ON A COMPACTNESS PROPERTY OF TOPOLOGICAL GROUPS

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ABSTRACT. A density theorem of semisimple analytic groups acting on locally compact groups is presented.

Let G and H be locally compact groups with G acting continuously on H as a group of automorphisms. An element h of H is said to be G-bounded if the orbit $Gh = \{g(h): g \in G\}$ has compact closure in H. We write $F_G(H)$ for the set of all G-bounded elements in H. It is very easy to verify that $F_G(H)$ is a G-invariant subgroup of H. However in general, $F_G(H)$ is not closed in H. In this paper, we shall study the group $F_G(H)$ for certain topological groups G. Our main result is the following

Theorem. Let G be a semisimple analytic group without compact factors acting on a locally compact group H continuously as a group of automorphisms. If the set $F_G(H)$ is dense in H, then G acts trivially on H.

The theorem generalizes some results in [2], [4] and is closely related to the density property of certain subgroups in semisimple analytic groups without compact factors. The result of Corollaries 4.1 and 4.2 is contained in [2], [4].

In the sequel, we shall use the term ‘‘G acts on H’’ for ‘‘G acts on H as a group of automorphisms’’.

1. Minimally almost periodic groups. Let G be a locally compact group. We recall that G is minimally almost periodic if there are no nontrivial continuous homomorphisms $f: G \rightarrow G'$ of locally compact groups such that the closure $\text{Cl}(f(G))$ of $f(G)$ in $G'$ is compact. Minimally almost periodic groups have been widely studied. Yet for our need, we shall establish some lemmas concerning minimally almost periodic groups.

Lemma 1.1. Let G be a minimally almost periodic group acting continuously on a locally compact group H, and N be a closed G-invariant normal subgroup of H.

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If the set $F_G(H)$ is dense in $H$, and $G$ acts trivially on both $N$ and $H/N$, then $G$ acts trivially on $H$.

**Proof.** Let $h$ be any fixed element of $F_G(H)$. As $G$ acts trivially on $H/N$, there is a continuous function $f: G \to N$ such that $g(h) = h f(g)$, $g \in G$. Since $G$ acts trivially on $N$, we have
\[
bf'(g'h) = (g'h)b = g'(gb) = g'(h f(g))
\]
\[
= g'(b)f(g) = h f(g')(f(g)) \quad (g', g \in G).
\]
Hence $f$ is a continuous homomorphism. We know that $Cl(Gb)$ is compact. $Cl(f(G))$, being contained in $b^{-1}Cl(Gb)$, is evidently compact. Therefore $f$ has to be trivial; equivalently $g(b) = b$ for every $g$ in $G$. Since the set $F_G(H)$ is dense in $H$, it follows readily that $G$ acts trivially on $N$.

**Lemma 1.2.** Let $G$ be a connected minimally almost periodic group acting continuously on a locally compact abelian group $A$. If the set $F_G(A)$ is dense in $A$, then $G$ acts trivially on $A$.

**Proof.** First we assume that $A$ is compactly generated. In this case, $A$ has a unique maximal compact subgroup $K$. Obviously $K$ is characteristic, hence $G$-invariant. By a well-known theorem of Iwasawa [3], the automorphism group Aut($K$) of $K$ with compact-open topology is totally disconnected, hence $G$ acts trivially on $K$. By Lemma 1.1, we may assume that $K = \{ e \}$ and $A$ is an abelian Lie group. Let $A^0$ be the identity component of $A$. Since $G$ is connected, $G$ acts trivially on $A/A^0$. Again by Lemma 1.1, we may even assume that $A$ is connected. Under these additional assumptions, $A = R^l$ for some positive integer $l$. Now we pick out a basis $\{ e_i, \ldots, e_l \}$ of $R^l$ from $F_G(R^l)$. This is possible because $F_G(R^l)$ is dense in $R^l$. With respect to this basis, for every $g$ in $G$, we write
\[
g(e_i) = \sum_{i=1}^{l} g_{ji} e_j \quad (1 \leq i \leq l),
\]
with $g_{ji}$ in $R$. It is easy to show that the map $g \mapsto (g_{ij})$ $(g \in G)$ is a continuous homomorphism $f$ of $G$ into $GL(l, R)$. Since all the entries $g_{ij}$ $(1 \leq i, j \leq l, g \in G)$ are bounded, we conclude $f(G)$ has compact closure in $GL(l, R)$. Hence $f$ has to be trivial, and the lemma is proved in case that $A$ is compactly generated. For the general case, $G$ acts trivially on $A/A^0$, hence $G$ leaves any open subgroup of $A$ invariant. Let $N$ be a compactly generated open subgroup of $A$. Clearly $F_G(N) = F_G(A) \cap N$ is still dense in $N$. By what we have just proved, $G$ acts trivially on $N$ and by Lemma 1.1, the proposition follows.

