TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 187, Issue 1, 1974

A TOPOLOGY FOR A LATTICE-ORDERED GROUP(1)

BY

R. H. REDFIELD

ABSTRACT. Let G be an arbitrary lattice-ordered group. We define a topology on G, called the $\mathfrak{J}$-topology, which is a group and lattice topology for G and which is preserved by cardinal products. The $\mathfrak{J}$-topology is the interval topology on totally ordered groups and is discrete if and only if G is a lexico-sum of lexico-extensions of the integers. We derive necessary and sufficient conditions for the $\mathfrak{J}$-topology to be Hausdorff, and we investigate $\mathfrak{J}$-topology convergence.

1. **Introduction.** Several ways have been developed for topologizing lattices in general and lattice-ordered groups (l-groups) in particular. The method of Lefschetz [13] and Šmarda [19] defines a group and lattice topology on an l-group by using filters of convex l-subgroups to generate the neighbourhoods of 0. For any lattice, Rennie's L-topology [18] takes as a subbase for its open sets convex sets whose intersection with any maximal chain is an open set of the chain's interval topology. However, the L-topology is not always a group topology on an l-group [17].

In any lattice L, a set $\{x_\beta \mid \beta \in B\} \subseteq L$ is said to *order-converge* [3] to $x \in L$ if there exist nets $\{a_\beta\} \subseteq L$ and $\{b_\beta\} \subseteq L$ such that (1) if $\eta \leq \gamma$, then $a_\eta \leq a_\gamma$ and $b_\eta \geq b_\gamma$, (2) $a_\beta \leq x_\beta \leq b_\beta$ for all $\beta \in B$, and (3) $\bigvee_{\beta \in B} a_\beta = x = \bigwedge_{\beta \in B} b_\beta$. The *order topology* [3] then takes as its closed sets those sets which contain the the limit of any order-convergent net which is itself contained in the set. Floyd [6] gives an example of a conditionally complete vector lattice whose order topology is not a group topology. The most successful convergence topology as far as l-groups are concerned is the topology of α-convergence (Papangelou [15], [16]) which is a group and lattice topology on completely distributive l-groups ([5], [14]). (This topology is defined in §7.)

The idea of generalizing the interval topology of a totally ordered set has

Received by the editors April 21, 1972.

AMS (MOS) subject classifications (1970). Primary 06A55; Secondary 06A45, 54F05, 02A99.

Key words and phrases. Lattice-ordered group, topological group, topological lattice, α-convergence, lexico-sum, lexico-extension, cardinal product, interval topology.

(1) This paper is based on material in the author's doctoral dissertation, written under the direction of N. R. Reilly at Simon Fraser University. The author wishes to thank Dr. Reilly for his advice and encouragement during the preparation of the thesis.

gone through several metamorphoses. Frink's original idea [7] of taking closed intervals (or, equivalently, the principal ideals and dual ideals) as a subbase for the closed sets never produces a group topology on a nontotally ordered abelian l-group [12]. His *ideal topology* [8], which takes the set of completely irreducible ideals and dual ideals as a subbase for its open sets, is much better in this respect: it is a group and lattice topology for every l-group (Holley [10](2)). However as pointed out by Alo and Frink [1], it is not necessarily preserved by infinite cardinal products. The *new interval topology*, introduced in [2], takes as closed sets those sets whose intersection with an arbitrary intersection of finite unions of closed, bounded intervals is also an intersection of finite unions of closed, bounded intervals. Holley [10] has given an example of an Archimedean l-group which is not a topological group with respect to its new interval topology.

The topology introduced in this paper also generalizes the interval topology of a totally ordered set. However, the emphasis here, rather than being on the form of the interval, is on the idea that in a totally ordered set an interval "spans" the set. With this idea for a guide, we define a topology, which we call the $\mathfrak{J}$-topology, on an arbitrary l-group G. We prove that, with respect to this topology, the l-group is both a topological group and a topological lattice (§2). If G is a cardinal product of l-groups, then the $\mathfrak{J}$-topology on G is the product of the $\mathfrak{J}$-topologies on the factors (§3). The $\mathfrak{J}$-topology on a totally ordered group is the interval topology (§4). The $\mathfrak{J}$-topology is discrete if and only if G is a lexico-sum of lexico-extensions of the integers (§5). Also, in §5, we derive necessary and sufficient conditions for the $\mathfrak{J}$-topology to be Hausdorff. There are some cases when the $\mathfrak{J}$-topology is not Hausdorff; a particular example is constructed in §6. Convergence with respect to the $\mathfrak{J}$-topology is investigated in §7, and the $\mathfrak{J}$-topology is compared with the topology derived from α-convergence when G is completely distributive.

Unless otherwise mentioned, we adopt the notation of Birkhoff [3]. In particular, for G an l-group and $a \in G$, $a^+ = a \vee 0$, and $a^- = a \wedge 0$. If G is an l-group, $A, B \subseteq G$, and $a, b \in G$ with $a \leq b$, then $[a, b] = \{x \mid a \leq x \leq b\}$, $(a, b) = \{x \mid a < x < b\}$; $[a]$ is the l-subgroup generated by a, $G(a)$ is the convex l-subgroup generated by a; $A + B = \{x + y \mid x \in A, y \in B\}$, $a + B = \{a\} + B$, $A \wedge B = \{x \wedge y \mid x \in A, y \in B\}$, $A \vee B = \{x \vee y \mid x \in A, y \in B\}$, $a \wedge B = \{a\} \wedge B$, $a \vee B = \{a\} \vee B$; $A^+ = \{x \mid x \in A, x \geq 0\}$, and $A^- = \{x \mid x \in A, x \leq 0\}$. For $A \subseteq G$, $A^\perp = \{x \mid |x| \wedge |a| = 0$ for all $a \in A\}$ is a convex l-subgroup of G, and if $g \in G$, $g^\perp = \{g\}^\perp$. We denote the additive l-group of the real numbers by R, that of the rationals by Q, and that of the integers by Z. We let N denote the natural numbers.

Let G be an l-group, and T a totally ordered group. The *lexico-graphic product*, $G \overleftarrow{\times} T$, is the group $G \times T$ with order defined by: $(g, t) \leq (a, b)$ if and only

(2) The author thanks Paul Conrad for bringing this reference to his attention.

if $t < b$ or $t = b$ and $g \leq a$. If $\{G_\lambda | \lambda \in \Lambda\}$ is a collection of l-groups, the *cardinal product*, $|\Pi_{\lambda\in\Lambda}| G_\lambda$, is the group $\Pi_{\lambda\in\Lambda} G_\lambda$, with order defined by: $f \leq g$ if and only if $\lambda f \leq \lambda g$ for all $\lambda \in \Lambda$. If Λ is finite, we denote the cardinal product by $G_1 |\times| \cdots |\times| G_n$. The *cardinal sum*, $|\Sigma_{\lambda\in\Lambda}| G_\lambda$, is the convex l-subgroup of $|\Pi_{\lambda\in\Lambda}| G_\lambda$ consisting of those functions which are 0 for all but a finite number of $\lambda \in \Lambda$. By the *product topology* on a product of topological spaces, we mean the usual (Tychonoff) product topology (see [11, p. 14]). We use T_1 and T_2, or Hausdorff, to denote the usual separation axioms (see [11, p. 7]). Let $\{A_\gamma | \gamma \in \Gamma\}$ be a filter-base or a filter subbase. We denote the filter generated by $\{A_\gamma | \gamma \in \Gamma\}$ by $F(\{A_\gamma | \gamma \in \Gamma\})$. We say that two l-groups are *l-isomorphic* if there is a bijective map between them which is both a lattice and a group isomorphism. We denote l-isomorphism by $\simeq$. An l-group G is said to be *completely distributive* if, whenever $\{l_{\alpha\beta} | \alpha \in A, \beta \in B\} \subseteq L$ for arbitrary index sets A and B, the equality

$$\bigwedge_{\alpha\in A} \bigvee_{\beta\in B} l_{\alpha\beta} = \bigvee_{f\in B^A} \bigwedge_{\alpha\in A} l_{\alpha(\alpha f)}$$

holds, provided that all the indicated joins and meets exist.

Let $(G, +)$ be a group (we do not assume G abelian), and let $\mathfrak{U}$ be a topology on G. Then $(G, \mathfrak{U})$ is a *topological group* (or $\mathfrak{U}$ is a *group topology* for G) if $+: (G \times G, \mathfrak{U} \times \mathfrak{U}) \to (G, \mathfrak{U})$ and $-: (G, \mathfrak{U}) \to (G, \mathfrak{U})$ are both continuous. Let $(L, \vee, \wedge)$ be a lattice, and let $\mathfrak{U}$ be a topology on L. Then $(L, \mathfrak{U})$ is a *topological lattice* (or $\mathfrak{U}$ is a *lattice topology* for L) if $\vee: (L \times L, \mathfrak{U} \times \mathfrak{U}) \to (L, \mathfrak{U})$ and $\wedge: (L \times L, \mathfrak{U} \times \mathfrak{U}) \to (L, \mathfrak{U})$ are both continuous.

We assume that the reader is familiar with the basic properties of l-groups found in Birkhoff [3, Chapter XIII], especially those described in §§3 and 4. In particular, we will use without comment such observations as $|a| \wedge |b| = 0$ implies $a + b = b + a$, $a = a^+ + a^-$, $|a| = a^+ - a^- = a \vee (-a)$, and $a^+ \wedge (-a)^+ = 0$. The proof that an l-group G is a topological group in the $\mathfrak{T}$-topology relies on the following theorem from Husain [11, p. 46]:

Theorem A. *Let G be a group with a filter-base $\mathfrak{N}(0)$ satisfying*

(i) *each $H \in \mathfrak{N}(0)$ is symmetric,*

(ii) *for each $H \in \mathfrak{N}(0)$, there is a $K \in \mathfrak{N}(0)$ such that $K + K \subseteq H$,*

(iii) *for each $H \in \mathfrak{N}(0)$ and each $a \in G$, there is a $K \in \mathfrak{N}(0)$ such that $a + K - a \subseteq H$.*

Then there exists a unique topology $\mathfrak{T}$ on G such that, with respect to $\mathfrak{T}$, G is a topological group, and $F(\mathfrak{N}(0))$ is the set of $\mathfrak{T}$-neighbourhoods of 0.

We also use the following theorem from Husain [11, p. 48]:

Theorem B. *For a topological group G, the following statements are equivalent:*

(a) *G is a T_0-space,*
(b) *G is a T_1-space,*
(c) *G is a T_2-space,*
(d) $\bigcap \mathfrak{U} = \{0\}$, *where $\mathfrak{U}$ is a fundamental system of neighbourhoods of* 0.

2. Definition of the topology and basic theorems. We first establish some notation for the definition of the topology. Throughout this section let G be an arbitrary l-group.

For $g \in G^+$, let

$$\mathfrak{J}(g) = \{h \in G^+ \mid \text{there exists } h' \in G \text{ such that } h \wedge h' = 0 \text{ and } h \vee h' = g\}.$$

For $g \in G^+ \setminus \{0\}$, let

$$N(0, g) = [-g, g] + g^{\perp}.$$

Let

$$\mathfrak{A} = \{h \in G^+ \setminus \{0\} \mid \mathfrak{J}(h) = \{0, h\}\},$$

and

$$\mathfrak{D}_1 = \{h \in \mathfrak{A} \mid \text{there exist } h_1, h_2, \cdots \in \mathfrak{A} \text{ such that } h_1 = h,\ h_{n+1} + h_{n+1} \leq h_n, \text{ and } h_n \in h_{n+1}^{\perp\perp}\}.$$

We note that if $h \in \mathfrak{D}_1$, then there is an $l \in \mathfrak{D}_1$ such that $0 < l < l + l \leq h$ and $h \in l^{\perp\perp}$: let $l = h_2$ in the definition of $\mathfrak{D}_1$.

