ABSTRACT. The author generalizes A. Denjoy's theory of flows on a torus to compact orientable 2-manifolds of higher genus. Natural extensions of A. Denjoy's hypotheses are made and necessary conditions that a flow satisfy the new hypotheses are given.

I. Introduction. In 1932, A. Denjoy [1] studied differential equations with no singular points on a 2-torus $T$ satisfying the following hypotheses:

$h_1$. There is a closed curve $\Gamma$, transverse to the flow, which is non-null-homotopic.

$h_2$. Every trajectory of the flow intersects $\Gamma$.

It is well known (i.e. see [2, p. 196]) that every differential equation with no singular points on a torus satisfies $h_1$. It is not difficult to give examples of such differential equations that do not satisfy hypothesis $h_2$.

After defining the rotation number $\alpha$ of a flow to be the rotation number of the induced orientation preserving homeomorphism $S$ of $\Gamma$ onto itself, Denjoy obtained the following results:

Theorem A. If $T'$ is a flow on $T$ satisfying hypotheses $h_1$ and $h_2$, $\alpha$ is rational if and only if the flow contains a periodic orbit.

Theorem B. If $T'$ is a flow on $T$ of class $C^2$, satisfying hypotheses $h_1$ and $h_2$ and $\alpha$ is irrational, then every trajectory is dense on $T$.

In this paper, the author generalizes Denjoy's theory to arbitrary compact orientable 2-manifolds of class $C^k$, $k \geq 1$.

A flow will be denoted by $T'$, $T_n$ will denote a compact orientable 2-manifold of genus $n$, the positive and negative trajectories with initial point $p$ will be denoted $p(+t)$ and $p(-t)$ respectively and the $\omega$-limit set of $p(+t)$ will be denoted $\Omega(p)$.

In [3], given a flow $T'$ of class $C^k$, $k \geq 2$, on $T_2$, with only stationary points.
of negative index, the author proves the existence of a $C^k$, closed, non-null-ho-
matopic curve $\Gamma$ that does not separate $T_n$ and is transverse to the flow. He
further conjectures that such a transversal exists for any $T_n$. Certainly, for any
$n$, it is easy to give examples for which such a transversal exists. Hence, we make the following hypotheses analogous to Denjoy's:

Given a $C^k$ flow $T^k$ on $T_n$, $k \geq 1$:

$H_1$. There exists a $C^k$ closed curve $\Gamma$ transverse to the flow which is
non-null-homotopic and does not separate $T_n$.

$H_2$. Every trajectory of the flow not a separatrix entering a singular point
for positive (negative) time intersects $\Gamma$ for positive (negative) time.

These two hypotheses will enable the author to show that there is a natural
one to one map $S'$ induced by the flow from that subset of $\Gamma$ lying on trajectories
that do not enter singular points of flow for positive time to that subset of $\Gamma$ lying
on trajectories that do not enter singular points of the flow for negative time. It
will be clear that $S'$ always has an extension to all of $\Gamma$. In fact $S'$ has many
extensions to all of $\Gamma$, but there exist flows for which no extension $S$ of $S'$
will be orientation preserving. Hence, we make the following hypothesis:

$H_3$. The flow is one for which an orientation preserving extension $S$ of $S'$
exists.

Definition 1.1. Flows on $T_n$ with a finite number of critical points all of
nonzero index satisfying hypotheses $H_1$, $H_2$, and $H_3$ will be called Denjoy-type
flows.

Note. $S$ will be seen to be unique for Denjoy-type flows.

Necessary conditions that flows be Denjoy-type will be shown to be that all
stationary points be simple singularities of negative index and that each sing-
ular point be connected to another (possibly the same one) by at least one tra-
jectory.

The following analogues to Denjoy's Theorems A and B will be proven.

Theorem $A_1$. The rotation number $\alpha$ of a Denjoy-type flow on $T_n$ is rational
if and only if the flow contains a periodic orbit or a closed curve consisting en-
tirely of trajectories.

