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SMOOTH PARTITIONS OF UNITY ON MANIFOLDS

BY

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ABSTRACT. This paper continues the study of the smoothness properties of (real) topological linear spaces. First, the smoothness results previously obtained about various important classes of locally convex spaces, such as Schwartz spaces, are improved. Then, following the ideas of Bonic and Frampton, we use these results to give sufficient conditions for the existence of smooth partitions of unity on manifolds modelled on topological linear spaces.

1. **Preliminaries.** In order to make this paper self-contained we include the definition of the two types of derivative used and the definition of an $\mathcal{S}$-category. We will employ the definitions of the derivative in topological linear spaces investigated in detail by Averbukh and Smolyanov [2], [3]. See also [14], [15] and [19]. Let TLS denote the class of all Hausdorff topological linear spaces over the real field $\mathbf{R}$. Let $\mathfrak{L}_1(E, F) = \mathfrak{L}(E, F)$ denote the set of all continuous linear maps from E into F, where $E, F \in$ TLS. We define by induction $\mathfrak{L}_p(E, F) = \mathfrak{L}(E, \mathfrak{L}_{p-1}(E, F))$. Each $\mathfrak{L}_p(E, F)$ is given the topology of uniform convergence on bounded subsets of E. If X is a topological space, $\mathcal{O}(X)$ will denote the class of all open subsets of X.

Let $f: U \to V$, where $U \in \mathcal{O}(E)$, $V \in \mathcal{O}(F)$, $E \in$ TLS and $F \in$ TLS. Then we say f is *Fréchet differentiable at* $x \in U$, if there exists $u \in \mathfrak{L}(E, F)$ such that for each bounded subset B of E and for each o-neighbourhood W in F, there exists $\delta > 0$ such that $f(x + tb) - f(x) - u \cdot tb \in tW$, whenever $b \in B$ and $|t| \leq \delta$.

If we replace "bounded set" by "sequentially compact set" in the above definition, then we say f is *Hadamard differentiable at* $x \in U$.

In each case the mapping u is uniquely determined and is denoted by $f'(x)$. If f is Fréchet (resp. Hadamard) differentiable at each $x \in U$, then we say f is *Fréchet* (resp. *Hadamard*) *differentiable*. The map $f': U \to \mathfrak{L}(E, F)$ defined by $x \to f'(x)$ is called the *Fréchet* (resp. *Hadamard*) *derivative of* f. Higher order derivatives are defined in the usual way [2, p. 227], [16, p. 8]. The nth order Fréchet (resp. Hadamard) derivative is denoted by $f^{(n)}$ and is a map from U into

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$\mathcal{L}_n(E, F)$. In case E and F are normed linear spaces, the Hadamard derivative coincides with the "quasi-derivative" [3, p. 91], [7, p. 157] and the Fréchet derivative coincides with the standard definition of derivative in normed linear spaces [2, p. 214], [7, p. 149]. Note that a Fréchet differentiable mapping defined on a nonmetrizable topological linear space may not be continuous [2, p. 213], [3, p. 105].

We now discuss the difficulties associated with the definition of C^k-mappings in topological linear spaces. First we note that the two important properties we will require of C^k-mappings are that they have the *composition property* (that is, the composite, when it is defined, of two C^k-mappings is again a C^k-mapping) and that the definition reduces to the standard definition in normed spaces.

Of course, in normed spaces there is no difficulty. A C^k-mapping is a k-times Fréchet differentiable mapping such that each derivative is continuous. One easily proves by induction that the composition property holds for such mappings [7, p. 183]. However, an examination of this proof reveals a grave difficulty if one attempts to generalise it to topological linear spaces. The fact that the composition mapping, comp: $\mathcal{L}(E, F) \times \mathcal{L}(F, G) \to \mathcal{L}(E, G)$ defined by $\operatorname{comp}(u, v) = v \circ u$ is continuous, when F is normed, is used. Unfortunately, as Dieudonné and Schwartz [8, p. 76] have shown, comp is not continuous in general.