Proposition 2.1. *Let $h, l \in G^+ \setminus \{0\}$ be such that $h < l$. Then the following statements are equivalent:*

(i) $l \in h^{\perp\perp}$,
(ii) $l^{\perp} = h^{\perp}$,
(iii) $l^{\perp} \supseteq h^{\perp}$.

Proof. Suppose (i) holds, and let $k \in h^{\perp}$. Then since $l \in h^{\perp\perp}$, $l \wedge |k| = 0$. Hence $k \in l^{\perp}$. Thus (iii) holds.

Suppose (iii) holds. Since $0 < h < l$, $0 \leq |k| \wedge h \leq |k| \wedge l$ for all $k \in G$. If $k \in l^{\perp}$, then $|k| \wedge l = 0$. Hence $|k| \wedge h = 0$, and thus $k \in h^{\perp}$. By hypothesis, $h^{\perp} \subseteq l^{\perp}$. Thus (ii) holds.

Suppose (ii) holds. If $l^{\perp} = h^{\perp}$, $l^{\perp\perp} = h^{\perp\perp}$ and hence $l \in l^{\perp\perp} = h^{\perp\perp}$. Thus (i) holds.

Therefore,

$$\mathfrak{D}_1 = \{h \in \mathfrak{A} \mid \text{there exist } h_1, h_2, \cdots \in \mathfrak{A} \text{ such that } h_1 = h,\ h_{n+1} + h_{n+1} \leq h_n, \text{ and } h_{n+1}^{\perp} = h_n^{\perp}\}.$$

Let

$$\mathfrak{D}_2 = \{h \in G^+ \setminus \{0\} \mid [0, h] = \{0, h\}\}.$$

Clearly, $\mathfrak{D}_1 \cap \mathfrak{D}_2 = \varnothing$. Let $\mathfrak{D}^* = \mathfrak{D}_1 \cup \mathfrak{D}_2$. Clearly $\mathfrak{D}^* \subseteq \mathfrak{A}$. If $\mathfrak{D}^* \neq \varnothing$, let

$$\mathfrak{N}_1(0) = \{N(0, g) \mid g \in \mathfrak{D}_1\},$$

$$\mathfrak{N}_2(0) = \{g^\perp \mid g \in \mathfrak{D}_2\},$$

$$\mathfrak{N}_3(0) = \mathfrak{N}_1(0) \cup \mathfrak{N}_2(0),$$

$$\mathfrak{N}(0) = \left\{ \bigcap_{i=1}^{n} H_i \,\middle|\, H_i \in \mathfrak{N}_3(0) \text{ for all } i = 1, \cdots, n \right\}.$$

If $\mathfrak{D}^* = \varnothing$, let $\mathfrak{N}(0) = \{G\}$.

For the l-group, $Z \mid\times\mid R$,

$$\mathfrak{A} = \{(a, 0) \mid a > 0\} \cup \{(0, b) \mid b > 0\},$$

$$\mathfrak{D}_1 = \{(0, b) \mid b > 0\}, \qquad \mathfrak{D}_2 = \{(1, 0)\},$$

$$\mathfrak{A} \supseteq \mathfrak{D}^*, \qquad \mathfrak{A} \neq \mathfrak{D}^*.$$

$\mathfrak{N}_1(0)$ consists of sets of the form $N(0, (0, b)) = Z \times [-b, b]$. $\mathfrak{N}_2(0)$ is the set $\{\{0\} \times R\}$. So besides sets in $\mathfrak{N}_1(0)$ and $\mathfrak{N}_2(0)$, $\mathfrak{N}(0)$ includes sets of the form $N(0, (0, b)) \cap (\{0\} \times R) = \{0\} \times [-b, b]$.

Lemma 2.2. *$\mathfrak{N}(0)$ is a filter-base satisfying*

(i) *each $H \in \mathfrak{N}(0)$ is symmetric;*

(ii) *for each $H \in \mathfrak{N}(0)$, there is a $K \in \mathfrak{N}(0)$ such that $K + K \subseteq H$;*

(iii) *for each $H \in \mathfrak{N}(0)$ and each $a \in G$, there is a $K \in \mathfrak{N}(0)$ such that $a + K - a \subseteq H$.*

For the proof of Lemma 2.2, we need the following three lemmas:

Lemma 2.3. *Let $g \in G^+ \setminus \{0\}$. Let $a \in [-g, g]$ and $b \in g^\perp$.*

(i) *If $a + b \in G^+$, then $a \wedge b = 0$.*

(ii) *$a + b = b + a$.*

Proof. (i) Let $b \in [-g, g] \cap g^\perp$. Since $b \in g^\perp$, $b^+ \wedge g = 0$. Since $b \in [-g, g]$, $b^+ \wedge g = b^+$. Thus $b \leq 0$. Hence $-b \geq 0$ and, as before, $-b = (-b) \wedge g = 0$. Thus $b = 0$, and hence $[-g, g] \cap g^\perp = \{0\}$.

Since $a + b \geq 0$, $a \geq -b$ so that $a^+ \geq (-b)^+ \geq 0$. Then $(-b)^+ \in [-g, g] \cap g^\perp$. Hence $(-b)^+ = 0$ so that $b \geq 0$. Similarly, $b \geq -a$ so that $a \geq 0$. Thus $a \wedge b \geq 0$, and hence $a \wedge b \in [-g, g] \cap g^\perp$. Therefore, $a \wedge b = 0$.

(ii) Since $a \in [-g, g]$, $|a| = a \vee (-a) \leq g \vee g = g$. Thus, since $|b| \wedge g = 0$, $|a| \wedge |b| = 0$. Hence $a + b = b + a$.

Lemma 2.4. *For all* $a, k \in G$,

(i) $-a + |k| + a = |-a + k + a|$,

(ii) $a + (-a + k + a)^{\perp} - a = k^{\perp}$.

Proof.

$$-a + |k| + a = -a + k \vee (-k) + a = (-a + k + a) \vee (-a - k + a)$$

$$= (-a + k + a) \vee (-(-a + k + a)) = |-a + k + a|.$$

$$k^{\perp} = \{l \mid |l| \wedge |k| = 0\}$$

$$= \{l \mid (-a + |l| + a) \wedge (-a + |k| + a) = 0\}$$

$$= \{l \mid |-a + l + a| \wedge |-a + k + a| = 0\} \quad \text{(by (i))}$$

$$= \{l \mid -a + l + a \in (-a + k + a)^{\perp}\} = a + (-a + k + a)^{\perp} - a.$$

Lemma 2.5. *For all* $a \in G$, $g \in \mathfrak{A}$ *if and only if* $-a + g + a \in \mathfrak{A}$.

Proof. Let $g \in \mathfrak{A}$, and $b = -a + g + a$. If $l \in \mathfrak{J}(b) \setminus \{0\}$, then for some l', $l \wedge l' = 0$ and $l \vee l' = b$. Thus

$$(a + l - a) \wedge (a + l' - a) = a + l \wedge l' - a = 0,$$

$$(a + l - a) \vee (a + l' - a) = a + l \vee l' - a = a + b - a.$$

Hence $a + l - a \in \mathfrak{J}(g) \setminus \{0\}$. Since $g \in \mathfrak{A}$, $a + l - a = g$, and thus $l = b$. Therefore $b \in \mathfrak{A}$.

The converse is proved similarly.

Proof of Lemma 2.2. If $\mathfrak{D}^* = \emptyset$, clearly Lemma 2.2 holds. Hence suppose $\mathfrak{D}^* \neq \emptyset$. We note that for all $g \in G^+ \setminus \{0\}$, $0 \in g^{\perp} \subseteq [-g, g] + g^{\perp}$. Hence, for all $H \in \mathfrak{N}(0)$, $H \neq \emptyset$. By definition $\mathfrak{N}(0)$ is closed under finite intersections. Thus, $\mathfrak{N}(0)$ is a filter-base.

(i) If $H \in \mathfrak{N}_2(0)$, clearly H is symmetric.

If $H \in \mathfrak{N}_1(0)$, then $H = N(0, g)$ for $g \in \mathfrak{D}_1$. If $p \in N(0, g)$, $p = p_1 + p_2$ where $p_1 \in [-g, g]$ and $p_2 \in g^{\perp}$. By Lemma 2.3(ii), $p = p_1 + p_2 = p_2 + p_1$. Thus $-p = -p_1 - p_2$. Clearly $-p_2 \in g^{\perp}$. Since $-g \leq p_1 \leq g$, then $g \geq -p_1 \geq -g$. Thus $-p \in N(0, g)$ and hence H is symmetric.

Thus, for all $H \in \mathfrak{N}_3(0)$, H is symmetric, and hence for all $H \in \mathfrak{N}(0)$, H is symmetric.

(ii) If $L \in \mathfrak{N}_2(0)$, let $L^* = L$. Then clearly, $L^* + L^* \subseteq L$.

If $L \in \mathfrak{N}_1(0)$, $L = N(0, g)$ for $g \in \mathfrak{D}_1$. Since $g \in \mathfrak{D}_1$, there is an $b \in \mathfrak{D}_1$ such that $b + b \leq g$ and $b^{\perp} \subseteq g^{\perp}$. Let $L^* = N(0, b)$. Since $b \in \mathfrak{D}_1$, $L^* \in \mathfrak{N}_1(0)$. If $p \in L^* + L^*$, then $p = a + b + c + d$ where $a, c \in [-b, b]$, and $b, d \in b^{\perp}$. By Lemma 2.3(ii), $p = (a + c) + (b + d)$. Since $b^{\perp} \subseteq g^{\perp}$, $b + d \in g^{\perp}$. And since $a, c \in [-b, b]$, $a + c \in [-b - b, b + b] \subseteq [-g, g]$. Thus, $p \in L$. Therefore, $L^* + L^* \subseteq L$.

If $H \in \mathfrak{N}(0)$, $H = \bigcap_{i=1}^{n} L_i$ for $L_i \in \mathfrak{N}_3(0)$. Let $K = \bigcap_{i=1}^{n} L_i^*$. Then $K \in \mathfrak{N}(0)$. If $l \in K + K$, $l = l_1 + l_2$ for $l_1, l_2 \in \bigcap_{i=1}^{n} L_i^*$. Then $l_1, l_2 \in L_i^*$ for all $i = 1, \cdots, n$, and by definition of L_i^*, $l_1 + l_2 \in L_i^* + L_i^* \subseteq L_i$ for all $i = 1, \cdots, n$. Thus $l \in H$. Therefore, $K + K \subseteq H$.

(iii) Let $a \in G$.

If $H \in \mathfrak{N}_2(0)$, then $H = b^\perp$ for $b \in \mathfrak{D}_2$. Let $H^a = (-a + b + a)^\perp$. Clearly $-a + b + a \in \mathfrak{D}_2$. Thus, $H^a \in \mathfrak{N}_2(0)$. By Lemma 2.4(ii), $a + H^a - a = H$.