**Remark.** In the preceding lemma, we assume only that the set $F_G(A)$ is dense in $A$. In general we do not know whether the set $F_G(A^0) = F_G(A) \cap A^0$ is dense in $A^0$. That is why we consider first compactly generated open subgroups of $A$ rather than the subgroup $A^0$. 
Corollary 1.3. Let $G$ be a connected minimally almost periodic group and $L$ a closed subgroup of $G$ with compact quotient $G/L$. Let $A$ be a locally compact abelian group such that $G$ acts continuously on $A$. If $L$ leaves an element $x$ of $A$ fixed, then $x$ is fixed by $G$.

Proof. Consider the group $C_1(F_G(A))$. By Lemma 1.2, $G$ acts trivially on $C_1(F_G(A))$. Clearly $x$ lies in $F'_G(A)$ and the corollary follows.

Corollary 1.3 reveals at least some density property of those subgroups $L$ of $G$ with compact quotient $G/L$. In general, the structure of minimally almost periodic groups is not entirely clear. However for connected groups, we have the following criterion. The result must be known but we offer a proof here for completeness.

Lemma 1.4. Let $G$ be a connected locally compact group. The following statements are equivalent:

(i) $G$ is minimally almost periodic.

(ii) $G$ is an analytic group such that $[G, G]$ is dense in $G$ and $G/R(G)$ has no compact factors where $R(G)$ is the radical of $G$.

Proof. (i) $\Rightarrow$ (ii) Since $G$ is a connected locally compact group, locally $G$ is the direct product of a compact group and a local Lie group. However $G$ is minimally almost periodic, hence $G$ is a Lie group. Consider then the groups $G/Cl([G, G])$ and $G/R(G)$. $G/R(G)$ (resp. $G/Cl([G, G])$ is minimally almost periodic semisimple (resp. abelian minimally almost periodic). (ii) follows immediately.

(ii) $\Rightarrow$ (i) By a well-known theorem of von Neumann, any topological group $G$ contains a unique minimal closed normal subgroup $N$ such that $G/N$ is maximally almost periodic, i.e., there is a continuous injection of $G/N$ into a compact group $K$. Hence it suffices to show that $G/N$ is trivial in our case. Clearly $G/N$ still satisfies all the assumptions in (ii). But, by a theorem of Freudenthal, a connected maximally almost periodic locally compact group is the direct product of a compact group and a vector group. Hence one concludes readily that $G/N$ is trivial, i.e., $G = N$ is minimally almost periodic.

2. Cross homomorphisms. Let $G$ be a locally compact group acting on a locally compact abelian group $A$ continuously. A continuous map $f: G \to A$ is called a cross homomorphism if $f$ satisfies the condition

$$f(gg') = g/f(g') + f(g)$$

for all $g, g' \in G$. Given any $v$ in $A$, the map $d_v: G \to A$, defined by $d_v(g) = gv - v$ ($g \in G$) clearly is a cross homomorphism. A cross homomorphism $f$ is said to be homologous to $0$ if $f = d_v$ for some $v$ in $A$.