Several attempts have been made to overcome this difficulty. For example, Frölicher and Bucher [9] have developed the calculus on pseudo-topological linear spaces. However we prefer to stay within the framework of topological linear spaces. This leads us to adopt a definition of C^k-mappings, which includes an extra condition ("local boundedness") on each derivative. Such a definition has also been proposed by Penot [16]. We show this definition has the composition property and reduces to the standard definition in normed spaces.

However, before we give the modified definition, it is interesting to see what can be done with the standard one. Thus let us suppose *temporarily* that a C^k-mapping is a k-times Fréchet differentiable continuous mapping, whose derivatives are all continuous. If U and V are open subsets of topological linear spaces, we let $C^k(U, V)$ denote the collection of all C^k-mappings from U to V. Then we have

1.1. (*Using the standard definition of* C^k-*mappings.*) *Let* $f \in C^k(U, V)$ *and* $g \in C^k(V, W)$, *where* $k \in \{1, 2, \cdots, \infty\}$. *Then* $g \circ f \in C^k(U, W)$ *in each of the following cases*:

(A) V *is an open subset of a normed linear space.*

(B) U *is a* k-*space and* V *and* W *are open subsets of topological linear spaces, which are separated by their duals.*

(C) f *or* g *is a translate of a continuous linear mapping.*

Proof. (A) In this case, comp is a continuous bilinear mapping, and so the standard proof given in [7, p. 183] applies.

(B) Since the second and third spaces are separated by their duals, the derivatives $f^{(n)}$ and $g^{(n)}$, for each n, are symmetric. The proof then follows using the formula for $(g \circ f)^{(n)}$ [16, p. 9], the symmetry of the derivatives and the fact that to prove a mapping defined on a k-space is continuous, it suffices to prove its restriction to each compact subset is continuous. This proof is essentially the same as the proof of the composition property for the modified definition of C^k-mappings below.

(C) Suppose $f\colon U \to V$ is defined by $f(x) = u(x) + b$, where $u \in \mathcal{L}(E, F)$ and $b \in F$ $(U \in \mathcal{O}(E), V \in \mathcal{O}(F))$. Then one can prove directly that $(g \circ f)^{(n)}(x) \cdot h_1 \cdots h_n = g^{(n)}(f(x)) \cdot (u \cdot h_1) \cdots (u \cdot h_n)$. That $(g \circ f)^{(n)}$ is continuous follows immediately.

Now suppose $g\colon V \to W$ is defined by $g(y) = v(y) + c$, where $v \in \mathcal{L}(F, G)$ and $c \in G$ $(W \in \mathcal{O}(G))$. Then one proves that $(g \circ f)^{(n)}(x) \cdot h_1 \cdots h_n = v(f^{(n)}(x) \cdot h_1 \cdots h_n)$, and so $(g \circ f)^{(n)}$ is continuous.

We remark that when [13] was written we were unaware of the difficulty associated with the composition of C^k-mappings. However all the compositions of C^k-mappings employed in [13] are of the type (A) or (C) in 1.1. Thus all the results and proofs there concerning C^k-mappings are valid.

The reason for the introduction of $\mathcal{S}$-categories by Bonic and Frampton was to enable them to prove smoothness results for different types of smoothness *in a unified manner.* Since an $\mathcal{S}$-category is, in particular, a *category*, we cannot fit the standard definition of C^k-mappings into the framework of $\mathcal{S}$-categories. Thus, although one can prove C^k-smoothness results without the full strength of the composition property (as in [13]), it is obviously desirable to have a definition of C^k-mappings which can be covered by the general theory of $\mathcal{S}$-categories. A more important reason for modifying the definition of C^k-mappings to get the composition property is that the composition property becomes essential as soon as one attempts to apply the smoothness results to obtain the existence of smooth partitions of unity on manifolds. The axiom of the compatibility of charts cannot be used unless one has the composition property!

Now we give the modified definition of C^k-mappings which will be used throughout the remainder of this paper. Let $f\colon U \to V$ be a continuous mapping, where $U \in \mathcal{O}(E)$, $V \in \mathcal{O}(F)$, $E \in$ TLS and $F \in$ TLS. We say f is a *C^k-mapping* $(k = 1, 2, \cdots)$ if the following two conditions are satisfied:

(a) f is k-times Fréchet differentiable and each derivative $f^{(n)}\colon U \to \mathcal{L}_n(E, F)$ $(n = 1, \cdots, k)$ is continuous.