Theorem $B_1$. If the rotation number $\alpha$ of a Denjoy-type flow of class $C^2$ is
irrational and $S$ and $S^{-1}$ are of class $C^2$, then every trajectory not entering a
singular point for positive (negative) time is dense on $T_n$ for positive (negative)
time.

2. Definition of the map $S$ induced by a flow on $T_n$. The following two
lemmas, which will not be used until the proofs of Theorems $A_1$ and $B_1$, are
stated here as a means of motivating the definition of $S$ and the necessity for
hypotheses $H_1$, $H_2$ and $H_3$. 

C.S. HARTZMAN
Familiarity with the definition of discrete flow on a closed Jordan curve $\Gamma$ will be assumed (i.e., see [2, pp. 190—195]). Such a discrete flow will be denoted \{$S^n$\}, where $S$ is the underlying homeomorphism of $\Gamma$ onto itself. $\alpha$ will denote the rotation number of $S$. The following two lemmas are well known:

**Lemma 2.1.** Let \{$S^n$\} be a flow on a Jordan curve $\Gamma$. Then $\alpha$ is rational if and only if $S^k$ has a fixed point for some $k > 0$.

**Lemma 2.2.** Let \{$S^n$\} be a flow on a Jordan curve $\Gamma$ having an irrational rotation number. For any $y \in \Gamma$, let $\Omega(y)$ be the set of limit points of \{$S^k y$: $k = 0, 1, 2, \ldots$\}. Then $\Omega(y)$ is independent of $y$ and if $S$ and $S^{-1}$ are of class $C^2$, $\Omega(y) = \Gamma$.

The map $S$ we are to define will be one that may serve as the underlying homeomorphism for a discrete flow on the transversal curve $\Gamma$ of hypothesis $H_1$. We first prove

**Lemma 2.3.** The singular points of any Denjoy-type flow of class $C^k$, $k \geq 1$, on $T_n$ are simple and of negative index.

**Remark.** A simple singular point of negative index will be one which has no elliptic or parabolic sectors. (See [5] for a discussion of these sectors.)

**Proof.** Let $p$ be a singular point. Either $p$ is a spiral point or small neighborhoods of $p$ can be decomposed into parabolic, elliptic and hyperbolic sectors. If $p$ is a spiral point, hypothesis $H_2$ is violated, since trajectories entering $p$ are not separatrices and any trajectory close enough to $p$ will enter any small enough neighborhood of $p$ for positive (or negative) time and never leave, thus not intersecting $\Gamma$ for positive (or negative) time; a contradiction of hypothesis $H_2$. If $p$ is not a spiral point and $p$ has elliptic or parabolic sectors, in these sectors there are trajectories which are not separatrices that enter any small enough neighborhood of $p$ and do not leave for positive (or negative) time, leading to the same contradiction as before. Hence singular points are simple.

It is well known that

\begin{equation}
2i = 2 + n_e - n_h,
\end{equation}

where $i$ is the index of the singular point, $n_e$ and $n_h$ are the number of elliptic and hyperbolic sectors respectively.

(2.1) implies that the indices of singularities of Denjoy-type flows are simple and of negative index since there are no elliptic sectors or parabolic sectors.

Q.E.D.

We now proceed to define an orientation preserving homeomorphism $S$ of $\Gamma$ onto itself induced by a Denjoy-type flow on $T_n$. 

Let $S'$ be the natural map from that subset of $\Gamma$, consisting of points of the trajectories through which do not enter singular points before intersecting $\Gamma$ again, for positive time, to $\Gamma$. Specifically, if $p \in \Gamma$ such that $p(+t)$ intersects $\Gamma$ again. Let $t_p$ be the first nonzero time that $p(+t)$ intersects $\Gamma$ again. Define $S'p = p(t_p)$. Let $Q$ be the set of points not in the domain of $S'$ and let $R$ be the set of points not in the range of $\Gamma$. It is clear that $Q$ consists of those points $q \in \Gamma$ such that $q(+t)$ enters a singular point before intersecting $\Gamma$ again and $R$ consists of those points $r \in \Gamma$ such that $r(-t)$ enters a singular point before intersecting $\Gamma$ again. We use the Poincaré-Hopf theorem to determine more about the sets $Q$ and $R$. (See [4] for a discussion of this theorem.)

