakly and μ_r can be considered as an element of $J(\Pi_1^r G_i)$. (2) still holds for μ_r which is canonical by Theorem 6.1. We will show that $\|\mu_r\| = 1$ for all r. It then follows that $\|\mu\| = 1$ so that, by [3, Theorem 2.1.4], μ is canonical. Since $1 \in E(\mu)$, where 1 is the trivial representation, μ is a Haar measure.

Since μ_r is canonical it can be written as $\mu_r = \Sigma_1^s \nu_j$ where $\nu_j \in F(G)$ and $\nu_j << \lambda_j$, the Haar measure of a closed normal subgroup H_j. We show first that $s = 1$. We can assume the λ_j are mutually singular and that $\Pi_{H_1} \lambda_j = 0$ for $j > 1$. Now if $\alpha \in E(\mu_r)$ then

(3) $$\|(\Psi_\alpha - 1)\nu_j\| \leq \|(\Psi_\alpha - 1)\mu_r\| < 1/300.$$

Fix $\gamma \in E(\nu_j)$ then by (3)

(4) $$\left|\int (\overline{\Psi}_\alpha - 1)\overline{\Psi}_\gamma \, d\nu_j\right| = \left|\int \overline{\Psi}_\alpha \overline{\Psi}_\gamma \, d\nu_j - \hat{\nu}_j(\gamma)\right| < 1/300.$$

Since $\hat{\nu}_j(\gamma)$ is a nonzero integer, (4) implies that the decomposition of $\alpha \otimes \gamma$ contains an element of $E(\nu_j)$. Thus $E(\mu_r) \subset \overline{\gamma} E(\nu_j)$ where $\overline{\gamma}$ is the representation conjugate to γ and $\overline{\gamma} E(\nu_j)$ consists of the $\beta \in \Gamma$ that appear in the decomposition of $\overline{\gamma} \otimes \theta$ for some $\theta \in E(\nu_j)$. Now $\nu_j << \lambda_j$ which implies that $E(\nu_j)$ is the union of finitely many hypercosets of $H_j^\perp$. Thus $E(\mu_r)$ is contained in the union of finitely many hypercosets of $H_j^\perp$ so that there is $\omega_j \in J(G)$ with $\omega_j << \lambda_j$ and $\mu_r * \omega_j = \mu_r$. Let $j > 1$ and apply Π_{H_1} to this last equality. Since $\Pi_{H_1} \lambda_j = 0$ it follows that $0 = \Pi_{H_1} \mu_r = \lambda_1$. This is a contradiction so that $s = 1$ and $\mu_r = \lambda_1$.

We can thus write $\mu_r = Q\lambda_1$ where Q is a central idempotent polynomial on H_1 and also $Q = \Sigma_E a_\alpha \chi_\alpha|H_1$ where $a_\alpha > 0$ and $E \subset E(\mu_r)$. It follows from (2) that

(5) $$\int |(Q/Q(e) - 1)Q| d\lambda_1 \leq \sum_E \frac{a_\alpha d_{(\alpha)} \|(\Psi_\alpha - 1)\mu_r\|}{\Sigma_E a_\alpha d_{(\alpha)}} < \frac{1}{300}.$$

Thus since $\int |Q|^2 d\lambda_1 = Q(e)$ it follows from (5) that

(6) $$\|\mu_r\| = \int |Q| d\lambda_1 < 1 + 1/300$$

so that, by Lemma 5.2, $\|\mu_r\| = 1$ and the proof is complete.

Lemma 6.3. *Let p be a positive integer and $M < \infty$. There is $\delta = \delta(p, M) > 0$ such that if G is as in 6.2 and $\mu \in F(G)$ satisfies the following:*

(a) $\|\mu\| \leq M$.

(b) *There is a normal Borel subgroup T of G with $|\mu|(T^c) = 0$.*

(c) *There are Borel homomorphisms $f_1, f_2, \cdots, f_p$ of T into the unit circle with $f_1 \equiv 1$, $f_i\mu \neq f_j\mu$ for $i \neq j$ and for each i there is an irreducible sequence $\{\alpha\} \subset E(\mu)$ with $\Psi_\alpha \to f_i$ pointwise on T.*

(d) *$\alpha \in E(\mu)$ if and only if $\|(\Psi_\alpha - \bar{f}_i)\mu\| < \delta$ for some i,*

then $\omega = f_1\mu * f_2\mu * \cdots * f_p\mu = A\mathfrak{M}_H$ for some integer $A \neq 0$ and some closed normal subgroup H.