If $H \in \mathfrak{N}_1(0)$, then $H = N(0, g)$ for $g \in \mathfrak{D}_1$. Let $b = -a + g + a$. Since $g \in \mathfrak{D}_1$, there are $g_1, g_2, \cdots \in \mathfrak{A}$ such that $g = g_1$, $g_{n+1} + g_{n+1} \leq g_n$, and $g_{n+1}^\perp \subseteq g_n^\perp$. Consider $-a + g_1 + a, -a + g_2 + a, \cdots$. By definition, $-a + g_1 + a = b$. By Lemma 2.5, $-a + g_n + a \in \mathfrak{A}$ for all n. Since $g_{n+1} + g_{n+1} \leq g_n$, $(-a + g_{n+1} + a) + (-a + g_{n+1} + a) \leq -a + g_n + a$. Since $g_{n+1}^\perp \subseteq g_n^\perp$, $-a + g_{n+1}^\perp + a \subseteq -a + g_n^\perp + a$. Hence by Lemma 2.4(ii), $(-a + g_{n+1} + a)^\perp \subseteq (-a + g_n + a)^\perp$. Thus $b \in \mathfrak{D}_1$, and therefore, $H^a = N(0, b) \in \mathfrak{N}_1(0)$. Let $p \in a + H^a - a$. Then $p = a + b_1 + b_2 - a$ for $b_1 \in [-b, b]$, and $b_2 \in b^\perp$. Since $b_1 \in [-b, b]$ and $b = -a + g + a$, then $a + b_1 - a \in [-g, g]$. By Lemma 2.4(ii), $g^\perp = a + b^\perp - a$. Since $b_2 \in b^\perp$, $a + b_2 - a \in g^\perp$. Therefore, $p = (a + b_1 - a) + (a + b_2 - a) \in H$. Hence $a + H^a - a \subseteq H$.

Let $H \in \mathfrak{N}(0)$. Then $H = \bigcap_{i=1}^{n} H_i$ for $H_i \in \mathfrak{N}_3(0)$. Let $K = \bigcap_{i=1}^{n} H_i^a$. Then $K \in \mathfrak{N}(0)$. Since $\bigcap_{i=1}^{n} H_i^a \subseteq H_i^a$ for all $i = 1, \cdots, n$,

$$a + K - a = a + \bigcap_{i=1}^{n} H_i^a - a \subseteq \bigcap_{i=1}^{n} (a + H_i^a - a) \subseteq \bigcap_{i=1}^{n} H_i = H.$$

For $g \in G$, let $\mathfrak{N}(g) = \{g + H \mid H \in \mathfrak{N}(0)\}$. Let $\mathfrak{T} = \{W \subseteq G \mid \text{for all } x \in W,\ W \in F(\mathfrak{N}(x))\}$.

Theorem 2.6. *G is a topological group in the topology $\mathfrak{T}$.*

Proof. The theorem follows immediately from Lemma 2.2 and Theorem A.

We call $\mathfrak{T}$ the *$\mathfrak{T}$-topology* on G.

Turning to the question of whether G is a topological lattice, we first establish a criterion for an l-group to be a topological lattice.

Let G be an l-group. Let $\mathfrak{V}$ be a collection of subsets of G such that $0 \in V$ for all $V \in \mathfrak{V}$. Let

$$\mathfrak{U} = \{W \subseteq G \mid \text{for all } x \in W, \text{ there exists } V \in \mathfrak{V} \text{ such that } x \in x + V \subseteq W\}.$$

Lemma 2.7. *If $\mathfrak{V}$ is a filter-base, $\mathfrak{U}$ is a topology on G.*

Proof. Let $\{U_\alpha \mid \alpha \in A\} \subseteq \mathfrak{U}$. If $x \in \bigcup_{\alpha \in A} U_\alpha$, then $x \in U_\beta$ for some $\beta \in A$. Hence there is a $V \in \mathfrak{V}$ such that $x \in x + V \subseteq U_\alpha \subseteq \bigcup_{\alpha \in A} U_\alpha$. Let $\{U_1, \cdots, U_n\} \subseteq \mathfrak{U}$. If $x \in \bigcap_{i=1}^{n} U_i$, then $x \in U_i$ for all i. Hence for all i, there is a $V_i \in \mathfrak{V}$ such that $x \in x + V_i \subseteq U_i$. Since $\mathfrak{V}$ is a filter-base, there is a $V^* \in \mathfrak{V}$ such that $V^* \subseteq \bigcap_{i=1}^{n} V_i$. Thus

$$x \in x + V^* \subseteq x + \bigcap_{i=1}^{n} V_i \subseteq \bigcap_{i=1}^{n} (x + V_i) \subseteq \bigcap_{i=1}^{n} U_i.$$

Thus $\mathfrak{U}$ is a topology on G.

Lemma 2.8. *Suppose every $V \in \mathfrak{V}$ is a convex sublattice of G. Then for all $r, s \in G$ and $V \in \mathfrak{V}$,*

(i) $(r + V) \wedge (s + V) = (r \wedge s) + V$,

(ii) $(r + V) \vee (s + V) = (r \vee s) + V$.

Proof. (i) Let $x \in V$. Then

$$(r \wedge s) + x = (r + x) \wedge (s + x) \in (r + V) \wedge (s + V).$$

Thus $(r \wedge s) + V \subseteq (r + V) \wedge (s + V)$.

Let $x, y \in V$. Then

$$\begin{aligned}(r + (x \wedge y)) \wedge (s + (x \wedge y)) &\leq (r + x) \wedge (s + y) \\ &\leq (r + (x \vee y)) \wedge (s + (x \vee y)).\end{aligned}$$

Hence

$$x \wedge y \leq -(r \wedge s) + (r + x) \wedge (s + y) \leq x \vee y.$$

Since V is a convex sublattice of G, $-(r \wedge s) + (r + x) \wedge (s + y) \in V$, i.e., $(r + x) \wedge (s + y) \in (r \wedge s) + V$. Thus $(r + V) \wedge (s + V) \subseteq (r \wedge s) + V$. Therefore, (i) holds.

Similarly (ii) holds.

Theorem 2.9. *Suppose $\mathfrak{V}$ is a filter-base satisfying*

(a) *each $V \in \mathfrak{V}$ is a convex sublattice of G,*

(b) *if $V \in \mathfrak{V}$, there is a $U \in \mathfrak{U}$ such that $0 \in U \subseteq V$.*

Then G is a topological lattice with respect to $\mathfrak{U}$.

Proof. Let $W \in \mathfrak{U}$. Let $g, h \in G$ be such that $g \wedge h \in W$. Since $W \in \mathfrak{U}$, there exists $V \in \mathfrak{V}$ such that $(g \wedge h) + V \subseteq W$. By (b) there exists $U \in \mathfrak{U}$ such that $0 \in U \subseteq V$. Let $W(g) = q + U$ and $W(h) = h + U$. Clearly $W(g), W(h) \in \mathfrak{U}$. Since $0 \in U$, we have that $g \in W(g)$ and $h \in W(h)$. By (a) and Lemma 2.8,

$$W(g) \wedge W(h) \subseteq (g + V) \wedge (h + V) = (g \wedge h) + V \subseteq W.$$

Similarly, if $W \in \mathfrak{U}$ and $g, h \in W$ are such that $g \vee h \in W$, then we may construct sets $W(g), W(h) \in \mathfrak{U}$ such that $g \in W(g)$, $h \in W(h)$, and $W(g) \vee W(h) \subseteq W$.

We now apply Theorem 2.9 in the case of the $\mathfrak{J}$-topology. Let $\mathfrak{J}$ be the $\mathfrak{J}$-topology on an l-group G. For $A \subseteq G$ let the interior of A be denoted by $\operatorname{Int}(A)$. We note that the proof of the following result depends on Lemma 2.2(ii).

Proposition 2.10. *For all $A \subseteq G$,*

$$\operatorname{Int}(A) = \{r \in G \mid \textit{there exists } H \in \mathfrak{N}(0) \textit{ such that } r + H \subseteq A\}.$$

Proof. Let $S = \{r \in G \mid \text{there exists } H \in \mathfrak{N}(0) \text{ such that } r + H \subseteq A\}$. Since $0 \in H$ for all $H \in \mathfrak{N}(0)$, $S \subseteq A$. Let $r \in S$ and $H \in \mathfrak{N}(0)$ be such that $r + H \subseteq A$. Let $H^* \in \mathfrak{N}(0)$ be such that $H^* + H^* \subseteq H$. Then for all $h \in H^*$,

$$r + h + H^* \subseteq r + H^* + H^* \subseteq r + H \subseteq A.$$

Hence $r + h \in S$, and so $r + H^* \subseteq S$. Thus $S \in \mathfrak{T}$, and hence $S \subseteq \operatorname{Int}(A)$. If $s \in \operatorname{Int}(A)$, then there is an $H \in \mathfrak{N}(0)$ such that $s + H \subseteq \operatorname{Int}(A) \subseteq A$. Thus $s \in S$. Therefore, $S = \operatorname{Int}(A)$.

Corollary 2.11. *For all $H \in \mathfrak{N}(0)$, $0 \in \operatorname{Int}(H)$.*

Proof. For all $H \in \mathfrak{N}(0)$, $0 + H \subseteq H$.

Lemma 2.12. *Let $0 \in A \subseteq B \subseteq G$. Suppose A is a convex sublattice of G. Then $A + B^{\perp}$ is a convex sublattice. In particular, for all $H \in \mathfrak{N}(0)$, H is a convex sublattice of G.*

Proof. Suppose $0 \leq y \leq l_1 + l_2$ for $l_1 \in A^+$, $l_2 \in (B^{\perp})^+$. Since $A \subseteq B$, $A^{\perp} \supseteq B^{\perp}$, and hence $l_1 \wedge l_2 = 0$. Since A is convex, $y \wedge l_1 \in A^+$. Similarly, $y \wedge l_2 \in (B^{\perp})^+$. Further,

$$(y \wedge l_1) \wedge (y \wedge l_2) = y \wedge (l_1 \wedge l_2) = 0.$$

Hence

$$(y \wedge l_1) + (y \wedge l_2) = (y \wedge l_1) \vee (y \wedge l_2) = y \wedge (l_1 \vee l_2) = y.$$

Thus $y = y_1 + y_2$ where $y_1 \in A^+$, $y_2 \in (B^{\perp})^+$. Similarly, if $l_1 + l_2 \leq y \leq 0$ for $l_1 \in A^-$, $l_2 \in (B^{\perp})^-$, then $y = y_1 + y_2$ for $y_1 \in A^-$ and $y_2 \in (B^{\perp})^-$.

Let $y \in G$ be such that $p \leq y \leq q$, for $p, q \in A + B^{\perp}$. Then $p = a + b$ and $q = c + d$, for $a, c \in A$ and $b, d \in B^{\perp}$. Then

$$a^- + b^- \leq p^- \leq y^- \leq 0 \leq y^+ \leq q^+ \leq c^+ + d^+.$$

Since $0, a, c \in A$ and A is a sublattice, $a^- \in A^-$ and $c^+ \in A^+$. Similarly, $b^- \in (B^{\perp})^-$ and $d^+ \in (B^{\perp})^+$. Therefore,

$$y^+ = u + v \quad \text{for } u \in A^+ \text{ and } v \in (B^{\perp})^+,$$

$$y^- = x + w \quad \text{for } x \in A^- \text{ and } w \in (B^{\perp})^-.$$

By Lemma 2.3 (ii), $y = u + v + w + x = (u + w) + (v + x)$. Clearly $v + x \in B^{\perp}$. Since $x \leq 0 \leq u$, $x \leq u + x \leq u$, and since A is convex and $x, u \in A$, then $u + x \in A$. Therefore, $y \in A + B^{\perp}$ and hence $A + B^{\perp}$ is convex.

We note that for $a, b, c, d \in G$,

$$(a \wedge c) + (b \wedge d) \leq (a + b) \wedge (c + d) \leq (a + b) \vee (c + d)$$
$$\leq (a \vee c) + (b \vee d).$$

Thus, since A and $B^{\perp}$ are convex sublattices, $A + B^{\perp}$ is a sublattice.

If $H \in \mathfrak{N}_1(G)$, $H = [-g, g] + g^{\perp}$ for $g \in \mathfrak{D}_1$. Let $A = [-g, g] = B$. Then $0 \in A \subseteq B$. Clearly A is a convex sublattice. Since $B^{\perp} = g^{\perp}$, H is a convex sublattice of G. Clearly if $H \in \mathfrak{N}_2(0)$, H is a convex sublattice. If $\mathfrak{D}^* \neq \emptyset$ and if $H \in \mathfrak{N}(0)$, then $H = \bigcap_{i=1}^{n} H_i$ for $H_i \in \mathfrak{N}_3(0)$. Since each H_i is a convex sublattice of G, H is a convex sublattice. If $\mathfrak{D}^* = \emptyset$, $\mathfrak{N}(0) = \{G\}$ and clearly G is a convex sublattice of G.