(b) For each $x \in U$ and for each derivative $f^{(n)}$ $(n = 1, \cdots, k)$, there exists an open neighbourhood N of x contained in U such that $f^{(n)}(N)$ is a bounded subset of $\mathfrak{L}_n(E, F)$. That is, each derivative is "locally bounded".

We say f is a C^∞*-mapping* if it is a C^k-mapping, for each $k = 1, 2, \cdots$.

Notice that if E and F are normed, then the local boundedness of each $f^{(n)}$ follows immediately from its continuity, and so our definition reduces to the standard definition in normed spaces.

Next we state the composition property for C^k-mappings. Let us denote by $C^k(U, V)$, the class of all C^k-mappings from U into V.

1.2. *Let $f \in C^k(U, V)$ and $g \in C^k(V, W)$, where $k \in \{1, 2, \cdots, \infty\}$, $U \in \mathcal{O}(E)$, $V \in \mathcal{O}(F)$ and $W \in \mathcal{O}(G)$. Suppose F and G are both separated by their duals. Then $g \circ f \in C^k(U, W)$.*

Proof. The proof uses the symmetry of the higher derivatives and the formula for $(g \circ f)^{(n)}$. Since it is long but straightforward we omit it. A detailed proof can be found in [15].

Seminorms have the following useful property. Let LCS denote the class of all (Hausdorff) locally convex spaces over $\mathbb{R}$.

1.3. *Let $E \in LCS$ and p be a continuous seminorm on E. Suppose p is Gâteaux (= weakly) differentiable on $\omega \subset E$. Then $\{p'(x) \mid x \in \omega\}$ is bounded in E' (the strong dual of E).*

Proof. This follows immediately from the result that for each $x \in \omega$ and $y \in E$, $|p'(x) \cdot y| \le p(y)$ [10, p. 348].

1.3 shows that the "local boundedness" condition in the definition of C^1-mappings is automatically satisfied by seminorms.

Let $E, F \in$ TLS and $U \in \mathcal{O}(E)$, $V \in \mathcal{O}(F)$. Then we define collections of mappings $C^0(U, V)$, $D_F^k(U, V)$ and $D_H^k(U, V)$ from U into V as follows:

(i) $C^0(U, V)$ is the class of all continuous mappings from U into V.

(ii) $D_F^k(U, V)$ (resp. $D_H^k(U, V)$) ($k = 1, 2, \cdots, \infty$) is the class of all continuous mappings from U into V, which are k-times Fréchet (resp. Hadamard) differentiable.

Throughout the remainder of this paper we restrict attention to the class of topological linear spaces, which are *separated by their duals*. TLS will denote the class of all such spaces.

We then define the categories C^0, D_H^k, D_F^k and C^k ($k = 1, 2, \cdots, \infty$) as follows:

C^0 is the category whose objects are all open subsets of all topological linear spaces (separated by their duals). For each pair of objects U and V, the morphisms from U into V are defined to be $C^0(U, V)$, with the usual composition of mappings as their product.

D_H^k (resp. D_F^k, resp. C^k) ($k = 1, 2, \cdots, \infty$) has the same objects as C^0 and for each pair of objects U and V, the morphisms from U into V are defined to be $D_H^k(U, V)$ (resp. $D_F^k(U, V)$, resp. $C^k(U, V)$).

The composition property for the categories D_H^k and D_F^k is proved in [16, p. 9]. (Actually Penot only gives the proof for D_F^k, but since the higher Hadamard derivatives are symmetric, it can easily be modified to cover D_H^k also [15].) See also [2, p. 234].

We now give the definition of an $\mathcal{S}$-category. This definition was first given in Banach spaces by Bonic and Frampton [5, p. 878].

An $\mathcal{S}$-*category* is a category S, whose objects are all open subsets of all topological linear spaces (separated by their duals). For each pair of objects U and V, the morphisms $S(U, V)$ are mappings from U into V with the usual composition as their product. We suppose also that the morphisms satisfy the following conditions:

S1. $C^\infty(U, V) \subset S(U, V) \subset C^0(U, V)$, for each pair of objects U and V.