Proof. δ is chosen so that

$$p\delta^{1/2}M^{p-1}(pM + 2) < 1/300. \tag{7}$$

Clearly each $f_i\mu \in F(G)$ so that $\omega \in F(G)$. Also, by (c),

$$\int f_i\, d\mu = \lim \int \bar{\Psi}_\alpha\, d\mu \neq 0 \text{ so that } \hat{\omega}(1) = \prod (f_i\mu)\hat{\ }(1) \neq 0.$$

Let β be a fixed element of $E(\omega)$. Then $\beta \in E(f_i\mu)$ for all i so that, by using (c), $\lim \int \bar{\Psi}_\beta \bar{\Psi}_\alpha d\mu = \int \bar{\Psi}_\beta f_i d\mu \neq 0$. Since $\{\alpha\}$ is an irreducible sequence we must have $\beta \otimes \alpha$ irreducible eventually and $\beta\alpha \in E(\mu)$. Thus (d) implies that

$$\|(\Psi_\beta \Psi_\alpha - \bar{f}_j)\mu\| < \delta \quad \text{for some } j. \tag{8}$$

Since μ is carried by T it follows from (c) that

$$\|(\Psi_\beta - f_i\bar{f}_j)\mu\| = \|(\Psi_\beta \bar{f}_i - \bar{f}_j)\mu\| \leq \delta. \tag{9}$$

Now since $f_1 \equiv 1$, $E(\omega) \subset E(\mu)$ so that there is k with

$$\|(\Psi_\beta - \bar{f}_k)\mu\| < \delta. \tag{10}$$

We will show for such a k that

$$f_k\omega = \omega. \tag{11}$$

Since $\bar{f}_k\mu$ and $f_i\bar{f}_j\mu$ are both in $F(G)$ it follows from (9) and (10) that (since $\delta < \frac{1}{2}$) $\bar{f}_k\mu = f_i\bar{f}_j\mu$. Hence for this k and all i there is j with $f_i f_k\mu = f_j\mu$. Since $\{f_i\mu\}$ are distinct it follows that

$$\{f_i f_k\mu : 1 \leq i \leq p\} = \{f_i\mu : 1 \leq i \leq p\}. \tag{12}$$

Now the $f_i\mu$ are central measures so that if σ is a permutation of $(1, 2, \cdots, p)$ then

$$\omega = f_{\sigma(1)}\mu * \cdots * f_{\sigma(p)}\mu. \tag{13}$$

Let $\lambda = \mu \times \mu \times \cdots \times \mu$ on $G \times G \times \cdots \times G$. Then, for $\gamma \in \Gamma$, it follows from (12) and (13) that for some permutation σ

$$\begin{aligned}(f_k\omega)\hat{}(\gamma) &= \int \overline{\Psi}_\gamma f_k \, d\omega \\ &= \int \cdots \int \overline{\Psi}_\gamma(g_1 \cdots g_p) f_k(g_1 \cdots g_p) \prod_1^p f_i(g_i)\, d\lambda \\ &= \int \cdots \int \overline{\Psi}_\gamma(g_1 \cdots g_p) \prod f_{\sigma(i)}(g_i)\, d\lambda = \int \overline{\Psi}_\gamma d\omega = \hat{\omega}(\gamma).\end{aligned} \tag{14}$$

(11) then follows from (14).

Now let $R = \{g \in T : |\Psi_\beta(g) - \overline{f}_k(g)| < \delta^{1/2}\}$. It follows from (10) that

$$|\mu|(R^c) \leq \delta^{1/2}. \tag{15}$$

It also follows from Lemma 5.1 that if $g_i \in R$ $(1 \leq i \leq p)$

$$|\Psi_\beta(g_1 \cdots g_p) - \overline{f}_k(g_1 \cdots g_p)| < p^2\delta^{1/2}. \tag{16}$$

Then by (7), (11), (15), (16) and (a)

$$\begin{aligned}\|(\Psi_\beta - 1)\omega\| &= \|(\Psi_\beta - \overline{f}_k)\omega\| \\ &\leq \int \cdots \int |\Psi_\beta(g_1 \cdots g_p) - \overline{f}_k(g_1 \cdots g_p)| d(|\mu| \times \cdots \times |\mu|) \\ &\leq p^2\delta^{1/2}M^p + 2pM^{p-1}\delta^{1/2} < 1/300.\end{aligned} \tag{17}$$

The last inequality is obtained by integrating over $S = R \times \cdots \times R$ and S^c separately.