Theorem 2.13. *If G is an l-group, then the $\mathfrak{T}$-topology on G is a lattice topology.*

Proof. By Lemma 2.2, $\mathfrak{N}(0)$ is a filter-base. By Lemma 2.12, each $H \in \mathfrak{N}(0)$ is a convex sublattice of G, and thus $\mathfrak{N}(0)$ satisfies (a) of Theorem 2.9. By Corollary 2.11, if $H \in \mathfrak{N}(0)$, then $0 \in \operatorname{Int}(H) \subseteq H$. Thus, since $\operatorname{Int}(H) \in \mathfrak{T}$, we have that $\mathfrak{N}(0)$ satisfies (b) of Theorem 2.9. Therefore, by Theorem 2.9, the $\mathfrak{T}$-topology on G is a lattice topology.

3. Cardinally ordered products. The main result of this section is the following:

Theorem 3.1. *Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a collection of l-groups. For $\lambda \in \Lambda$, let $\mathfrak{T}_\lambda$ be the $\mathfrak{T}$-topology on G_λ. Let $\mathfrak{G}$ be an l-subgroup of $|\Pi_{\lambda\in\Lambda}|\, G_\lambda$, which contains $|\Sigma_{\lambda\in\Lambda}|\, G_\lambda$, and let $\mathfrak{T}$ be the $\mathfrak{T}$-topology on $\mathfrak{G}$. Let $\mathfrak{P}$ be the topology that $\mathfrak{G}$ inherits from the product topology on $|\Pi_{\lambda\in\Lambda}|\, G_\lambda$. Then $\mathfrak{T} = \mathfrak{P}$.*

The proof of Theorem 3.1 is very long, but straightforward. We will not give it here, but instead we refer the interested reader to [17], where the same result is proved for a somewhat more general topology (cf. §8). For use in the sequel, we will introduce some notation and list four of the lemmas which are used in the proof of Theorem 3.1.

In addition to the notation introduced in the statement of the theorem, we let $p_\gamma\colon \mathfrak{G} \to G_\gamma$ denote the γth projection. Then $U \in \mathfrak{P}$ if and only if for all $f \in U$, there exist $U_{\gamma_i} \in \mathfrak{T}_{\gamma_i}$, $i = 1, \cdots, n$, such that $f \in \bigcap_{i=1}^{n} p_{\gamma_i}^{-1}(U_{\gamma_i}) \subseteq U$. To denote the sets $\mathfrak{A}$, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}^*$, $\mathfrak{N}_1(0)$, $\mathfrak{N}_2(0)$, $\mathfrak{N}_3(0)$, and $\mathfrak{N}(0)$ in G_λ, we use $\mathfrak{A}_\lambda$, $\mathfrak{D}_{1\lambda}$, $\mathfrak{D}_{2\lambda}$, etc. To denote them in $\mathfrak{G}$, we use $\mathfrak{A}$, $\mathfrak{D}_1$, $\mathfrak{D}_2$, etc. For $g \in G_\lambda$, we denote $N(0, g)$ by $N_\lambda(0, g)$ and for $f \in \mathfrak{G}$, we use $N(0, f)$. Then $U \in \mathfrak{T}$ if and only if for all $f \in U$, there exists $\mathfrak{H} \in \mathfrak{N}(0)$ such that $f \in f + \mathfrak{H} \subseteq U$. For $h_\gamma \in G_\gamma$, let $\overline{h_\gamma} \in |\Sigma_{\lambda\in\Lambda}|\, G_\lambda \subseteq \mathfrak{G}$ be defined by

$$\lambda\overline{h_\gamma} = \begin{cases} 0 & \text{if } \lambda \neq \gamma, \\ h_\gamma & \text{if } \lambda = \gamma. \end{cases}$$

Lemma 3.2. *If $h_\gamma \in \mathfrak{D}_{2\gamma}$, then $\overline{h_\gamma} \in \mathfrak{D}_2$.*

Lemma 3.3. *If* $b_\gamma \in \mathfrak{D}_{1\gamma}$, *then* $\overline{b_\gamma} \in \mathfrak{D}_1$.

Lemma 3.4. *Let* $b \in \mathfrak{D}_2$. *Then there exist* $\gamma \in \Lambda$ *and* $b_\gamma \in \mathfrak{D}_{2\gamma}$ *such that* $\overline{b_\gamma} = b$ *and* $p_\gamma^{-1}(b_\gamma^\perp) = b^\perp$.

Lemma 3.5. *Let* $b \in \mathfrak{D}_1$. *Then there exist* $\gamma \in \Lambda$ *and* $b_\gamma \in \mathfrak{D}_{1\gamma}$ *such that* $\overline{b_\gamma} = b$ *and* $p^{-1}(N_\gamma(0, b_\gamma)) = N(0, b)$.

4. Totally ordered groups. Let T be a totally ordered group. Let $\mathfrak{L}$ denote the interval topology on T. Then $U \in \mathfrak{L}$ if and only if for all $x \in U$, there exist $a, b \in T$ such that $x \in (a, b) \subseteq U$. In this section we prove that $\mathfrak{L}$ is the $\mathfrak{T}$-topology on T.

Lemma 4.1. $T^+\backslash\{0\} = \mathfrak{A}$.

Proof. Clearly $\mathfrak{A} \subseteq T^+\backslash\{0\}$. Let $t \in T^+\backslash\{0\}$. If $s \in \mathfrak{T}(t)$, there is an $s' \in T$ such that $s \vee s' = t$ and $s \wedge s' = 0$. Since T is totally ordered, $s \wedge s' = s$ or $s \vee s' = s$. Hence either $s = 0$ or $s = t$. Thus, $t \in \mathfrak{A}$.

Lemma 4.2. *Let* $g \in T^+\backslash\{0\}$. *Then there exists an* $h \in T$ *such that* $0 < h < h + h \leq g$ *if and only if* $(0, g) \neq \emptyset$.

Proof. If there exists an $h \in T$ such that $0 < h + h \leq g$, then clearly $(0, g) \neq \emptyset$. Otherwise, let $h' \in (0, g)$. If $h' + h' \leq g$, we are done. Suppose $h' + h' \not\leq g$. Then since T is totally ordered, $h' + h' > g$. Let $h = g - h'$. Since $h' + h' > g$, $-h' + g - h' < 0$, and hence $h + h = g - h' + g - h' \leq g$.

Theorem 4.3. *If* $\mathfrak{T}$ *is the* $\mathfrak{T}$-*topology on* T, *then* $\mathfrak{T} = \mathfrak{L}$.

Proof. (i) Suppose there is an $h \in T^+\backslash\{0\}$ such that $[0, h] = \{0, h\}$. Then $h \in \mathfrak{D}_2$. Thus $h^\perp \in \mathfrak{N}(0)$. Since T is totally ordered and $h \neq 0$, $h^\perp = \{0\}$. Since $\{0\} \subseteq \operatorname{Int}(h^\perp) \subseteq h^\perp = \{0\}$, $\mathfrak{T}$ is discrete. But $\{0\} = (-h, h) \in \mathfrak{L}$ and hence $\mathfrak{L}$ is discrete. So $\mathfrak{T} = \mathfrak{L}$.

(ii) Suppose that for all $t \in T^+\backslash\{0\}$, $(0, t) \neq \emptyset$. If $t \in T^+\backslash\{0\}$, by Lemma 4.2, there is a $t_2 \in T$ such that $0 < t_2 < t_2 + t_2 \leq t$. By Lemma 4.1, $t_2 \in \mathfrak{A}$ and $t \in \mathfrak{A}$. Since T is totally ordered and $t_2 \neq 0 \neq t$, $t_2^\perp = \{0\} = t^\perp$. Therefore, since for all $t \in T^+\backslash\{0\}$ $(0, t) \neq \emptyset$, for all $t \in T^+\backslash\{0\}$ there exist $t_1, t_2, \cdots \in \mathfrak{A}$ such that $t_1 = t$, $t_{n+1} + t_{n+1} \leq t_n$, and $t_n^\perp = t_{n+1}^\perp$. That is, $T^+\backslash\{0\} \subseteq \mathfrak{D}_1$. Thus $T^+\backslash\{0\} = \mathfrak{D}_1$. Hence

$$\mathfrak{N}(0) = \mathfrak{N}_1(0) = \{[-t, t] \mid t \in T^+\backslash\{0\}\}.$$

Let $U \in \mathfrak{L}$. Let $u \in U$. Then there exist $a, b \in T$ such that $u \in (a, b) \subseteq U$. Let $c = (-u + b) \wedge (-a + u)$. Since $a < u < b$, then $c > 0$. Hence there is a $c' \in T$ such that $0 < c' < c$. Then $u \in [u - c', u + c'] \subseteq (u - c, u + c)$. But $u + c \leq u + (-u + b) = b$ and $u - c \geq u + (-u + a) = a$. Thus $u \in u + [-c', c'] \subseteq (a, b) \subseteq U$. Since $c' > 0$, $c' \in \mathfrak{D}_1$. So $u + [-c', c'] \in \mathfrak{N}(u)$. Since u was arbitrary in U, $U \in \mathfrak{T}$.

Let $U \in \mathfrak{T}$. Let $u \in U$. Then there is a $g \in T^+ \setminus \{0\}$ such that $u \in u + [-g, g] \subseteq U$. Hence $u \in (u - g, u + g) \subseteq U$. Since u was arbitrary in U, $U \in \mathfrak{L}$.

The following corollary is an immediate consequence of Theorem 3.1 and Theorem 4.3.

Corollary 4.4. *If G is a cardinal product of totally ordered groups, then the $\mathfrak{T}$-topology on G is the product of the interval topologies on the factors.*

5. Groups with Hausdorff $\mathfrak{T}$-topology. Recall (Conrad [4]) that an l-group L is a *lexico-extension* of an l-group S if S is an l-ideal of L, L/S is a totally ordered group, and every positive element in $L \setminus S$ exceeds every element in S. If $A_1, \cdots, A_n$ are totally ordered groups, then by a finite alternating sequence of cardinal summations and lexico-extensions, we can construct l-groups from the A_i, in which each A_i is used exactly once to make a cardinal extension and in which the lexico-extensions are arbitrary. Such l-groups are *lexico-sums* of the A_i.

The first theorem of this section characterizes those l-groups having discrete $\mathfrak{T}$-topology as lexico-sums of lexico-extensions of the integers.

Lemma 5.1. $\{0\} \in \mathfrak{N}(0)$ *if and only if there exists* $\{b_1, \cdots, b_n\} \subseteq \mathfrak{D}_2$ *such that* $(\bigvee_{i=1}^n b_i)^{\perp\perp} = G$.

Proof. Suppose there exists $\{b_1, \cdots, b_n\} \subseteq \mathfrak{D}_2$ such that $(\bigvee_{i=1}^n b_i)^{\perp\perp} = G$. Then $b_i^\perp \in \mathfrak{N}_2(0)$ for all $i = 1, \cdots, n$ and hence $\bigcap_{i=1}^n (b_i^\perp) \in \mathfrak{N}_3(0)$. But $\bigcap_{i=1}^n (b_i^\perp) = (\bigvee_{i=1}^n b_i)^\perp = \{0\}$ since $(\bigvee_{i=1}^n b_i)^{\perp\perp} = G$. Thus $\{0\} \in \mathfrak{N}_3(0)$.