Lemma 2.1. Let $G$ be a semisimple analytic group acting on a locally compact abelian group $A$ continuously. Then any cross homomorphism $f: G \to A$ is homologous to $0$. 
Proof. Let $e$ be the identity element of $G$. Since $f$ is a cross homomorphism, $f(e) = 0$. Hence $f(G)$ is contained in $A^0$ because $f$ is continuous. Therefore we may even assume that $A$ is connected. Let $K$ be the unique maximal compact subgroup of $A$. Clearly $K$ is $G$-invariant and $G$ acts trivially on $K$. $f$ induces then a cross homomorphism $\overline{f} : G \rightarrow A/K$. $A/K$ is isomorphic to $R^t$ for some positive integer $t$. It is well known that $\overline{f}$ is homologous to 0. Hence there exists an element $v$ in $A$ such that

$$f(g) = gv - v \pmod{K}, \quad g \in G.$$ 

Let $f_1 : G \rightarrow K$ be the map defined by

$$f_1(g) = f(g) - gv + v, \quad g \in G.$$ 

One verifies readily that $f_1$ is a cross homomorphism. Since $G$ acts trivially on $K$, $f_1$ is a homomorphism, hence $f_1(G) = \{0\}$. Thus $f = f_1$ is homologous to 0.

3. Linear Lie groups. Let $GL(n, C)$ (resp. $\mathfrak{gl}(n, C)$) be the group of all $n$ by $n$ nonsingular complex matrices (resp. the Lie algebra of all $n$ by $n$ complex matrices). Clearly $\mathfrak{gl}(n, C)$ is the Lie algebra of $GL(n, C)$ and the exponential map $\exp : \mathfrak{gl}(n, C) \rightarrow GL(n, C)$ is just the usual one. Let $\lambda$ be any positive number. We denote by $\mathfrak{g}_\lambda(n, C)$ the set of all elements $X$ in $\mathfrak{gl}(n, C)$ such that the imaginary parts of all the eigenvalues of $X$ lie in the open interval $(-\lambda, \lambda)$. Let $G$ be any Lie subgroup of $GL(n, C)$ and $\mathfrak{g}$ its Lie algebra. We write $g_\lambda$, $G_\lambda$ and $\exp_\lambda$ for $\mathfrak{g}_\lambda(n, C; \lambda)$, $\exp(\mathfrak{g}_\lambda)$ and the restriction of $\exp_\lambda$ on $\mathfrak{g}_\lambda$ respectively.

Lemma 3.1 [4]. The maps $\exp_\lambda$ ($0 < \lambda \leq n$) are diffeomorphisms.

Proposition 3.2. Let $G$ be a semisimple analytic subgroup of $GL(n, C)$ and $H$ a Lie subgroup of $GL(n, C)$. Suppose that

(i) $G$ has no compact factors,

(ii) $G$ normalizes $H$, and

(iii) $F_G(H)$ is dense in $H$ where $G$ acts on $H$ through conjugation.

Then $G$ centralizes $H$.

Proof. Let $\lambda$ be any positive number smaller than $n$. By Lemma 3.1, $\exp_\lambda : \mathfrak{h} \rightarrow H_\lambda$ is a diffeomorphism. Clearly $H_\lambda$ is $G$-invariant under conjugation. Since $F_G(H)$ is dense in $H$, there is a basis $\{X_1, \ldots, X_r\}$ of $\mathfrak{h}$ such that $X_i \in H_\lambda$ and $\exp X_i \in F_G(H)$ ($1 \leq i \leq r$). Let $Ad$ be the adjoint representation of $GL(n, C)$ on $\mathfrak{gl}(n, C)$. Then with respect to this basis, all elements in the group $Ad(G)|_\mathfrak{h}$ have bounded entries because $\exp X_i \in F_G(H)$ ($1 \leq i \leq n$) and $\exp_\lambda$ is a diffeomorphism. Hence $Ad(G)|_\mathfrak{h}$ has compact closure. By (i) and Lemma 1.4, $G$ centralizes $H^0$. Clearly $G$ acts trivially on $H/H^0$ for $H/H^0$ is discrete and $G$ is connected. By Lemma 1.1, $G$ acts trivially on $H$, therefore $G$ centralizes $H$. 
4. Proof of the theorem. We prove the theorem in several steps.