(17) holds for all $\beta \in E(\omega)$. Lemma 6.2 can then be applied to show that $\omega/\hat{\omega}(1)$ is a Haar measure.

7. **Some more technical lemmas.** The main purpose of this section is to prove Lemma 7.6.

Lemma 7.1. *Let H be a closed normal subgroup of G. Suppose $\{\alpha\}$ is a sequence in Γ such that*

$$|\Psi_\alpha(g)| \to 1 \quad a.e.\ (\mathfrak{M}_H). \tag{1}$$

Then, for α large enough, $|\Psi_\alpha(g)| \equiv 1$ on H.

Proof. Since H is closed and normal we can write $\chi_\alpha|_H = a\sum_1^p \chi_{\beta_i}$ where $\beta_i \in \Gamma(H)$ and $ap\, d(\beta_i) = d(\alpha)$ for all i. Then

$$\int_H |\Psi_\alpha|^2 d\mathfrak{M}_H = a^2 p(d(\alpha))^{-2} = (pd^2(\beta_1))^{-1}. \tag{2}$$

By (1) and the Lebesgue dominated convergence theorem $(pd^2(\beta_1))^{-1} \to 1$. Thus eventually $p = d(\beta_1) = 1$. That is $\chi_\alpha|_H = d(\alpha)\gamma$ where $\gamma \in \Gamma(H)$ and $d(\gamma) = 1$ so that $|\Psi_\alpha| \equiv 1$ on H.

Lemma 7.2. *Let H and K be closed normal subgroups of G with H open in K. Let f be the characteristic function of H. Suppose $\{\alpha\}$ is a sequence in Γ such that*

$$|\Psi_\alpha(g)| \to f(g) \quad a.e. \ (\mathfrak{M}_k). \tag{3}$$

Then, for α large enough, $|\Psi_\alpha| \equiv f$ on K.

Proof. From Lemma 7.1 we can assume $|\Psi_\alpha| \equiv 1$ on H. Write $\chi_\alpha|_K = a\Sigma_1^p \chi_{\beta i}$ where $\beta_i \in \Gamma(K)$. It follows as in the previous proof that

$$(pd^2(\beta_1))^{-1} = \int_K |\Psi_\alpha|^2 d\mathfrak{M}_K = \mathfrak{M}_K(H) + o(1). \tag{4}$$

Thus eventually $(pd^2(\beta_1))^{-1} = \mathfrak{M}_K(H)$ and so $\int_{K-H} |\Psi_\alpha|^2 d\mathfrak{M}_K = 0$; that is $\Psi_\alpha = 0$ on $K - H$.

Lemma 7.3. *Let H and K be closed normal subgroups of G with H open in K. Let $E \subset \Gamma$ be such that $\{\Psi_\alpha|_H : \alpha \in E\}$ is finite. Then $\{\Psi_\alpha|_K : \alpha \in E\}$ is also finite.*

Proof. If the lemma is false there is a sequence $\{\alpha\} \subset \Gamma$ such that $\Psi_\alpha|_K$ are distinct and $\Psi_\alpha|_H$ are all the same. By Lemma 4.4 $\Psi_{\alpha_0}\mathfrak{M}_H$, being a weak limit point of $\{\Psi_\alpha \mathfrak{M}_H\}$, is singular to $\mathfrak{M}_K$. But $\mathfrak{M}_H << \mathfrak{M}_K$ which gives a contradiction.

The following lemma was proved by Amemiya and Itô [1] although not stated in this form:

Lemma 7.4. *Let μ and ν be nonzero regular Borel measures on some space such that $f_n\mu \to \nu$ weakly where $\|f_n\|_\infty \leq 1$. Then given $k < \|\nu\|/\|\mu\| \leq 1$ there is N such that*

$$\|(f_n - f_m)\mu\| < 2\|\mu\|(1 - k + (1 - k^2)^{1/2}) \quad for\ n, m \geq k. \tag{5}$$

Lemma 7.5. *If, in the above, $\|\mu\| = \|\nu\|$ then $f_n\mu \to \nu$ in norm.*

Proof. Letting k be close to 1 in (5) shows that $\{f_n\mu\}$ is a Cauchy sequence.