Suppose that for all $\{b_1, \cdots, b_n\} \subseteq \mathfrak{D}_2$, $(\bigvee_{i=1}^n b_i)^{\perp\perp} \neq G$. Then for all $\{b_i, \cdots, b_n\} \subseteq \mathfrak{D}_2$, $((\bigvee_{i=1}^n b_i)^\perp)^+ \setminus \{0\} \neq \varnothing$. Let $\{H_1, \cdots, H_n\} \subseteq \mathfrak{N}_3(0)$ and consider $\bigcap_{i=1}^n H_i$. Without loss of generality, $H_1 = b^\perp$ for $b = \bigvee_{j=1}^m b_i$ where $\{b_1, \cdots, b_m\} \subseteq \mathfrak{D}_2$, and for $i = 2, \cdots, n$, $H_i = N(0, f_i)$ for $f_i \in \mathfrak{D}_1$. Let $k \in (b^\perp)^+ \setminus \{0\}$, and define $l_2, \cdots, l_n \in G$ as follows:

$$l_2 = \begin{cases} k \wedge f_2 & \text{if } k \wedge f_2 > 0, \\ k & \text{if } k \wedge f_2 = 0, \end{cases}$$

$$l_i = \begin{cases} l_{i-1} \wedge f_i & \text{if } l_{i-1} \wedge f_i > 0, \\ l_{i-1} & \text{if } l_{i-1} \wedge f_i = 0, \end{cases}$$

for $i = 3, \cdots, n$. Clearly $0 < l_n \leq l_{n-1} \leq \cdots \leq l_2 \leq k$. Thus $l_n \in H_1$. Let $3 \leq i \leq n$. Then if $l_{i-1} \wedge f_i > 0$, $l_i = l_{i-1} \wedge f_i \in [-f_i, f_i] \subseteq H_i$, and if $l_{i-1} \wedge f_i = 0$, $l_i = l_{i-1} \in f_i^\perp \subseteq H_i$. If $k \wedge f_2 > 0$, $l_2 \in [-f_2, f_2] \subseteq H_2$; if $k \wedge f_2 = 0$, $l_2 \in f_2^\perp \subseteq H_2$. Thus, since $l_n \leq l_i$ for all $i = 2, \cdots, n$, $0 < l_n \in \bigcap_{i=1}^n H_i$ and therefore $\bigcap_{i=1}^n H_i \neq \{0\}$.

Corollary 5.2. *G has discrete $\mathfrak{J}$-topology if and only if there exists $\{h_1, \cdots, h_n\} \subseteq \mathfrak{D}_2$ such that $G = (\bigvee_{i=1}^n h_i)^{\perp\perp}$.*

Lemma 5.3. *If $h \in \mathfrak{D}_2$, then $h^{\perp\perp}$ is a lexico-extension of $[h] \simeq Z$.*

Proof. Since $h \in \mathfrak{D}_2$, $[h] \simeq Z$ and also $[h]$ is a normal l-subgroup of $h^{\perp\perp}$. Let $k, l \in h^{\perp\perp}$. If $(l-k)^+ > 0$, $(l-k)^+ \wedge h = h$, and if $(-(l-k))^+ > 0$, $(-(l-k))^+ \wedge h = h$. Thus, since $(l-k)^+ \wedge (-(l-k))^+ = 0$, $(l-k)^+ = 0$ or $(-(l-k))^+ = 0$. Thus, either $l \leq k$ or $k \leq l$, i.e., $h^{\perp\perp}$ is totally ordered. Thus, $G(h)$ is totally ordered so that since $h \in \mathfrak{D}_2$, $G(h) = [h]$ and $[h]$ is an l-ideal of $h^{\perp\perp}$. Further, since $h^{\perp\perp}$ is totally ordered, each element $0 < a \in h^{\perp\perp} \backslash [h]$ exceeds every element of $[h]$. Hence, by [4, Lemma 1.1], $h^{\perp\perp}$ is a lexico-extension of $[h]$.

Lemma 5.4. *If $\{h_1, \cdots, h_n\} \subseteq \mathfrak{D}_2$, then $(\bigvee_{i=1}^n h_i)^{\perp\perp}$ is a lexico-sum of $h_1^{\perp\perp}, \cdots, h_n^{\perp\perp}$.*

Proof. If $x \in ((\bigvee_{i=1}^n h_i)^{\perp\perp})^+ \backslash \{0\}$, then clearly $x \wedge h_i > 0$ for at least one h_i. Hence $(\bigvee_{i=1}^n h_i)^{\perp\perp}$ cannot contain more than n disjoint elements. But $\{h_1, \cdots, h_n\} \subseteq (\bigvee_{i=1}^n h_i)^{\perp\perp}$ is disjoint since $\{h_1, \cdots, h_n\} \subseteq \mathfrak{D}_2$. For each $i = 1, \cdots, n$, let A_i be the subgroup of $(\bigvee_{i=1}^n h_i)^{\perp\perp}$ generated by $\{x \in (\bigvee_{i=1}^n h_i)^{\perp\perp} \mid x \wedge h_j = 0$ for all $j \neq i\} = \mathfrak{Y}$. Then by [2, Theorem 1], $(\bigvee_{i=1}^n h_i)^{\perp\perp}$ is a lexico-sum of the totally ordered groups A_i.

If $x \in (h_i^{\perp\perp})^+$, then $x \wedge h_j = 0$ for $j \neq i$. Hence $x \in A_i$ and thus $h_i^{\perp\perp} \subseteq A_i$.

If $x \wedge h_j = 0 = y \wedge h_j$ for all $j \neq i$, then $(x+y) \wedge h_j = 0$ for all $j \neq i$. Thus $\mathfrak{Y}$ is a convex subsemigroup of positive elements of $(\bigvee_{i=1}^n h_i)^{\perp\perp}$ that contains 0 and by [4, Lemma 2.3], $A_i^+ = \{x \in (\bigvee_{i=1}^n h_i)^{\perp\perp} \mid x \wedge h_j = 0$ for all $j \neq i\}$. If $y \in (h_i^\perp \cap (\bigvee_{i=1}^n h_i)^{\perp\perp}) \backslash \{0\}$, then clearly $|y| \wedge h_j > 0$ for some $j \neq i$. Hence $A_i \cap h_i^\perp = \{0\}$. Let $x \in A_i^+$. If $k \in h_i^\perp$, then $|k| \wedge x \in A_i$ since A_i is convex and $|k| \wedge x \in h_i^\perp$ since $h_i^\perp$ is convex. Thus $|k| \wedge x = 0$, and hence $x \in h_i^{\perp\perp}$. Therefore $A_i \subseteq h_i^{\perp\perp}$, and hence $h_i^{\perp\perp} = A_i$.

Theorem 5.5. *An l-group G has discrete $\mathfrak{J}$-topology if and only if G is l-isomorphic to a lexico-sum of lexico-extensions of Z.*

Proof. Suppose G is l-isomorphic to a lexico-sum of lexico-extensions of Z. Then there exists $n \in N$ and l-ideals $A_1, \cdots, A_n \subseteq G$ such that G is a lexico-sum of $A_1, \cdots, A_n$ and each A_i is l-isomorphic to a lexico-extension of Z. Let $h_i \in A_i$ correspond to $1 \in Z$ under the l-isomorphism. Then $h_i \in \mathfrak{D}_2$. Clearly $(\bigvee_{i=1}^n h_i)^\perp = 0$, and hence $(\bigvee_{i=1}^n h_i)^{\perp\perp} = G$. Then by Corollary 5.2, G has discrete $\mathfrak{J}$-topology.

Suppose G has discrete $\mathfrak{J}$-topology. Then by Corollary 5.2, $G = (\bigvee_{i=1}^n h_i)^{\perp\perp}$ for $\{h_1, \cdots, h_n\} \subseteq \mathfrak{D}_2$. By Lemma 5.3, $h_i^{\perp\perp}$ is l-isomorphic to a lexico-extension of

Z. By Lemma 5.4, G is a lexico-sum of $b_1^{\perp\perp}, \cdots, b_n^{\perp\perp}$.

The last two results of this section give necessary and sufficient conditions for the $\mathfrak{T}$-topology to be Hausdorff.

Proposition 5.6. *If G is an l-group with $\mathfrak{T}$-topology $\mathfrak{T}$, then the following are equivalent*:

(a) $\mathfrak{T}$ *is* T_2,

(b) *for all* $g \in G^+\backslash\{0\}$, *there exists* $H \in \mathfrak{N}(0)$ *such that* $g \notin H$.

Proof. By Theorem B, (a) is equivalent to

(b′) $\bigcap \mathfrak{N}(0) = \{0\}$.

Clearly (b′) implies (b). Conversely, suppose that (b) holds, and let $g \in \bigcap \mathfrak{N}(0)$. If $g \neq 0$, $|g| > 0$. Hence $|g| \notin \bigcap \mathfrak{N}(0)$ by (b). But by Lemmas 2.2 and 2.12, every $H \in \mathfrak{N}(0)$ is a symmetric sublattice of G. Hence since $g \in \bigcap \mathfrak{N}(0)$, $|g| = g \vee (-g) \in \bigcap \mathfrak{N}(0)$. This is a contradiction. Therefore $g = 0$, and (b′) holds.

Corollary 5.7. *If $\mathfrak{D}^* \cap [0, g] \neq \emptyset$ for all $g \in G^+\backslash\{0\}$, then G has Hausdorff $\mathfrak{T}$-topology.*

Proof. Let $g \in G^+\backslash\{0\}$. Let $b' \in \mathfrak{D}^* \cap [0, g]$. If $b' \in \mathfrak{D}_1$, let $b \in \mathfrak{D}_1$ be such that $b + b \leq b'$ and $b' \in b^{\perp\perp}$, and let $H = N(0, b) \in \mathfrak{N}(0)$. If $b' \in \mathfrak{D}_2$, let $b = b'$ and $H = b^\perp \in \mathfrak{N}(0)$. Then by our choice of b, $g \notin H$. Thus, by Proposition 5.6, G has Hausdorff $\mathfrak{T}$-topology.

Example 6.2 in the next section shows that the converse of Corollary 5.7 fails to hold in general. However, we do have Theorem 5.9 below.

Let $g \in G^+\backslash\{0\}$. For $b \in \mathfrak{T}(g)$, let

$$M(g, b) = G^+\backslash[(b, \infty) \cap (b')^\perp]$$

where $b \wedge b' = 0$ and $b \vee b' = g$. We note that $M(g, 0) = G^+\backslash(g^\perp)$, and $M(g, g) = G^+\backslash(g, \infty)$.

Lemma 5.8. *Let $g \in G^+\backslash\{0\}$. If $\mathfrak{D}^* \cap (\bigcap_{b \in \mathfrak{T}(g)} M(g, b)) \neq \emptyset$, then there exists $H \in \mathfrak{N}_3(0)$ such that $g \notin H$.*

Proof. Let $l \in \mathfrak{D}^* \cap (\bigcap_{b \in \mathfrak{T}(g)} M(g, b))$. If $l \in \mathfrak{D}_2$, let $H = l^\perp$. If $g \wedge l = 0$, $l \notin M(g, 0)$. So $g \wedge l > 0$, i.e., $g \notin H$. Suppose $l \in \mathfrak{D}_1$. Then there exists $b \in \mathfrak{D}_1$ such that $b + b \leq l$ and $b^\perp = l^\perp$. Let $H = N(0, b) \in \mathfrak{N}_3(0)$. Suppose $g \in H$. By Lemma 2.3(i), since $g > 0$, $g = a + b$ for $a \in [0, b]$, $b \in (b^\perp)^+$. Then $a \in \mathfrak{T}(g)$, and $b = a'$. Since $a \leq b < l$, $l \in (a, \infty)$, and since $b \in (l^\perp)^+$, $l \in (b^\perp)^+$. Thus, $l \in (a, \infty) \cap (a')^\perp$, and hence $l \notin M(g, a) \supseteq \bigcap_{b \in \mathfrak{T}(g)} M(g, b)$. This contradicts our choice of l. Therefore, $g \notin H$.