S2. If $f \in S(U, V)$ and W is an open subset of V containing $f(U)$, then $f \in S(U, W)$.

S3. If for each $x \in U$, there is an open set W with $x \in W \subset U$ such that $f|W \in S(W, V)$, then $f \in S(U, V)$.

S4. If $f_1 \in S(U_1, V_1)$ and $f_2 \in S(U_2, V_2)$, then $f_1 \times f_2 \in S(U_1 \times U_2, V_1 \times V_2)$.

The $\mathcal{S}$-categories which are of interest to us are C^0, D_H^k, D_F^k and C^k ($k = 1, 2, \cdots, \infty$). The verification of axioms S1 to S4 for each of these categories is straightforward.

2. Smoothness of some classes of locally convex spaces. In this section, we improve the smoothness results obtained in [13].

Suppose $E \in$ TLS and S is an $\mathcal{S}$-category. Then we say E is S-*smooth*, if given $V \in \mathcal{O}(E)$ and $a \in V$, there exists $f \in S(E, \mathbf{R})$ such that $f(a) > 0$, $f \geq 0$ and $\{x \in E \mid f(x) > 0\} \subset V$.

If p is a seminorm on a vector space, we denote by N_p, the set $\{x \in E \mid p(x) = 0\}$.

Suppose $E \in$ LCS and S is an $\mathcal{S}$-category. We say E is *strongly* S-*smooth* if there exists a collection $P(E)$ of continuous seminorms on E which generate the topology on E and satisfy $p \in S(E \setminus N_p, \mathbf{R})$, for each $p \in P(E)$.

In [13, p. 230], it was shown that a strongly S-smooth locally convex space is S-smooth. Also the following result was proved. For the definition of locally convex kernels, we refer the reader to [10, §19]. Products and subspaces are special cases of locally convex kernels.

2.1 **(Kernel theorem)** [13, p. 232]. *Let* $E[\mathfrak{T}] = K_\alpha A_\alpha^{-1}(E_\alpha[\mathfrak{T}_\alpha])$ *be the locally convex kernel of the locally convex spaces* $\{E_\alpha\}_{\alpha \in A}$. *Let* S *be an* $\mathcal{S}$-*category. Then* E_α *is* S-*smooth (resp. strongly* S-*smooth), for each* $\alpha \in A$, *implies* E *is* S-*smooth (resp. strongly* S-*smooth).*

Then, as corollaries of the kernel theorem, we have

2.2. *Every Schwartz space (and, in particular, every nuclear space) is strongly C^∞-smooth.*

Proof. Randtke [17, p. 6] has shown that every Schwartz space E is topologically isomorphic to a subspace of a product $(c_0)^A$, for some index set A (depending on E). Since c_0 is strongly C^∞-smooth [5, p. 896], the result follows from the kernel theorem.

2.3. *Every separable or Lindelöf locally convex space is strongly D_H^1-smooth.*

Proof. In [13, p. 232], it was shown that every separable normed space is strongly D_H^1-smooth. But every separable or Lindelöf locally convex space is topologically isomorphic to a subspace of a topological product of separable normed spaces [10, p. 208]. The result follows by applying the kernel theorem.

Before we can prove the next result (2.7), we will need three lemmas (2.4, 2.5 and 2.6).

2.4. *Let $E \in LCS$ and p be a continuous seminorm on E. Then p is Gâteaux differentiable at $x \in E$ if and only if there exists $u \in E'$ such that for every Cauchy net $(h_\alpha)_{\alpha\in A}$ in E and for every net $(t_\alpha)_{\alpha\in A}$ in $\mathbf{R}$ with $t_\alpha \to 0$ $(t_\alpha \neq 0)$, we have $t_\alpha^{-1} \cdot [p(x + t_\alpha h_\alpha) - p(x)] - u \cdot h_\alpha \to 0$.*

Proof. Clearly the condition implies p is Gâteaux differentiable at x. For the converse, suppose the negation of the condition holds. Hence, for all $u \in E'$, there exists a Cauchy net $(h_\alpha)_{\alpha\in A}$ in E, there exists $t_\alpha \to 0$ and there exists $\epsilon > 0$ such that

$$|t_\alpha^{-1} \cdot [p(x + t_\alpha h_\alpha) - p(x)] - u \cdot h_\alpha| > \epsilon,$$

for every $\alpha \in A$.