It also follows immediately from Lemma 7.4 that if $1 \leq A \leq \|\nu\| \leq \|\mu\| < A + 1/100A$ then

$$\|(f_n - f_m)\mu\| < 1/4 \quad \text{for large } n, m. \tag{6}$$

Lemma 7.6. *Let G_i $(1 \le i < \infty)$ be compact groups satisfying condition* II *and let $G = \Pi G_i$. Let $\mu \in F(G)$ have support group L. If μ satisfies the following two conditions then μ is canonical.*

(a) *There is an integer N so that if P is a normal Borel subgroup of G and $\Pi_1^N G_i' \subset P$ (G_i' is the commutator subgroup of G_i) then $\Pi_P \mu \neq 0$ implies $P \cap L$ is open in L.*

(b) *For each i there is $M_i < \infty$ so that if $\alpha \in E(\mu)$ and $\alpha = \alpha_1 \cdots \alpha_n$ where $\alpha_i \in \Gamma(G_i)$ then $d(\alpha_i) < M_i$.*

Proof. The proof is by induction on $\|\mu\|$. Assume it is true for $1 \le \|\mu\| < A$ and let $A \le \|\mu\| < A + 1/100A$. We will show that $T = \{\Psi_\alpha|_L : \alpha \in E(\mu)\}$ is finite; μ is then canonical by Lemma 4.1.

We will assume T is infinite in order to obtain a contradiction. By using a diagonal process and the fact that each G_i satisfies condition II it follows from (b) that there is a sequence $\{\alpha_n\} \subset E(\mu)$ such that

(i) $\alpha_n = \alpha_{n,1}\alpha_{n,2} \cdots$ where $\alpha_{n,i} \in \Gamma(G_i)$,

(ii) there are $\beta_i \in \Gamma(G_i)$ such that $\alpha_{n,i} = \beta_i \gamma_{n,i}$ for $n \ge n(i)$ where $\gamma_{n,i} \in \Gamma(G_i)$ and $d(\gamma_{n,i}) = 1$, and

(iii) $\Psi_{\alpha_n}|_L$ are distinct.

By taking a subsequence of $\{\alpha_n\}$ we can write

$$\alpha_n = \beta_1 \cdots \beta_n \gamma_n \lambda_n \tag{7}$$

where $d(\gamma_n) = 1$ and $\lambda_n \in \Gamma(\Pi_{n+1}^\infty G_i)$.

We will first show that there is $i_0 > N$ such that

$$|\Psi_{\beta_i}|_L \equiv 1 \quad \text{for } i \ge i_0. \tag{8}$$

Let $P = \{g: \lim_{i \to \infty} |\Psi_{\beta_i}(g)| = 1\}$. P is clearly a normal Borel subgroup and by (7) $\Psi_{\alpha_n} \to 0$ on P^c as $n \to \infty$. Since $\hat{\mu}(\alpha_n) \neq 0$ it follows that $|\mu|(P) \neq 0$. Thus since $\Pi_1^N G_i' \subset P$ it follows from (a) that $P \cap L$ is open in L. In particular $P \cap L$ is closed so that by Lemma 7.1 there is $i_0 > N$ such that $|\Psi_{\beta_i}| \equiv 1$ on $P \cap L$ for $i \ge i_0$. We will show $P \cap L = L$.

If $P \cap L \neq L$ then, by Lemma 4.2,

$$|\mu|(L - P \cap L) \ge 1/2. \tag{9}$$

Also $\lim_{p \to \infty} |\Psi_{\beta_{i_0}} \cdots \Psi_{\beta_p}| = 0$ on $L - P \cap L$ so that if p is large enough it follows from Lemma 7.2 that

$$\Psi_{\beta_{i_0}} \cdots \Psi_{\beta_p} \equiv 0 \quad \text{on } P \cap L. \tag{10}$$

Let $\theta = \beta_i \cdots \beta_p$ and let $K = \Pi_1^p G_i'$. Define $\phi = \bar{\Psi}_\theta \mu * \mathfrak{M}_K$. It is easily seen that $\phi \neq 0$, $\phi \in F(G)$ and

(11) $$\sigma_n = \beta_{p+1} \cdots \beta_n \gamma_n \lambda_n \in E(\phi) \quad \text{for } n > p.$$

Also since Ψ_θ vanishes on $L - (L \cap P)$ it follows from (9) that $\|\phi\| \le \|\mu\| - \frac{1}{2} < A$. Also ϕ satisfies (a) and (b). (b) is easy. To see (a) (with p in place of N) suppose $K \subset Q$ and $\Pi_Q \phi \neq 0$. Then $\Pi_Q \mu \neq 0$ so that $Q \cap L$ is open in L. Now the support group $L^* = L(\phi) \subset (L \cap P)K$. Thus $L^* Q \subset LQ$. Q is open in LQ and thus is also open in $L^* Q$; that is $L^* \cap Q$ is open in L^*. It follows from the induction hypothesis that ϕ is canonical.