Theorem 2.1 (Poincaré-Hopf). Let $M$ be a $C^k$, $k \geq 1$, compact manifold with boundary. Let $T^d$ be a $C^k$ flow on $M$. Then $\Sigma = \chi(M)$ where $\Sigma$ is the sum of the indices of the singular points of $T^d$ and $\chi(M)$ is the Euler characteristic of the manifold.

Comparing this theorem with equation (2.1) and the fact that our singular points have only hyperbolic sectors, we see that the maximum number of trajectories that may enter singularities is $2(2n - 2)$ since the Euler characteristic of $T_n$ is $2 - 2n$. Hence, $Q$ contains at most $2(2n - 2)$ points. Suppose $Q = \{q_1, \ldots, q_k\}$, $0 \leq k \leq 2(2n - 2)$. It is clear that $R$ has the same number of points, $R = \{r_1, \ldots, r_k\}$. Order the set $Q$ according to an orientation for $\Gamma$.

Given hypothesis $H_3$, that there is an orientation preserving extension $S$ of $S'$, it is clear how $S$ must be defined. Consider the arc $Q_i = (q_{i-1}, q_i)$ on $\Gamma$ (take $Q_1$ to be $(q_0, q_1)$). Let $R_i = \{r: r = p(t_p), p \in Q_i\}$, where $t_p$ is the first time that $p(+t)$ intersects $\Gamma$. $R_i$ is also an arc on $\Gamma$. $S'$ is orientation preserving on each arc $Q_i$. The arcs $Q_i$ inherit an ordering on $\Gamma$ from the ordering of $Q$. Under any orientation preserving extension $S$ of $S'$ the arcs $R_i$ would inherit the same ordering. $Q_i$ is adjacent to $Q_{i+1}$ for each $i$ and are separated by the point $q_i$. Then under any choice for $S, R_i$ is adjacent to $R_{i+1}$ and is separated by one point of $R$, call it $r_i$. Then for any orientation preserving extension $S$ of $S'$, $Sq_i$ must be defined to be $r_i$. $S$ is readily seen to be one to one and onto. The continuity of $S$ and $S^{-1}$ follow easily from the continuous dependence of the trajectories of the flow on initial conditions and hypothesis $H_2$. Hence $S$ is an orientation preserving homeomorphism of $\Gamma$ onto itself and is uniquely determined by the flow.

The following example (Diagram 2.1) of a flow on $T^2$ illustrates an $S'$ that has no orientation preserving extension; hence the necessity of hypothesis $H_3$. In this example, $T_2$ is obtained by identifying opposite sides of the rectangle with the same orientation and identifying the boundaries of the discs with opposite orientation; $p_1$ and $p_2$ represent two simple saddle points.
and identifying the boundaries of the discs with opposite orientation; $p_1$ and $p_2$ represent two simple saddle points.

3. Necessary conditions that a flow $T^t$ on $T^n$ be Denjoy-type. Let $T^t$ be a Denjoy-type flow of class $C^k$, $k \geq 1$. Let $O_i = \{ p : p = q(t) \}$ for some $q \in Q_i$, $0 < t < t_q$, where $t_q$ is the first time such that $q(t)$ intersects $\Gamma$ again. By hypothesis $H_2$, the $O_i$'s taken together with $\Gamma$, the singular points of $T^t$ and the trajectories that enter singular points for positive or negative time constitute all of $T_n$. Hence the boundary of $O_i$ consists of arcs of $\Gamma$, singular points and arcs of trajectories that enter singular points. The following lemma describes $O_i$ more precisely.