Now $(u \cdot h_\alpha)_{\alpha\in A}$ is a Cauchy net in $\mathbf{R}$. Hence there exists $\beta \in A$ such that $\alpha \geq \beta$ implies $|u \cdot h_\alpha - u \cdot h_\beta| \leq \epsilon/3$ and $p(h_\alpha - h_\beta) \leq \epsilon/3$. Further

$$|t_\alpha^{-1} \cdot [p(x + t_\alpha h_\alpha) - p(x)] - u \cdot h_\alpha|$$
$$\leq |t_\alpha^{-1} \cdot [p(x + t_\alpha h_\beta) - p(x)] - u \cdot h_\beta| + p(h_\alpha - h_\beta) + |u \cdot (h_\alpha - h_\beta)|.$$

Thus, for every $\alpha \geq \beta$, we have

$$|t_\alpha^{-1} \cdot [p(x + t_\alpha h_\beta) - p(x)] - u \cdot h_\beta| \geq \epsilon - [\epsilon/3 + \epsilon/3] = \epsilon/3.$$

Thus p is not Gâteaux differentiable at x.

2.5. *Let $E \in LCS$ and p be a continuous seminorm on E. Suppose bounded subsets of E are precompact. Then p is Gâteaux differentiable at $x \in E$ if and only if p is Fréchet differentiable at x.*

Proof. This follows easily from 2.4.

The next lemma is a special case of a result of Asplund and Rockafellar [1, p. 459].

2.6. *Let $E \in LCS$ and p be a continuous seminorm on E. Suppose that p is Fréchet differentiable on $\omega \subset E$. Then $p': \omega \to E'$ is continuous.*

Now we can prove

2.7. *Let E be a separable or Lindelöf locally convex space such that bounded subsets of E are precompact. Then E is strongly C^1-smooth.*

Proof. This follows directly from 2.3, 2.5, 2.6 and 1.3.

3. **Partitions of unity.** This section contains the application of the smoothness results of §2, to proving the existence of smooth partitions of unity on manifolds.

We begin with the definition of a manifold of class S. This is a straightforward generalisation of the usual notion of a manifold of class C^k [11, p. 16], [5, p. 882]. As before, we consider only those topological linear spaces which are separated by their duals.

Let M be a regular topological space, $E \in$ TLS and S an $\mathcal{S}$-category. A pair (U, ϕ), where U is an open subset of M, $\phi(U)$ is an open subset of E and $\phi: U \to \phi(U)$ is a homeomorphism, will be called a *chart*. An *atlas of class* S is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ such that $\{U_\alpha\}$ covers M and, whenever $U_\alpha \cap U_\beta \neq \emptyset$, the mapping $\phi_\beta \circ \phi_\alpha^{-1} \in S(\phi_\alpha(U_\alpha \cap U_\beta), \phi_\beta(U_\alpha \cap U_\beta))$. One then defines the usual equivalence relation on the class of all atlases of class S on M. An equivalence class of atlases of class S on M is said to define a structure of *manifold of class* S on M. We then say M is a *manifold of class* S *modelled on* $E \in$ TLS. A chart belonging to an atlas in the equivalence class is called a *chart of* M. If (U, ϕ) is a chart of M and $a \in U$, we say (U, ϕ) is a *chart at* a.

An open subset of a topological linear space E is a manifold of class S modelled on E, for any $\mathcal{S}$-category S, in the natural way. Products of manifolds are defined in the usual way [11, p. 17]. We may now enlarge the class of objects in S to include all manifolds of class S. Given manifolds M and N of class S, the morphisms $S(M, N)$ are defined as follows: $S(M, N)$ consists of all continuous maps $f: M \to N$ such that for each $x \in M$, there exists a chart (U, ϕ) at x and a chart (V, ψ) at $f(x)$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1} \in S(\phi(U), \psi(V))$. Thus enlarged S is a category and satisfies the obvious extensions of axioms S2, S3 and S4 to manifolds.