Theorem 5.9. *An l-group G has Hausdorff $\mathfrak{T}$-topology if and only if for all $g \in G^+\backslash\{0\}$,*

$$\mathfrak{D}^* \cap \left(\bigcap_{h \in \mathfrak{T}(g)} M(g, h) \right) \neq \emptyset.$$

Proof. Suppose that for all $g \in G^+ \backslash \{0\}$, $\mathfrak{D}^* \cap (\bigcap_{h \in \mathfrak{T}(g)} M(g, h)) \neq \emptyset$. By Lemma 5.8 and Proposition 5.6, G has Hausdorff $\mathfrak{T}$-topology.

Suppose G has Hausdorff $\mathfrak{T}$-topology. Let $g \in G^+ \backslash \{0\}$. Then by Proposition 5.6, there exists $H \in \mathfrak{N}(0)$ such that $g \notin H$. Thus, by definition of $\mathfrak{N}(0)$, there exists $L \in \mathfrak{N}_3(0)$ such that $g \notin L$.

If $L \in \mathfrak{N}_2(0)$, $L = l^\perp$ for $l \in \mathfrak{D}_2$. If $l \notin \bigcap_{h \in \mathfrak{T}(g)} M(g, h)$, then $l \notin M(g, h)$ for some $h \in \mathfrak{T}(g)$. Hence $l \in (h, \infty) \cap (h')^\perp$. Then $l > h$ and since $l \in \mathfrak{D}_2$, $h = 0$. Then $h' = g$ and since $l \in (h')^\perp$, $g \in l^\perp = L$. This contradicts $g \notin L$. Thus $l \in \bigcap_{h \in \mathfrak{T}(g)} M(g, h)$.

If $L \in \mathfrak{N}_1(0)$, $L = N(0, l)$ for $l \in \mathfrak{D}_1$. If $l \notin \bigcap_{h \in \mathfrak{T}(g)} M(g, h)$, then $l \notin M(g, h)$ for some $h \in \mathfrak{T}(g)$. So $l \in (h, \infty) \cap (h')^\perp$. Since $0 \leq h < l$, $h \in [-l, l]$. Since $l \in (h')^\perp$, $h' \in l^\perp$. Thus $g = h + h' \in [-l, l] + l^\perp = L$. This contradicts $g \notin L$. So $l \in \bigcap_{h \in \mathfrak{T}(g)} M(g, h)$.

Example 5.10. Let $A(R)$ be the set of order-preserving permutations of the real numbers. $A(R)$ is a group under composition and an l-group under the partial order defined: $f \leq g$ if and only if $rf \leq rg$ for all $r \in R$. Since $A(R)$ is doubly transitive, $A(R)$ is divisible (Holland [9]). Then clearly $\mathfrak{A} = \mathfrak{D}_1$. For $f \in A(R)$, let $S(f) = \{r \mid rf \neq r\}$ denote the support of f. Clearly $f \in \mathfrak{A}$ if and only if $f > i$ (where i is the identity permutation in $A(R)$) and there exist $r, s \in R \cup \{-\infty, +\infty\}$ such that $S(f) = (r, s)$.

Let $g \in A(R)^+ \backslash \{i\}$. For $r \in R$, as in [9], let

$$I(r) = \{s \in R \mid \text{there exist integers } m, n \text{ such that } rg^n \leq s \leq rg^m\}.$$

Since $g > i$, there is a $t \in R$ such that $I(t)$ contains more than one point. If $p \in I(t)$, then there is an integer n such that $tg^n \leq p < tg^{n+1}$. Thus if $pg = p$, then $p < tg^{n+1} \leq pg = p$, which is a contradiction. Hence for all $p \in I(t)$, $pg > p$. Clearly $I(t)$ is convex. Thus, there exist $r, s \in R \cup \{-\infty, \infty\}$ such that $(r, s) \subseteq I(t) \subseteq [r, s]$. If $r \in I(t)$, $rg^{-1} \in I(t)$ but $rg^{-1} < r$. Hence $r \notin I(t)$. Similarly $s \notin I(t)$. So $I(t) = (r, s)$.

Define $g' \in A(R)$ by

$$xg' = \begin{cases} xg & \text{if } x \in I(t), \\ x & \text{otherwise.} \end{cases}$$

Clearly $S(g') = I(t) = (r, s)$. Hence $g' \in \mathfrak{A} = \mathfrak{D}_1$. Clearly $g' \in [i, g]$. Since g was an arbitrary element in $A(R)^+ \backslash \{i\}$, by Corollary 5.7, $A(R)$ has Hausdorff $\mathfrak{T}$-topology. Alternately, we note that if $f \in [i, g] \backslash M(g, h)$ for some $h \in \mathfrak{T}(g)$, then $f > h$ and $f \wedge h' = 0$ so that $h = h + h' < f + h' = f \vee h' \leq g \vee g = g$, a contradiction. Hence

$[i, g] \subseteq \bigcap_{h \in \mathfrak{T}(g)} M(g, h)$. Thus, by Theorem 5.9, $A(R)$ has Hausdorff $\mathfrak{T}$-topology.

6. An l-group with indiscrete $\mathfrak{T}$-topology.

Example 6.1. We will construct an l-group which is a (nonconvex) l-subgroup of a cardinal product of totally ordered groups and has $\mathfrak{A} = \emptyset$.

Let $G = |\Pi_1^\infty| R$. Let

$$L = \{f \in G \mid \text{there is an } n \in N \text{ such that for all } i,\ (i+n)f = (i)f\}.$$

Clearly L is an l-subgroup of G. Let $f \in L^+ \setminus \{0\}$. Let $n \in N$ be such that for all i, $(i+n)f = (i)f$. Define $h_1, h_2 \in G$ by

$$(i)h_1 = \begin{cases} (i)f & \text{if } 2kn < i \leq (2k+1)n \text{ for some } k \in Z^+, \\ 0 & \text{if } (2k+1)n < i \leq (2k+2)n \text{ for some } k \in Z^+, \end{cases}$$

$$(i)h_2 = \begin{cases} 0 & \text{if } 2kn < i \leq (2k+1)n \text{ for some } k \in Z^+, \\ (i)f & \text{if } (2k+1)n < i \leq (2k+2)n \text{ for some } k \in Z^+. \end{cases}$$

Since $f > 0$, then $h_1 > 0$ and $h_2 > 0$. Clearly, $(i+2n)h_1 = (i)h_1$ and $(i+2n)h_2 = (i)h_2$. So $h_1, h_2 \in L$. It is also clear that $h_1 + h_2 = f$ and $h_1 \wedge h_2 = 0$. Hence $f \notin \mathfrak{A}$. Therefore $\mathfrak{A} = \emptyset$ and hence L has indiscrete $\mathfrak{T}$-topology.

Example 6.2. The l-group of this example has Hausdorff $\mathfrak{T}$-topology and an element g such that $\mathfrak{A} \cap [0, g] = \emptyset$.

Let L be the l-group of Example 6.1. Let $C(R)$ be the l-group of all continuous functions from R to R. Addition in $C(R)$ is then defined by $x(f+g) = xf + xg$. The order on $C(R)$ is like that on $A(R)$, i.e., $f \leq g$ if and only if $xf \leq xg$ for all $x \in R$.

Define a map $\pi: L \to C(R)$ as follows: for $f \in L$, $f\pi \in C(R)$ is the function defined by

$$(x)(f\pi) = \begin{cases} 2(if)x - 2(if)i & \text{if } x \in (i, i+\tfrac{1}{2}], \\ -2(if)x + 2(if)(i+1) & \text{if } x \in (i+\tfrac{1}{2}, i+1], \\ 0 & \text{otherwise}, \end{cases}$$

where $i = 1, 2, \ldots$. Clearly π is an l-isomorphism of L onto $L\pi \subseteq C(R)$. Let

$$H = \{f \in C(R) \mid xf = q \text{ for some } q \in Q, \text{ for all } x \in R\}.$$

Clearly H is an l-subgroup of $C(R)$. Let F be the l-subgroup of $C(R)$ generated by $(L\pi) \cup (H)$.

Let $p \in L$ be defined by $(i)p = 1$ for all i. Let $g = p\pi$. If $f \in [0, g] \cap F$, then $1f = 0$. Since $f \in F$, $f = \bigvee_{\alpha=1}^n \bigwedge_{\beta=1}^m (h_{\alpha\beta} + l_{\alpha\beta}\pi)$ where $h_{\alpha\beta} \in H$, $l_{\alpha\beta} \in L$ for all α and β. If $x \in R$, then $xf = \bigvee_{\alpha=1}^n \bigwedge_{\beta=1}^m (xh_{\alpha\beta} + x(l_{\alpha\beta}\pi))$. Since $1f = 0$, $0 = \bigvee_{\alpha=1}^n \bigwedge_{\beta=1}^m (1h_{\alpha\beta} + 1(l_{\alpha\beta}\pi)) = \bigvee_{\alpha=1}^n \bigwedge_{\beta=1}^m 1h_{\alpha\beta}$. Since $\bigvee_{\alpha=1}^n \bigwedge_{\beta=1}^m h_{\alpha\beta} \in H$, for all $x \in R$, $x(\bigvee_{\alpha=1}^n \bigwedge_{\beta=1}^m h_{\alpha\beta}) = 0$. For $1 \leq \alpha \leq n$, $1 \leq \beta \leq m$, let $n_{\alpha\beta} \in N$ be

such that $(i + n_{\alpha\beta})l_{\alpha\beta} = (i)l_{\alpha\beta}$. Let d be the least common multiple of the $n_{\alpha\beta}$. Let $l'_{\alpha\beta}, l''_{\alpha\beta} \in L$ be defined by

$$(i)l'_{\alpha\beta} = \begin{cases} (i)l_{\alpha\beta} & \text{if } 2kd < i \leq (2k+1)d \text{ for some } k \in Z^+, \\ 0 & \text{if } (2k+1)d < i \leq (2k+2)d \text{ for some } k \in Z^+, \end{cases}$$

$$(i)l''_{\alpha\beta} = \begin{cases} 0 & \text{if } 2kd < i \leq (2k+1)d \text{ for some } k \in Z^+, \\ (i)l_{\alpha\beta} & \text{if } (2k+1)d < i \leq (2k+2)d \text{ for some } k \in Z^+. \end{cases}$$

Let $f' = \bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} (b_{\alpha\beta} + (l'_{\alpha\beta}\pi))$, $f'' = \bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} (b_{\alpha\beta} + (l''_{\alpha\beta}\pi))$. If $x \in (1 \vee (2kd), (2k+1)d]$ for some $k \in Z^+$, then $(x)(l''_{\alpha\beta}\pi) = 0$ and hence $xf'' = x(\bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} b_{\alpha\beta}) = 0$. Also $(x)(l'_{\alpha\beta}\pi) = (x)(l_{\alpha\beta}\pi)$ and hence $xf' = xf \geq 0$. If $x \in ((2k+1)d, (2k+2)d]$ for some $k \in Z^+$, then $(x)(l'_{\alpha\beta}\pi) = 0$ and hence $xf' = 0$. Also $(x)(l''_{\alpha\beta}\pi)$ and hence $xf'' = xf > 0$. If $x \leq 1$, $(x)(l'_{\alpha\beta}\pi) = (x)(l''_{\alpha\beta}\pi) = (x)(l_{\alpha\beta}\pi) = 0$, and hence $xf' = x(\bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} b_{\alpha\beta}) = 0 = xf'' = xf$. Thus $f' \wedge f'' = 0$, and $f' \vee f'' = f$. Therefore $f \notin \mathfrak{A}$, and since f was an arbitrary element of $[0, g] \cap F$, then $\mathfrak{A} \cap [0, g] \cap F = \emptyset$.