Since ϕ is canonical it follows that for some closed normal subgroup H with $\Pi_H \phi \neq 0$ there is an infinite set I such that $\sigma_n \in \sigma_{n_0} H^\perp$ for all $n \in I$; that is $\Psi_{\sigma_n}|_H$ are the same for $n \in I$. Since $\phi * \mathfrak{M}_K = \phi$ we can assume $K \subset H$ so that $H \cap L$ is open in L. By Lemma 7.3 this implies that $\{\Psi_{\sigma_n}|_L : n \in I\}$ is finite. But $\alpha_n = \theta \sigma_n$ so that we have a contradiction to (iii). Hence $P \cap L = L$ and (8) is proved.

Now let $\theta_n = \beta_{i_0} \cdots \beta_n \gamma_n \lambda_n$. A subsequence of $\{\overline{\Psi}_{\theta_n} \mu\}$ converges weakly to some ω. Now $\omega \neq 0$ and $\omega \in F(G)$; this is because the Ψ_{β_i} are multiplicative on L for $i \ge i_0$ and because $\{\gamma_n \lambda_n\}$ is an irreducible sequence. Now ω clearly satisfies (b). To see (a) suppose $\Pi_P \omega \neq 0$; since $\omega << \mu$ it follows that $\Pi_P \mu \neq 0$. Hence if $\Pi_1^N G_i' \subset P$ we have $P \cap L$ open in L. But $L(\omega) \subset L$ so that $P \cap L(\omega)$ is open in $L(\omega)$. The remainder of the proof is divided into two cases.

Case 1. $\|\omega\| < A$. Then ω is canonical by the induction hypothesis. Let H be a closed normal subgroup of G such that $0 \neq \Pi_H \omega << \mathfrak{M}_H$. Since $\overline{\Psi}_{\theta_n} \mu|_H \to \omega|_H$ weakly it follows from Lemma 4.4 that $\{\Psi_{\theta_n}|_H\}$ is finite. Now $\Psi_{\theta_n} \equiv 1$ on $\Pi_1^N G_i'$ so that $\{\Psi_{\theta_n}|_S\}$ is finite where $S = H \cdot \Pi_1^N G_i'$. Also $\Pi_S \mu \neq 0$ so that $S \cap L$ is open in L. It follows from Lemma 7.3 that $\{\Psi_{\theta_n}|_L\}$ is finite and since $\alpha_n = \beta_1 \cdots \beta_{i_0 - 1} \theta_n$ this contradicts (iii).

Case 2. $\|\omega\| \ge A$. It follows here from the remark (6) after Lemma 7.5 that, if n, m are large enough,

(12) $$\|(\overline{\Psi}_{\theta_n} - \overline{\Psi}_{\theta_m})\mu\| < 1/4.$$

Fixing n and letting $m \to \infty$ in (12) then gives

(13) $$\|\overline{\Psi}_{\theta_n} \mu - \omega\| \le 1/4$$

and

(14) $$\|\overline{\Psi}_{\theta_n} \mu\| \ge \|\omega\| - 1/4 \ge A - 1/4.$$

By (13) and (14), using that $|\Psi_{\theta_n}| \le 1$,

$$\begin{aligned}\|\mu - \Psi_{\theta_n}\omega\| &\le \|(1-|\Psi_{\theta_n}|^2)\mu\| + \||\Psi_{\theta_n}|^2\mu - \Psi_{\theta_n}\omega\| \\ &\le 2[\|\mu\| - \|\overline{\Psi}_{\theta_n}\mu\|] + \|\overline{\Psi}_{\theta_n}\mu - \omega\| \qquad (15)\\ &\le 3/4 + 1/50A.\end{aligned}$$