It will be convenient to extend the definition of S-smoothness to manifolds.

A manifold M of class S is *S-smooth*, if given $V \in \mathcal{O}(M)$ and $a \in V$, there exists $f \in S(M, \mathbf{R})$ such that $f(a) > 0$, $f \geq 0$ and $\{x \in M \mid f(x) > 0\} \subset V$.

We omit the easy proofs of the next two results.

3.1. *Let M be a manifold of class S. Then the following are equivalent*:

(i) *M is S-smooth.*

(ii) *Given $V \in \mathcal{O}(M)$ and compact $A \subset V$, there exists $U \in \mathcal{O}(M)$ and $f \in S(M, \mathbf{R})$ such that $A \subset U$, $0 \leq f \leq 1$, $f|U = 1$ and $\{x \in M \mid f(x) > 0\} \subset V$.*

(iii) *The topology on M is the same as the coarsest topology on M such that every mapping in $S(M, \mathbf{R})$ is continuous.*

3.2. *Let M be a manifold of class S modelled on $E \in TLS$. Then M is S-smooth if and only if E is S-smooth.*

Let M be a manifold of class S. A family of mappings $\{\phi_\alpha \in S(M, \mathbf{R})\}_{\alpha \in A}$ is an *S-partition of unity* if

(i) $\phi_\alpha \geq 0$, for each $\alpha \in A$,

(ii) $\{\{x \in M \mid \phi_\alpha(x) > 0\}\}_{\alpha \in A}$ is a locally finite open covering of M and

(iii) $\Sigma_{\alpha \in A} \phi_\alpha(x) = 1$, for each $x \in M$.

If $\{U_\beta\}_{\beta \in B}$ is an open covering of M, we say that the S-partition of unity $\{\phi_\alpha\}_{\alpha \in A}$ is *subordinate* to $\{U_\beta\}_{\beta \in B}$, if for each $\alpha \in A$, there exists $\beta \in B$ such that $\{x \in M \mid \phi_\alpha(x) > 0\} \subset U_\beta$.

M admits S-partitions of unity if, for each open covering $\{U_\beta\}_{\beta \in B}$ of M, there is an S-partition of unity $\{\phi_\alpha\}_{\alpha \in A}$ subordinate to $\{U_\beta\}_{\beta \in B}$.

A manifold M of class S is *S-normal* if given $V \in \mathcal{O}(M)$ and closed subset A of M such that $A \subset V$, there exists $f \in S(M, \mathbf{R})$ such that $0 \leq f \leq 1$, $f|A = 1$ and $\{x \in M \mid f(x) > 0\} \subset V$.

The proofs of the nontrivial implications in the next result (with the exception of (C) implies (A), when M is paracompact, which is straightforward) are almost exactly the same as the corresponding proofs for Banach manifolds [5, pp. 883–885], and so are omitted. Details may be found in [15].

3.3. *Let M be a manifold of class S. Consider the following properties*:

(A) *M admits S-partitions of unity,*

(B) *M has the S-approximation property,*

(C) *M is S-normal,*

(D) *M is S-smooth.*

Then (A) *implies* (B), (A) *implies* (C) *and* (C) *implies* (D). *If M is normal,* (B) *implies* (C). *If M is paracompact,* (C) *implies* (A). *If M is Lindelöf,* (D) *implies* (A).

Notice in connection with the Lindelöf assumption in 3.3 that there is a *separable* S-smooth locally convex space, which is not S-normal. The product space $\mathbf{R}^{\mathbf{R}}$ is separable and C^{∞}-smooth, since it is nuclear. But $\mathbf{R}^{\mathbf{R}}$ is not C^{∞}-normal, since it is not even normal [18].

The following lemma is needed to transfer closed sets between a manifold and the space it is modelled on.