We now show that F has Hausdorff $\mathfrak{J}$-topology. Let $g \in F^+ \backslash \{0\}$. Let $r \in R$ be such that $rg > 0$. Let $q \in Q$ be such that $rg > q > 0$. Let $f \in H^+ \backslash \{0\} \subseteq F^+ \backslash \{0\}$ be defined by $xf = q$ for all $x \in R$. Clearly $f^\perp = \{0\}$. Thus $f \in M(g, h)$ for all $h \in \mathfrak{J}(g) \backslash \{g\}$. But since $rg > q = rf$, $f \notin (g, \infty)$. Hence $f \in M(g, g)$. Thus $f \in \bigcap_{h \in \mathfrak{J}(g)} M(g, h)$. Clearly $f \in \mathfrak{A}$ and since H is divisible, clearly $f \in \mathfrak{D}_1$. Thus for all $g \in F^+ \backslash \{0\}$, $\mathfrak{D}^* \cap (\bigcap_{h \in \mathfrak{J}(g)} M(g, h)) \neq \emptyset$. Hence by Theorem 5.9, F has Hausdorff $\mathfrak{J}$-topology. Alternately, since $f^\perp = \{0\}$, $N(0, f) = [-f, f]$, and since $f \notin (g, \infty)$, $g \notin N(0, f)$. Thus, since $f \in \mathfrak{D}_1$, F has Hausdorff $\mathfrak{J}$-topology by Proposition 5.6.

7. **$\mathfrak{J}$-topology convergence.** Convergence with respect to the $\mathfrak{J}$-topology may be characterized as follows:

Theorem 7.1. *Let $\{x_\beta \mid \beta \in B\}$ be a net in an l-group G. Then $\{x_\beta\}$ converges to x with respect to the $\mathfrak{J}$-topology on G if and only if* (i) *for all $g \in \mathfrak{D}_1$, there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $g \wedge |-x + x_\beta| \in \mathfrak{J}(|-x + x_\beta|)$, and* (ii) *for all $g \in \mathfrak{D}_2$, there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $g \wedge |-x + x_\beta| = 0$.*

Proof. Suppose $\{x_\beta\}$ $\mathfrak{J}$-converges to x. (i) Let $g \in \mathfrak{D}_1$. Then there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $x_\beta \in x + \mathrm{Int}(N(0, g)) \subseteq x + N(0, g)$. Hence $-x + x_\beta \in N(0, g)$ for all $\beta \geq \alpha$. Since $N(0, g)$ is a symmetric sublattice (Lemmas 2.2 and 2.12),

$$|-x + x_\beta| = (-x + x_\beta) \vee (-(-x + x_\beta)) \in N(0, g).$$

Hence $|-x + x_\beta| = a + b$ for $a \in [0, g]$, $b \in g^\perp$. Thus, $|-x + x_\beta| \wedge g = (a \vee b) \wedge g = a \wedge g = a$. Since $a \wedge b = 0$, $|-x + x_\beta| \wedge g \in \mathfrak{T}(|-x + x_\beta|)$.

(ii) Let $g \in \mathfrak{D}_2$. Then there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $x_\beta \in x + \operatorname{Int}(g^\perp) = x + g^\perp$. Hence $-x + x_\beta \in g^\perp$. Thus, $|-x + x_\beta| \in g^\perp$, i.e., $|-x + x_\beta| \wedge g = 0$.

Conversely, suppose $\{x_\beta \mid \beta \in B\}$ is a net in G such that there is an $x \in G$ satisfying (i) and (ii). Let $H \in \mathfrak{N}_1(0)$. Then by (i) there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $|-x + x_\beta| \in H$, i.e. $x_\beta \in x + H$. Similarly, if $H \in \mathfrak{N}_2(0)$, by (ii), there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $x_\beta \in x + H$. Let $U \in \mathfrak{T}$, such that $x \in U$. Then there is an $H \in \mathfrak{N}(0)$ such that $x \in x + H \subseteq U$. $H = \bigcap_{i=1}^n H_i$ for $H_i \in \mathfrak{N}_3(0)$. As we noted above, for each i there is an $\alpha_i \in B$ such that whenever $\beta \geq \alpha_i$, $x_\beta \in x + H_i$. Let $\alpha \geq \alpha_i$ for all i. Then whenever $\beta \geq \alpha$, $x_\beta \in x + H$. Thus $\{x_\beta\}$ $\mathfrak{T}$-converges to x.

If X is a set with topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$, then $\mathfrak{T}_1 \supseteq \mathfrak{T}_2$ if every net which converges to x with respect to $\mathfrak{T}_1$ converges to x with respect to $\mathfrak{T}_2$. We use this fact in comparing the $\mathfrak{T}$-topology to the topology of α-convergence.

Let L be a lattice. Papangelou ([15], [16]) defined α-convergence in L as follows: A net $\{x_\beta \mid \beta \in B\}$ is said to *α-converge* to $x \in L$ if and only if x is the only element of L satisfying the following condition:

$$x = \bigvee_{\beta \geq \alpha} (x_\beta \wedge x) = \bigwedge_{\beta \geq \alpha} (x_\beta \vee x)$$

for all $\alpha \in B$. Ellis [5] showed that there is a topology $\mathfrak{S}$ on an l-group G such that convergence with respect to $\mathfrak{S}$ is equivalent to α-convergence if and only if G is completely distributive. When such a topology $\mathfrak{S}$ exists, it is called the *topology of α-convergence* and, as Ellis noted, it must be Hausdorff. Madell [14] proved that with respect to $\mathfrak{S}$, G is a topological group and a topological lattice. He also showed that any Hausdorff topology $\mathfrak{S}$ on G with respect to which G is both a topological group and a topological lattice lies between $\mathfrak{S}$ and the discrete topology. Therefore, we have

Theorem 7.2. *Let G be a completely distributive l-group. If $\mathfrak{S}$ is the topology of α-convergence on G and $\mathfrak{T}$ is the $\mathfrak{T}$-topology, then the following are equivalent:*

(i) $\mathfrak{S} \subseteq \mathfrak{T}$,

(ii) *$\mathfrak{T}$ is Hausdorff.*

Concerning inclusion the other way, we have the following theorem:

Theorem 7.3. *Let G be a completely distributive l-group with $\mathfrak{T}$-topology $\mathfrak{T}$ and topology of α-convergence $\mathfrak{S}$. Then $\mathfrak{S} \supseteq \mathfrak{T}$ if and only if for all nets $\{x_\beta \mid \beta \in B\} \subseteq G^+ \setminus \{0\}$ such that $\bigwedge_{\delta \in \Delta} x_\delta = 0$ whenever Δ is a cofinal subset of B, and for all $g \in \mathfrak{D}_1$, there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $g \wedge x_\beta \in \mathfrak{T}(x_\beta)$.*

Papangelou [16] proved the following lemma in the case when G is a completely distributive abelian l-group. Madell [14] noted that the lemma remained true when the assumption of commutativity was removed.

Lemma 7.4. *If G is completely distributive and $\{x_\beta\}$ is a net in G, the following statements are equivalent:*

(i) $\{x_\beta\}$ *α-converges to* 0,

(ii) *for each cofinal subset Δ of B,* $\bigwedge_{\delta\in\Delta}|x_\delta| = 0$.

Proof of Theorem 7.3. If $\mathcal{S} \supseteq \mathcal{T}$, then every net $\{x_\beta \mid \beta \in B\}$ which α-converges to x converges to x with respect to $\mathcal{T}$. Let $\{x_\beta\} \subseteq G^+\backslash\{0\}$ be a net such that $\bigwedge_{\delta\in\Delta} x_\delta = 0$ for all cofinal subsets $\Delta \subseteq B$. By Lemma 7.4, $\{x_\beta\}$ α-converges to 0. Hence $\{x_\beta\}$ converges to 0 with respect to $\mathcal{T}$. By Theorem 7.1, for all $g \in \mathfrak{D}_1$, there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $g \wedge x_\beta \in \mathcal{T}(x_\beta)$.

Conversely, suppose the condition of the theorem holds. Let $\{x_\beta\}$ be a net which α-converges to x. Then $\{-x + x_\beta\}$ α-converges to 0. By Lemma 7.4, $\bigwedge_{\delta\in\Delta}|-x + x_\delta| = 0$ for all cofinal subsets Δ of B. Hence for all $g \in \mathfrak{D}_1$, there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $g \wedge |-x + x_\beta| \in \mathcal{T}(|-x + x_\beta|)$. Let $g \in \mathfrak{D}_2$. Suppose for all $\alpha \in B$, there is a $\beta \geq \alpha$ such that $|-x + x_\beta| \wedge g > 0$. Then $|-x + x_\beta| \wedge g = g$, i.e. $|-x + x_\beta| \geq g$. Let $\Delta = \{\delta \in B \mid |-x + x_\delta| \geq g\}$. Then Δ is cofinal in B, but $\bigwedge_{\delta\in\Delta}|-x + x_\delta| \geq g > 0$. This is a contradiction, and hence there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $|-x + x_\beta| \wedge g = 0$. Thus by Theorem 7.1, $\{x_\beta\}$ converges to x with respect to $\mathcal{T}$.

We now apply Theorem 7.2 and Theorem 7.3 to two particular situations. These applications show that, at least in certain circumstances, the criterion established in Theorem 7.3 is a convenient one to use.

Proposition 7.5. *Let $\{T_\lambda \mid \lambda \in \Lambda\}$ be a collection of totally ordered groups. Suppose G is a completely distributive l-subgroup of $|\Pi_{\lambda\in\Lambda}|\, T_\lambda$ which contains $|\Sigma_{\lambda\in\Lambda}|\, T_\lambda$. Then $\mathcal{S} = \mathcal{T}$ on G.*

Proof. (We will use the notation of §3 throughout the proof.) Let $g \in G^+\backslash\{0\}$. Then there is an $\eta \in \Lambda$ such that $\eta g > 0$. If there exists $t \in \mathfrak{D}_{2\eta}$, then $\bar{t} \in \mathfrak{D}_2$ by Lemma 3.2. Since $\eta g \geq t$, we have that $\bar{t} \in [0, g]$. If $\mathfrak{D}_{2\eta} = \emptyset$, then by the proof of Theorem 4.3(ii), $T^+\backslash\{0\} = \mathfrak{D}_{1\eta}$. Hence $\eta g \in \mathfrak{D}_{1\eta}$. Then by Lemma 3.3, $\overline{\eta g} \in \mathfrak{D}_1$, and clearly $\overline{\eta g} \in [0, g]$. Therefore, by Corollary 5.7, G has Hausdorff $\mathcal{T}$-topology, and hence, by Theorem 7.2, $\mathcal{S} \subseteq \mathcal{T}$.