Since ω is carried by L it follows that (a subsequence of) $\{\Psi_{\theta_n}\omega\}$ converges weakly to some $\lambda \in F(G)$. (15) shows that $\|\mu - \lambda\| < 1$ so that $\mu = \lambda$. Since $\|\lambda\| \le \|\omega\|$ we must then have that $\|\omega\| = \|\mu\|$ so that by Lemma 7.5 $\overline{\Psi}_{\theta_n}\mu \to \omega$ in norm. It then follows that, for a subsequence,

$$|\Psi_{\theta_n}| \to 1 \quad \text{a.e. } (|\mu|). \tag{16}$$

Now (16) occurs on a normal Borel subgroup S which contains $\Pi_1^N G_i'$. Thus, by (a), $S \cap L$ is open in L. $S \cap L$ is then closed and carries μ so that $S \cap L = L$. This gives, by Lemma 7.1, that

$$|\Psi_{\theta_n}| \equiv 1 \quad \text{on } L \text{ for large } n. \tag{17}$$

But then $\overline{\Psi}_{\theta_n}\mu \in F(G)$ so that, because of (12),

$$\overline{\Psi}_{\theta_n}\mu = \overline{\Psi}_{\theta_m}\mu. \tag{18}$$

Finally (17) and (18) show that $\Psi_{\theta_n}|_L = \Psi_{\theta_m}|_L$ which implies that $\Psi_{\alpha_n}|_L = \Psi_{\alpha_m}|_L$ and this contradicts (iii).

8. Proof of Theorem 2.2.

Theorem 2.2. *Let G_i $(i \in A)$ be compact groups satisfying conditions* I *and* II *and let $G = \Pi_A G_i$. Then every measure in $F(G)$ is canonical.*

Proposition 8.1. *If the theorem is true for countable products it is true for any A.*

Proof. Let $\mu \in F(G)$. We can assume μ is irreducible and has support group L. If μ is not canonical then by Lemma 4.1 there is a sequence $\{\alpha_n\} \subset E(\mu)$ such that $\Psi_{\alpha_n}|_L$ are distinct. There is a countable set $B \subset A$ such that $\alpha_n \in \Gamma(\Pi_B G_i)$ for all n. Let $K = \Pi_{A-B}G_i$ and $\nu = \mu * \mathfrak{M}_K$. Then $E(\nu)$ contains $\{\alpha_n\}$ and ν is canonical since it can be considered as a measure on $\Pi_B G_i$. It is then easily seen that ν is irreducible so that $\{\Psi_{\alpha_n}|_{L(\nu)}\}$ is finite. But $L(\nu) \cap L$ is open in L so that $\{\Psi_{\alpha_n}|_L\}$ is finite by Lemma 7.3 which is a contradiction.

It remains to prove the theorem when A is countable. The proof is by induction on $\|\mu\|$. Assume it is true for $\|\mu\| < C$ and let $\|\mu\| < C + 1/100C$. We can also assume that μ is irreducible.

Proposition 8.2. *We can assume* μ *satisfies condition* (b) *of Lemma* 7.6 *and that* $1 \in E(\mu)$.

Proof. Suppose $E(\mu)$ contains a sequence $\{\alpha\}$ where $\alpha = \alpha_1 \alpha_2 \cdots$ with $\alpha_i \in \Gamma(G_i)$ and suppose, for some j, that $d(\alpha_j) \to \infty$ as $\alpha \to \infty$. Since $\hat{\mu}(\alpha) \neq 0$ and G_j satisfies condition I it follows that

$$|\mu|\left(\prod_{i \neq j} G_i \times Z(G_j)\right) \neq 0. \tag{1}$$

By Theorem 3.2 we then have that $L = L(\mu) \subset \Pi_{i \neq j} G_i \times K_j$ where K_j is a finite normal extension of $Z(G_j)$. Doing this for all such j we obtain that $L \subset \Pi K_i = G^*$ where $K_i = G_i$ or K_i is a finite normal extension of $Z(G_i)$. Now each K_i satisfies conditions I and II and μ satisfies condition (b) with respect to G^*. If μ is canonical with respect to G^* then by Lemma 4.7 it is canonical with respect to G. It can still be assumed that μ is irreducible; if μ is reducible as an element of $F(G^*)$ then it is canonical on G^* by the induction hypothesis and then it is canonical on G.