3.4 [6, p. 36]. *Let M be a manifold modelled on $E \in TLS$ and let $a \in M$. Then there exists a chart (U, ϕ) at a with the following property: a subset Y of U is closed in M if and only if $\phi(Y)$ is closed in E.*

Then we have

3.5. *Let M be a manifold of class S modelled on $E \in TLS$. If M is paracompact and E is S-normal, then M admits S-partitions of unity.*

Proof. Given an open covering of M, we can find an atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$, which is a locally finite open refinement of the covering and such that each chart in the atlas has the property in 3.4. By the normality of M, there exist locally finite open refinements $\{V_\alpha\}_{\alpha\in A}$ and $\{W_\alpha\}_{\alpha\in A}$ such that $\overline{W}_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$, for each $\alpha \in A$.

Now $\overline{W}_\alpha$ is a closed subset of M and a subset of U_α. Hence $\phi_\alpha(\overline{W}_\alpha)$ is a closed subset of E. Also $\phi_\alpha(V_\alpha) \in \mathcal{O}(E)$. Since E is S-normal, there exist $g_\alpha \in S(E, \mathbf{R})$ such that $0 \le g_\alpha \le 1$, $g_\alpha = 1$ on $\phi_\alpha(\overline{W}_\alpha)$ and $\{x \in E \mid g_\alpha(x) > 0\} \subset \phi_\alpha(V_\alpha)$.

For each $\alpha \in A$, define $f_\alpha: M \to \mathbf{R}$ by $f_\alpha(x) = g_\alpha(\phi_\alpha(x))$, if $x \in U_\alpha$ and $f_\alpha(x) = 0$, otherwise. Clearly $f_\alpha \in S(M, \mathbf{R})$, $f_\alpha = 1$ on $\overline{W}_\alpha$ and $\{x \in M \mid f_\alpha(x) > 0\} \subset V_\alpha$. Since $\{V_\alpha\}$ is locally finite, we can define $f = \Sigma_{\alpha\in A} f_\alpha$. Then $f \in S(M, \mathbf{R}^+)$ $(\mathbf{R}^+ = \{\xi \in \mathbf{R} \mid \xi > 0\})$. Put $\phi_\alpha = f_\alpha / f$. Then $\{\phi_\alpha\}_{\alpha\in A}$ is the required S-partition of unity.

3.5 includes as special cases a theorem of Leduc [12, p. 12-08], a theorem of Bolis [4, p. 9] and a theorem of Bourbaki [6, p. 40].

As a corollary of 3.5, we have

3.6. *Let M be a manifold of class S modelled on $E \in TLS$. If M is paracompact and E is separable and S-smooth, then M admits S-partitions of unity.*

Proof. By 3.5 and 3.3, it suffices to show that if M is a paracompact manifold modelled on a separable topological linear space E, then E is Lindelöf.

Let $a \in M$ and (U, ϕ) be a chart at a, with the property in 3.4. Since M is regular, there exists $V \in \mathcal{O}(M)$ such that $a \in V \subset \overline{V} \subset U$. Since $\overline{V}$ is closed in M, $\phi(\overline{V})$ is closed in E. Thus $\phi(\overline{V}) \supset \overline{\phi(V)}$ (closure in E!). On the other hand, $\phi(\overline{V})$ is the closure of $\phi(V)$ in $\phi(U)$. Hence $\phi(\overline{V}) \subset \overline{\phi(V)}$, and so $\phi(\overline{V}) = \overline{\phi(V)}$.

Now $\phi(V) \in \mathcal{O}(E)$, and so $\phi(V)$ is separable. Thus $\phi(\overline{V})$ is separable. But $\phi(\overline{V})$ is paracompact, and so $\phi(\overline{V})$ is Lindelöf. After translating $\phi(\overline{V})$ to make o an interior point, we can write E as a countable union of Lindelöf spaces. Thus E is Lindelöf.

Applying 3.5 to the various classes of locally convex spaces studied in §2, we obtain

3.7. (i) *Let M be a paracompact manifold of class C^k ($k \in \{1, 2, \cdots, \infty\}$) modelled on a Lindelöf Schwartz space. Then M admits C^k-partitions of unity.*

(ii) *Let M be a paracompact manifold of class D^1_H modelled on a Lindelöf locally convex space. Then M admits D^1_H-partitions of unity.*

(iii) *Let M be a paracompact manifold of class C^1 modelled on a Lindelöf locally convex space E with the property that bounded subsets of E are precompact. Then M admits C^1-partitions of unity.*

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