(i) $G$ leaves invariant any open subgroup of $H$. Since $H/H^0$ is discrete and $G$ is connected, $G$ acts trivially on $H/H^0$. Clearly $H^0$ is contained in any open subgroup of $H$. Hence (i) follows easily.

(ii) We may assume that $H$ is an analytic group. Let $H_1$ be an open subgroup of $H$ such that $H_1$ is a projective limit of Lie groups. Let $K$ be a normal compact subgroup of $H_1$ such that $H_1/K$ is a Lie group. Then consider $H_2 = H_1/K$. $H_2$ is again an open subgroup of $H$. It is well known that a connected locally compact group has a unique maximal compact normal subgroup. It follows that $H_2$ also has a unique maximal normal compact subgroup $L$. By (i) $H_2$ is $G$-invariant, hence $L$ is also $G$-invariant. Since $L$ is compact, Aut$(L)^0$ = the inner automorphism group by a theorem of Iwasawa [3]. Therefore Aut$(L)^0$ is compact. The action of $G$ on $L$ is induced by a continuous homomorphism $f : G \rightarrow Aut(L)$. Clearly $f(G)$, being contained in Aut$(L)^0$, has compact closure. By Lemma 1.4, $f(G)$ is trivial, i.e., $G$ acts trivially on $L$. Therefore by Lemma 1.1, we may even assume that $H = H_2/L$ is an analytic group.

(iii) By (ii) we assume further that $H$ is an analytic group. Let $M = G \cdot H$ be the semidirect product of $G$ and $H$. Let Ad be the adjoint representation of $M$ on its Lie algebra. Passing over to Ad$(M)$, by Proposition 3.2, one concludes that given any $h \in H$

$$g(h) = h s(g), \quad g \in G,$$

where $s(g)$ is in the center $Z(H)$ of $H$. By a direct calculation, $s : G \rightarrow Z(H)$ is a cross homomorphism. By Lemma 2.1, $s$ is homologous to 0. Hence there is $z \in Z(H)$ with $s(g) = g(z^{-1})z$ for all $g \in G$. Now consider the element $bz$. Clearly $g(bz) = bz$ for all $g \in G$. Let $F$ be the set of all fixed points of $H$. Clearly $F$ is a closed subgroup of $H$. By what we have just proved, $F \cdot Z(H) = H$. Hence $F$ is normal and $H/F$ is abelian. By Proposition 1.2, $G$ acts trivially on $H/F$. By Lemma 1.1, $G$ acts trivially on $H$. Therefore the proof of the theorem is hereby completed.

Corollary 4.1. Let $G$ be an analytic semisimple group without compact factors, and $g$ an element of $G$. If the conjugacy class $\{xgx^{-1} : x \in G\}$ has compact closure in $G$, $g$ is in the center $Z(G)$ of $G$.

Proof. $G$ acts on $G$ through conjugation. By the theorem $F_G(G) = Z(G)$. Clearly $g$ is in $F_G(G)$.

Corollary 4.2. Let $G$ be an analytic semisimple group without compact factors and $\alpha$ an automorphism of $G$. If the subset $\{\alpha(g)g^{-1} : g \in G\}$ has compact closure then $\alpha$ is the identity map.
Proof. Let $\omega: G \to \text{Aut}(G)$ be the homomorphism defined by $\omega(g)(x) = g \times g^{-1}$, $(g, x \in G)$. Clearly $G$ acts on $\text{Aut}(G)$ through $\omega$ and conjugation, and $\alpha \in F_G(\text{Aut}(G))$. By the theorem, $G$ leaves $\alpha$ fixed, i.e., $\omega(\alpha(g)) = \alpha\omega(g)\alpha^{-1} = \omega(g)$ for all $g$ in $G$. It follows then $\alpha(g)g^{-1}$ is in the center $Z(G)$ of $G$ and the map $g \to \alpha(g)g^{-1}$ ($g \in G$) is a homomorphism of $G$ into $Z(G)$. Since $G$ is semi-simple, this map has to be trivial. Therefore $\alpha(g) = g$ for all $g$ in $G$, i.e., $\alpha$ is the identity map of $G$.

REFERENCES


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