**Lemma 3.1.** $O_i$ is the $C^k$ diffeomorphic to $I \times I$, where $I = (0, 1)$.

**Proof.** Let $g$ be a diffeomorphism between $I$ and $Q_i$. Define a function $f : I \times I \to O_i$ by

$$f(x, s) = p(xt_q, g(s)),$$

where $q = g(s)$. That is, $f(x, s)$ is the point in $O_i$ on the trajectory through $g(s)$ at time $xt_q$. $f$ is clearly one to one and onto. The $C^k$ character of $t_q$ with respect to $q$ and of the trajectories of the flow with respect to initial conditions easily implies the $C^k$ differentiability of $f$ and $f^{-1}$. Q.E.D.
Note. If $Q_i = \Gamma$, then $Q_i$ is diffeomorphic to the circular cylinder $I \times C$.

Lemma 3.2. If $T^n$ is a Denjoy-type flow of class $C^k$, $k \geq 1$, there must be at least one point $q \in \Gamma$ such that $q(+)\_t$ enters a singular point without intersecting $\Gamma$ again. (Hence there is also one point $r \in \Gamma$ such that $r(-)\_t$ enters a singular point without intersecting $\Gamma$ again.)

Proof. Suppose there is no point $q$ such that $q(+)\_t$ enters a singular point without intersecting $\Gamma$ again. Then there is only one set $Q_i$ as above, call it $Q$, and $Q = \Gamma$. The corresponding set $O = \{p: p = q(t), q \in \Gamma, 0 < t < t_q\}$ is a circular cylinder whose closure is given by $cl\_O = \{p: p = q(t), q \in \Gamma, 0 \leq t \leq t_q\}$. It is clear that $cl\_O$ is a torus. Hence, there is a torus embedded as a close submanifold of $T^n$, a contradiction. Q.E.D.

Lemmas 3.1 and 3.2 are fundamental in giving us a workable description of $T^n$. $T^n$ can be thought of as $(\bigcup_i O_i) \cup (\bigcup_i \partial O_i)$. We already know that $\partial O_i$ consists of arcs of trajectories through points of $\Gamma$ entering singular points without intersecting $\Gamma$ again (i.e., trajectories $q_i(+)\_t$, $r_i(-)\_t$, singular points, trajectories connecting singular points and arcs of $\Gamma$). We also know that any such object is in $\partial O_i$. It is also clear that a portion of any separatrix entering a singular point will appear on some of the $\partial O_i$'s. $T^n$ is now decomposed into closed strips $O_i$ where the boundaries of these strips are to be identified suitably to yield $T^n$. A more careful examination of $O_i$ will show that one pair of opposite boundary arcs of $O_i$ consists of arcs of $\Gamma$ and the singular points and trajectories entering singular points lie on the other pair of opposite boundary arcs. Also, the only points that lie on both pairs of opposite boundary arcs are the points $q_j \in Q$ and $r_j \in R$. Each $q_j$ and each $r_j$ will have exactly two representations in the totality of strips $O_i$, namely on one on $O_{j-1}$ and one on $O_j$. Likewise, the trajectories $q_j(+)\_t$ and $r_j(-)\_t$ will have exactly two representations, one on $O_{j-1}$ and one on $O_j$. The singular points that these trajectories enter will have at least two representations one of which is on $O_{j-1}$ and another on $O_j$. Trajectories connecting singular points will have exactly two representations on the boundaries $\partial O_i$ which are to be identified. If there were three or more representations of the same trajectory, upon identifying them, the result cannot be a 2-manifold since any neighborhood of a point on such a trajectory would not be homeomorphic to an open set in $R^2$ (see Diagram 3.1).

Theorem 3.1. Necessary conditions that a flow be Denjoy-type on $T^n$ of class $C^k$, $k \geq 1$, are that all singular points be simple of negative index and that each singular point be connected to another (possibly the same one) by at least one trajectory.
Proof. To prove this theorem, it is only necessary to prove that given any singular point, there is at least one trajectory connecting this singular point to another (possibly the same one). The remaining part of this theorem is covered by Lemma 2.3.