Let $\alpha = \alpha_1 \cdots \alpha_N \in E(\mu)$ and let $K = \Pi_1^N G_i'$. Then $\overline{\Psi}_\alpha \mu * \mathfrak{M}_K = \lambda \in F(G)$, $1 \in E(\lambda)$ and $\|\lambda\| \leq \|\mu\|$. Also λ satisfies condition (b) since we can now assume that μ does. We will show that if λ is canonical then μ is also. Suppose $\lambda = \Sigma \nu_j$ where $\nu_j << \mathfrak{M}_{H_j}$ for some closed normal subgroup H_j. Let P be a normal Borel subgroup with $K \subset P$ and $\Pi_P \mu \neq 0$. Then since μ is irreducible $\Pi_P \mu = \mu$; also $\Pi_P \mathfrak{M}_K = \mathfrak{M}_K$ since $K \subset P$. Thus $\Pi_P \lambda = \lambda$ so that $\Pi_P \mathfrak{M}_{H_j} = \mathfrak{M}_{H_j}$ for all j. This implies that $P \cap H_j$ is open in H_j. But $\Pi_{H_j} \mu \neq 0$ so that $H_j \cap L$ is open in L. Hence $P \cap L$ is open in L. By Lemma 7.6 then μ is also canonical. We can thus assume μ, like λ, has $1 \in E(\mu)$.

The remainder of the proof involves using Lemma 6.3 to show that μ satisfies condition (a) of Lemma 7.6.

Call a sequence $\{\alpha_n\} \subset E(\mu)$ an *F-sequence* if $\alpha_n = \alpha_{n,1} \alpha_{n,2} \cdots$ with $\alpha_{n,i} \in \Gamma(G_i)$ and $d(\alpha_{n,i}) = 1$ for $i \leq n$. An F-sequence is an irreducible sequence so that a subsequence of $\{\overline{\Psi}_{\alpha_n} \mu\}$ will converge weakly to some nonzero $\omega \in F(G)$. Let B be the collection of all such ω. $\mu \in B$ since $\alpha_n = 1$ is an F-sequence. Now if $\omega \in B$ and $\|\omega\| < C$ then ω is canonical by induction and μ is then canonical by Lemma 4.5. We can thus assume that $\omega \in B$ implies

$$C \leq \|\omega\| \leq \|\mu\| < C + 1/100C. \tag{2}$$

It then follows, for $\omega \in B$, that, as in Case 2 at the end of the proof of Lemma 7.6, $\|\omega\| = \|\mu\|$ and

$$\|\overline{\Psi}_{\alpha_n} \mu - \omega\| \to 0 \quad \text{for some } F\text{-sequence}. \tag{3}$$

Then $\omega = f\mu$ for some $f \in L(\mu)$ with $|f| = 1$ a.e. $|\mu|$. We will identify B with the collection of such f.

Now another subsequence has

(4) $$\overline{\Psi}_{\alpha_n} \to f \quad \text{a.e. } |\mu|.$$

Let S_f be where (4) holds and $|f| = 1$. Then S_f is a normal Borel subgroup, $|\mu|(S_f^c) = 0$ and f is a Borel homomorphism of S_f into the unit circle.

Now B is finite. Otherwise we would have $\omega_i \in B$ with $\omega_i \to \omega$ weakly for some $\omega \neq 0$. Using (3) and a diagonal process on the sequences converging to ω_i we could then find an F-sequence $\{\alpha_i\}$ with

(5) $$\|\overline{\Psi}_{\alpha_i}\mu - \omega_i\| < 1/2 \quad \text{and} \quad \|\overline{\Psi}_{\alpha_i}\mu - \omega\| < 1/2.$$

Since $\omega_i \in F(G)$ this would give $\omega_i = \omega$.

Let $p = \text{card } B$, $M = \|\mu\|$ and $\delta = \delta(p, M)$ be the constant in Lemma 6.3. By using (3) there is an integer N such that if $\alpha = \alpha_1\alpha_2 \cdots \in E(\mu)$ and $d(\alpha_i) = 1$ for $i \leq N$ then

(6) $$\|(\overline{\Psi}_\alpha - f)\mu\| < \delta \quad \text{for some } f \in B.$$

Now let $T = \bigcap S_f$ over f with $f \in B$. Each f is a homomorphism on T and $|\mu|(T^c) = 0$. Let $K = \Pi_1^N G_i'$ and $\lambda = \mu * \mathfrak{M}_K$. Then $\lambda \neq 0$, $\|\lambda\| \leq M$ and, since $K \subset T$, $|\lambda|(T^c) = 0$. It also follows from (6) that $\alpha \in E(\lambda)$ if and only if

(7) $$\|(\overline{\Psi}_\alpha - f)\lambda\| < \delta \quad \text{for some } f\mu \in B.$$

We can now apply Lemma 6.3 to λ to obtain

(8) $$f_1\lambda * f_2\lambda * \cdots * f_q\lambda = A\mathfrak{M}_H$$

where the $f_i\lambda$ are the distinct elements of $\{f\lambda : f \in B\}$, $A \neq 0$ and H is a closed normal subgroup.