Let $\{x_\beta \mid \beta \in B\} \subseteq G^+\backslash\{0\}$ be a net such that $\bigwedge_{\delta\in\Delta} x_\delta = 0$ if Δ is a cofinal subset of B. Let $g \in \mathfrak{D}_1$ and suppose that for all $\alpha \in B$, there exists $\beta \geq \alpha$ such that $x_\beta > g$. Then $\Delta = \{\beta \in B \mid x_\beta > g\}$ is a cofinal subset of B, but $\bigwedge_{\delta\in\Delta} x_\delta \geq$

$g > 0$. This contradicts our choice of $\{x_\beta\}$. Hence there is an $\alpha \in B$ such that whenever $\beta \geq \alpha$, $x_\beta \not> g$. By Lemma 3.5, $g = \overline{h_\gamma}$ for some $\gamma \in \Lambda$ and some $h_\gamma \in \mathfrak{D}_{1\gamma}$. Then $\lambda x_\beta \geq 0 = \lambda g$ for all $\lambda \neq \gamma$, and hence since T_γ is totally ordered, $\gamma x_\beta \leq \gamma g$. Thus

$$\lambda(g \wedge x_\beta) = \begin{cases} \lambda x_\beta & \text{if } \lambda = \gamma, \\ 0 & \text{if } \lambda \neq \gamma. \end{cases}$$

Let $g' \in |\Pi_{\lambda \in \Lambda}| T_\lambda$ be defined by

$$\lambda g' = \begin{cases} \lambda x_\beta & \text{if } \lambda \neq \gamma, \\ 0 & \text{if } \lambda = \gamma. \end{cases}$$

Then $g' = x_\beta - g \wedge x_\beta$. Since $|\Sigma_{\lambda \in \Lambda}| T_\lambda \subseteq G$, we have that $g \wedge x_\beta \in G$ and hence $g' \in G$. Clearly $(g \wedge x_\beta) \vee g' = x_\beta$ and $(g \wedge x_\beta) \wedge g' = 0$. Hence $g \wedge x_\beta \in \mathfrak{T}(x_\beta)$. Thus, by Theorem 7.3, $\mathfrak{S} \supseteq \mathfrak{T}$.

In the next example, we show that in $A(R)$ the $\mathfrak{T}$-topology properly contains the topology of α-convergence.

Example. In Example 5.10, we noted that $A(R)$ is an l-group with $f \in \mathfrak{A} = \mathfrak{D}_1$ if and only if $f > i$ and there exist $r, s \in R \cup \{-\infty, +\infty\}$ such that $S(f) = \{x \in R \mid x \neq xf\} = (r, s)$. We showed that $A(R)$ has Hausdorff $\mathfrak{T}$-topology. Hence by Theorem 7.2, $\mathfrak{S} \subseteq \mathfrak{T}$.

Let $g \in A(R)$ be defined by

$$xg = \begin{cases} 2x & \text{if } x \in (0, 1], \\ x/2 + 3/2 & \text{if } x \in (1, 3], \\ x & \text{otherwise.} \end{cases}$$

Clearly $g > i$ and $S(g) = (0, 3)$. So $g \in \mathfrak{D}_1$. Define a net $\{f_n \mid n \in N\}$ in $A(R)^+ \setminus \{i\}$ by

$$xf_n = \begin{cases} (1 + (1/n))x - (1/n) & \text{if } x \in (1, 2], \\ (1 - (1/2n))x + (2/n) & \text{if } x \in (2, 4], \\ x & \text{otherwise.} \end{cases}$$

Clearly $f_n > i$ and $S(f_n) = (1, 4)$ for all n. Hence $f_n \in \mathfrak{A}$ for all n. Since $0 < f_n \wedge g < f_n$ for all n, $f_n \wedge g \notin \mathfrak{T}(f_n)$ for all n. Hence by Theorem 7.3, $\mathfrak{S} \not\supseteq \mathfrak{T}$.

8. A refinement of $\mathfrak{T}$. Let L be the l-group constructed in Example 6.1. Let $G = L \overleftarrow{\times} Z$. Then $\mathfrak{D}_1 = \emptyset = \mathfrak{D}_2$ so that G has indiscrete $\mathfrak{T}$-topology. In the next few paragraphs, we outline how the definition of the $\mathfrak{T}$-topology may be modified to give G a topology which has $\{[-g, g] \mid g \in G^+ \setminus (L \times \{0\})\}$ as a base for the neighbourhood filter about 0.

We modify the $\mathfrak{T}$-topology by enlarging the set $\mathfrak{D}_2$ as follows:

$$\mathfrak{D}_2' = \{h \in G^+\backslash\{0\} \mid \text{there exists a convex } l\text{-subgroup } C \subseteq G \text{ such that for all } c \in C,\ c < h \text{ and if } a < t < b \text{ for } a \in C,\ b \in h + C, \text{ then } t \in C \cup (h + C)\}.$$

Proposition 8.1. $\mathfrak{D}_2' \subseteq \mathfrak{A}$.

Proof. Let $h \in \mathfrak{D}_2'$. Let $a \wedge b = 0$, $a \vee b = h$. Let C be a convex l-subgroup satisfying the conditions of $\mathfrak{D}_2'$.

We first show that for all $c, d \in C$, $h + d > c$. Since $c - d \in C$, $(c - d) \wedge h = c - d$. Hence $c \wedge (h + d) = ((c - d) \wedge h) + d = c$. Thus $h + d \geq c$. But since $h \notin C$, $h + d > c$.

Without loss of generality, $a \notin C$. Since $a \in [0, h]$, $a \in h + C$, i.e., $a = h + a'$ for $a' \in C$. Thus, $0 = b + a'$ and $b \in C$. So by the above, $a > b$. Therefore, $a = h$ and $b = 0$. So $\mathfrak{T}(h) = \{0, h\}$.

Proposition 8.2. *For $h \in \mathfrak{D}_2'$, there is a unique convex l-subgroup C satisfying the conditions of $\mathfrak{D}_2'$.*

Proof. Suppose C and C' are two convex l-subgroups satisfying the conditions of $\mathfrak{D}_2'$. Let $d \in C'$. If $d \notin C$, either $d^+ \notin C$ or $d^- \notin C$. If $d^+ \notin C$, we let $b = d^+$. If $d^+ \in C$, then $d^- \notin C$ and we let $b = -d^-$. Then $b \in C'$, $b \notin C$, $0 < b < h$. Hence $b \in h + C$. As in Proposition 8.1, $b > c$ for all $c \in C$. Thus $(C')^+ \supseteq C^+$ and so $C' \supseteq C$. Consider $-b + h \in C$. Since $b \in C'$, $-b + h \notin C'$. Since $0 < b < h$, $0 < -b + h < h$. Let $b' = -b + h$. Then we have $b' \in C$, $b' \notin C'$, $0 < b' < h$. Hence $b' \in h + C'$, and therefore, $b' > c'$ for all $c' \in C'$. Thus $C \supseteq C'$. We conclude that $C = C'$. But we assumed $d \in C'\backslash C$. This is a contradiction. Hence $d \in C$. So $C' \subseteq C$. Similarly $C' \supseteq C$, and hence $C' = C$.

For $h \in \mathfrak{D}_2'$, let the unique convex l-subgroup satisfying the conditions of $\mathfrak{D}_2'$ be denoted by $C(h)$. Let

$$\mathfrak{N}_2'(0) = \{C(h) + h^\perp \mid h \in \mathfrak{D}_2'\}.$$

Lemma 2.2 holds with $\mathfrak{N}_2'(0)$ replacing $\mathfrak{N}_2(0)$. We may therefore define a topology

$$\mathfrak{T}' = \{W \subseteq G \mid \text{if } x \in W, \text{ then } W \in F(\mathfrak{N}'(x))\}$$

where $\mathfrak{N}'(x)$ is $\mathfrak{N}(x)$ modified in the obvious fashion. Call $\mathfrak{T}'$ the $\mathfrak{T}'$-topology on G. Theorems 2.6, 2.13, 3.1, and 4.3 can be proven for $\mathfrak{T}'$ [17].

However, this extra complication properly increases the $\mathfrak{T}$-topology only in special cases, such as the one mentioned at the beginning of the section:

Proposition 8.3. *If $C(h) \cap (\mathfrak{D}_1 \cup \mathfrak{D}_2) \neq \emptyset$ for all $h \in \mathfrak{D}_2'\backslash\mathfrak{D}_2$ and $l^\perp \subseteq C(h) +$*

$b^{\perp}$ *for all* $l \in C(b) \cap (\mathfrak{D}_1 \cup \mathfrak{D}_2)$, *then* $\mathfrak{T} = \mathfrak{T}'$.

Proof. Clearly for any l-group $\mathfrak{D}_2' \supseteq \mathfrak{D}_2$. Thus $\mathfrak{N}'(0) \supseteq \mathfrak{N}(0)$ and so $\mathfrak{N}'(x) \supseteq \mathfrak{N}(x)$ for all $x \in G$, i.e., $F(\mathfrak{N}'(x)) \supseteq F(\mathfrak{N}(x))$. Hence if $W \in \mathfrak{T}$, then for all $x \in W$, $W \in F(\mathfrak{N}(x)) \subseteq F(\mathfrak{N}'(x))$. Thus $W \in \mathfrak{T}'$. Therefore, $\mathfrak{T} \subseteq \mathfrak{T}'$ for any l-group.

If $l \in C(b) \cap (\mathfrak{D}_1 \cup \mathfrak{D}_2)$, then $l^{\perp} \subseteq N(0, l) \subseteq C(b) + b^{\perp}$. Hence $C(b) + b^{\perp} \in F(\mathfrak{N}(0))$. Thus, under the hypotheses of the proposition, $C(b) + b^{\perp} \in F(\mathfrak{N}(0))$ for all $b \in \mathfrak{D}_2' \backslash \mathfrak{D}_2$. Therefore $\mathfrak{T} \supseteq \mathfrak{T}'$.

The material outlined in this section is developed in more detail in [17].

REFERENCES

1. R. A. Alo and O. Frink, *Topologies of lattice products*, Canad. J. Math. 18 (1966), 1004–1014. MR 35 #98.

2. G. Birkhoff, *A new interval topology for dually directed sets*, Univ. Nac. Tucumán Rev. Ser. A 14 (1962), 325–331. MR 29 #3398.

3. ———, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R. I., 1967. MR 37 #2638.

4. P. Conrad, *The structure of a lattice-ordered group with a finite number of disjoint elements*, Michigan Math. J. 7 (1960), 171–180. MR 22 #6854.

5. J. Ellis, *Group topological convergence in completely distributive lattice-ordered groups*, Doctoral Dissertation, Tulane University, New Orleans, La., 1968.

6. E. E. Floyd, *Boolean algebras with pathological order topologies*, Pacific J. Math. 5 (1955), 687–689. MR 17, 450.

7. O. Frink, *Topology in lattices*, Trans. Amer. Math. Soc. 51 (1942), 569–582. MR 3, 313.

8. ———, *Ideals in partially ordered sets*, Amer. Math. Monthly 61 (1954), 223–234. MR 15, 848.

9. C. Holland, *The lattice-ordered group of automorphisms of an ordered set*, Michigan Math. J. 10 (1963), 399–408. MR 28 #1237.

10. F. K. Holley, *The ideal and new interval topologies on l-groups*, based on the doctoral dissertation, University of New Mexico, 1970 (preprint).

11. T. Husain, *Introduction to topological groups*, Saunders, Philadelphia, Pa., 1966. MR 34 #278.

12. J. Jakubik, *Interval topology of an l-group*, Colloq. Math. 11 (1963), 65–72. MR 28 #4041.

13. S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloq. Publ., vol. 27, Amer. Math. Soc., Providence, R. I., 1942. MR 4, 84.

14. R. Madell, *Complete distributivity and α-convergence*, Notices Amer. Math. Soc. 16 (1969), 239. Abstract #663–526.

15. F. Papangelou, *Order convergence and topological completion of commutative lattice groups*, Math. Ann. 155 (1964), 81–107. MR 30 #4699.

16. ———, *Some considerations on convergence in abelian lattice groups*, Pacific J. Math. 15 (1965), 1347–1364. MR 32 #7655.

17. R. H. Redfield, *A topology for a lattice-ordered group*, Doctoral Dissertation, Simon Fraser University, 1972.

18. B. C. Rennie, *Lattices*, Proc. London Math. Soc. (2) 52 (1951), 386–400. MR 13, 7.

19. B. Šmarda, *Topologies in l-groups*, Arch. Math. (Brno) 3 (1967), 69–81. MR 36 #6331.

DEPARMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, BURNABY, BRITISH COLUMBIA, CANADA