Consider the two edges of the $\partial O_i$'s bounded by the points $q_i$ and $r_i$. If either one of these edges contains more than one representation of a singular point the theorem is proven, for then there is a segment of an edge between the two singular points which represents a trajectory connecting them. If the both edges contain exactly one representation of a singular point it must be the same singular point $p$, since we must be able to identify the representations of the trajectories $q_i(t)$ and $r_i(-t)$ that appear on the two edges. Hence both edges bounded by $q_i$ and $r_i$ are completely identified. If these are the only representations of $p$, then $p$ has exactly two separatrices entering it, one for positive time and one for negative time. This implies that there are only two hyperbolic sectors associated with $p$, i.e., $p$ is a singularity of index zero (see Formula 2.1). There are, thus, other representations of $p$. But identifying all these representations of $p$ leads to an object that cannot be a 2-manifold for then $p$ does not have a neighborhood homeomorphic to an open set in $\mathbb{R}^2$. Q.E.D.

4. Analysis of Denjoy-type flows. In § 2, we saw how a Denjoy-type flow $T_t$ on $T_n$ induced a discrete flow $\{S^n\}$ on the transversal $\Gamma$ of hypothesis $H_1$. The rotation number $\alpha$ of this discrete flow was taken to be the rotation number
of the flow $T^t$. We prove the Theorems $A_1$ and $B_1$ as stated in the introduction.

**Theorem $A_1$.** The rotation number $\alpha$ of a Denjoy-type flow on $T^n$ is rational if and only if the flow contains a periodic orbit or a closed curve consisting entirely of trajectories.

**Proof.** The induced map $S$ of §2 is an orientation preserving homeomorphism of $\Gamma$. Suppose $\alpha$ is rational. By Lemma 2.1, $S^k$ has a fixed point for some $k > 0$. Call the fixed point $y_0$. If $y_0(+ t)$ does not enter a singular point, then by the way we defined $S$, $y_0(+ t)$ must be a periodic orbit which intersects $\Gamma$ $k$ times. If $y_0(+ t)$ enters a singular point, let $q$ be the last point on $\Gamma$ through which $y_0(+ t)$ goes before entering the singular point. Consider the point $r = Sq$. We know that $q$ is connected to $r$ by paths of trajectories and singular points, also by the way we defined $S$. Consider $r(+ t)$. $r(+ t)$ may or may not enter a singular point. If it does, carry out the process again. This process can be carried out a most finite number of times, since $k$ is finite. At any rate, after having intersected $\Gamma$ $k$ times, we are back to $y_0$. This completes the proof of the necessary condition; the proof of the sufficient condition is clear. Q.E.D.

**Theorem $B_1$.** If the rotation number $\alpha$ of a Denjoy-type flow of class $C^2$ is irrational and $S$ and $S^{-1}$ are of class $C^2$, then every trajectory not entering a singular point for positive (negative) time is dense on $T^n$ for positive (negative) time.

**Proof.** Since $S$ and $S^{-1}$ are $C^2$ and $\alpha$ is irrational, the set of points $S^n y$, $n = 0, 1, 2, \ldots$, is dense on $\Gamma$, by Lemma 2.2. Let $p(+ t)$ be a semiorbit that does not enter a singular point. Suppose $p(+ t)$ is not dense on $T^n$. Then there is a neighborhood $N$ on $T^n$ which $p(+ t)$ does not intersect. By hypothesis $H_1$, $p(+ t)$ intersects $\Gamma$ and hence $p(+ t) \cap \Gamma$ is dense on $\Gamma$. Now let $p_0 \in N$ such that $p_0(+ t)$ does not enter a singular point. Then $p_0(+ t)$ intersects $\Gamma$ at a point $q_0$ at time $t_{p_0}$. By continuity of the differential equation, we can choose an arc segment $l$ on $\Gamma$, containing $q_0$, which is so small that $p_q = q(t_{p_q})$ is contained in $N$ for all $q \in l$. Since $l$ is the image under trajectories of points $p_q$ in $N$ and $N$ contains no points of $p(+ t)$, $p(+ t)$ cannot intersect $l$. This contradicts the fact that $p(+ t) \cap \Gamma$ is dense on $\Gamma$. Q.E.D.

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DENJOY-TYPE FLOWS ON ORIENTABLE 2-MANIFOLDS


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