Now $\Pi_H\mu \neq 0$ so that $H \cap L$ is open in L by Theorem 3.2. On the other hand if P is a normal Borel subgroup, $K \subset P$ and $\Pi_P\mu \neq 0$ then $\mu = \Pi_P\mu$ and so $\lambda = \Pi_P\lambda$. Hence $\Pi_P\mathfrak{M}_H = \mathfrak{M}_H$ so that $P \cap H$ is open in H. This then implies $P \cap L$ is open in L so that μ is canonical by Lemma 7.6.

9. Connected groups. We can now use Theorem 2.2 to characterize $F(G)$ for connected G.

Lemma 9.1. *If G is a compact connected simple Lie group then G satisfies conditions* I *and* II.

Proof. Ragozin [4, Theorem 2.2] has shown that if n = dimension G and $\mu \in M^Z(G)$ is continuous then $\mu^n \in L_1(G)$. For $g \notin Z(G)$ let μ be given implicitly by

$$\int f \, d\mu = \int f(xgx^{-1}) \, d\mathfrak{M}_G(x). \tag{1}$$

It is easily seen that μ is a central continuous measure and $\hat{\mu}(\alpha) = \overline{\Psi}_\alpha(g)$ so that $(\mu^n)\hat{}(\alpha) = (\overline{\Psi}_\alpha(g))^n$. By Ragozin's result and the Riemann-Lebesgue lemma it follows that

$$\Psi_\alpha(g) \to 0 \quad \text{as } \alpha \to \infty \ (g \notin Z). \tag{2}$$

Condition I follows immediately. Also if $\{\alpha\}$ is a sequence with $d(\alpha) = t$ then (2) and the dominated convergence theorem show

$$1 = \int |\chi_\alpha|^2 \, d\mathfrak{M}_G = t^2 \mathfrak{M}_G(Z) + o(1) \quad \text{as } \alpha \to \infty. \tag{3}$$

But then $\mathfrak{M}_G(Z) \neq 0$ so that Z is open which is not possible since G is connected Thus there are only finitely many α with $d(\alpha) = t$ which implies condition II.

Theorem 9.2. *If G is a connected compact group then every measure in $F(G)$ is canonical.*

Proof. It is known (cf. [8, Chapitre V]) that a connected compact group G is a factor group of a group $G^* = \Pi G_i \times A$ where A is abelian and the G_i are connected simple Lie groups. By Lemma 9.1 and Theorem 2.2 every measure in $F(G^*)$ is canonical. Since a measure in $F(G)$ can be considered as a measure in $F(G^*)$ the theorem follows.

10. An example. Unfortunately the characterization of $F(G)$ does not hold for all G as the following simple example shows. Let $T \times T$ be the two dimensional torus and let G be the semidirect product of $T \times T$ and Z_2 where $\delta(t_1, t_2) = (t_2, t_1)\delta$ for $\delta \in Z_2, \delta \neq e$. Let μ_1 (resp. μ_2) be the Haar measure of $T \times e$ (resp. $e \times T$). Then $\mu = \mu_1 + \mu_2 \in F(G)$ but μ is not canonical. That μ is not canonical is seen by noting that μ is singular to $\mathfrak{M}_H$ for every closed normal subgroup H of G.

In this example μ is a sum of (noncentral) idempotents each of which is canonical with respect to its support group. It seems reasonable to conjecture that for any G this is always the case.

Also if μ is canonical or as in the example then the conclusion of Lemma 4.2 holds for any Borel subgroup H (whether or not it is closed and normal). It would be helpful to know whether this is true for any $\mu \in F(G)$.

It also seems likely that there is $\delta > 0$ so that if $\mu \in F(G)$ and $\|\mu\| > 1$ then $\|\mu\| > 1 + \delta$ (cf. Theorem 5.3). Otherwise some strange elements of F could be obtained by taking infinite products of measures on product groups.

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