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ISOLATED SINGULARITIES FOR SOLUTIONS OF THE NONLINEAR STATIONARY NAVIER-STOKES EQUATIONS

BY

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ABSTRACT. The notion for (u, p) to be a distribution solution of the nonlinear stationary Navier-Stokes equations in an open set is defined, and a theorem concerning the removability of isolated singularities for distribution solutions in the punctured open ball $B(0, r_0) - \{0\}$ is established. This result is then applied to the classical situation to obtain a new theorem for the removability of isolated singularities. In particular, in two dimensions this gives a better than expected result when compared with the theory of removable isolated singularities for harmonic functions.

1. **Introduction.** Let Ω be a bounded open set in Euclidean N-space, E_N, $N \geq 2$, and let $\mathbf{f} = (f_1, \cdots, f_N)$ be a fixed vector in $C^\alpha(\Omega)$, $0 < \alpha < 1$. $(\mathbf{u}, p)$ will be said to be a regular solution of the nonlinear stationary Navier-Stokes equations in Ω if $\mathbf{u}$ is in $C^{2+\alpha}(\Omega_1)$ and p is in $C^{1+\alpha}(\Omega_1)$ for every open subset Ω_1 whose closure is contained in Ω and if, furthermore, $(\mathbf{u}, p)$ satisfies

$$(1.1)\qquad \begin{aligned} &\nu\Delta u_j - u_k \partial u_j/\partial x_k - \partial p/\partial x_j + f_j = 0, \qquad j = 1, \cdots, N, \\ &\partial u_k/\partial x_k = 0, \end{aligned}$$

in Ω, where $\mathbf{u} = (u_1, \cdots, u_N)$, ν is a nonzero constant, and the summation convention is used.

We let $B(0, r_0)$ designate the open N-ball with center 0 and radius r_0 and obtain as a corollary to Theorems 2 and 3 the following result for regular solutions of (1.1) in $B(0, r_0) - \{0\}$.

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Theorem 1. *Let* $(\mathbf{u}, p)$ *be a regular solution of* (1.1) *in* $B(0, r_0) - \{0\}$ *where* f *is in* $C^{\alpha}[B(0, r_0)]$, $0 < \alpha < 1$. *Suppose that*

(i) *there is a* $\beta > N$ *such that* $\mathbf{u}$ *is in* $L^{\beta}[B(0, r_0)]$;

(ii) $|\mathbf{u}| = o(|x|^{-(N-1)/2})$ *as* $|x| \to 0$ *for the case* $N = 2$;

(iii) $|p| = o(|x|^{-(N-1)})$ *as* $|x| \to 0$.

Then $(\mathbf{u}, p)$ *can be defined at* 0 *so that* $(\mathbf{u}, p)$ *is a regular solution of* (1.1) *in* $B(0, r_0)$.

For $N \geq 3$, (ii) *is superfluous.*

We observe that for $N = 3$, Theorem 1 gives an improvement of the result mentioned in the next to the last paragraph of [10]. Also for $N = 2$, Theorem 1 gives a better than expected result when compared with the theory of removable isolated singularities for harmonic functions. (See the first paragraph of [10].)

If $(\mathbf{u}, p)$ is a regular solution of (1.1) in Ω, then it satisfies

$$\begin{aligned} &\nu\Delta u_j - \partial[u_j u_k]/\partial x_k - \partial p/\partial x_j + f_i = 0, \qquad j = 1, \cdots, N, \\ &\partial u_k/\partial x_k = 0 \end{aligned} \tag{1.2}$$

in Ω.

In view of this fact, we shall say that $(\mathbf{u}, p)$ is a distribution solution of the nonlinear stationary Navier-Stokes equations (1.1) in Ω [where we now only suppose that $\mathbf{f}$ is in $L^1(\Omega)$] if $\mathbf{u}$ is in $L^2(\Omega)$, p is in $L^1(\Omega)$, and

$$\begin{aligned} &\int_{\Omega} [\nu u_j \Delta\phi + u_j u_k \partial\phi/\partial x_k + p\partial\phi/\partial x_j + \phi f_j]\, dx = 0, \qquad j = 1, \cdots, N, \\ &\int_{\Omega} [u_k \partial\phi/\partial x_k]\, dx = 0 \end{aligned} \tag{1.3}$$

for all ϕ in $C_0^{\infty}(\Omega)$.

We intend to establish the following result for isolated singularities of distribution solutions of (1.1).

Theorem 2. *Let* $\mathbf{u}$ *be* $L^2[B(0, r_0)]$ *and* $\mathbf{f}$ *and* p *be in* $L^1[B(0, r_0)]$. *Suppose* $(\mathbf{u}, p)$ *is a distribution solution of the nonlinear stationary Navier-Stokes equations* (1.1) *in* $B(0, r_0) - \{0\}$. *Suppose furthermore that*

(i) $r^{-N} \int_{B(0,r)} |p|\, dx = o(r^{-(N-1)})$ *as* $r \to 0$, *and that*

(ii) $\{r^{-N} \int_{B(0,r)} |\mathbf{u}|^{\gamma} dx\}^{1/\gamma} = o(r^{-(N-1)/2})$ *as* $r \to 0$ *where* $\gamma = 2$ *for* $N \geq 3$ *and* $\gamma > 2$ *for* $N = 2$.

Then $(\mathbf{u}, p)$ *is a distribution solution of* (1.1) *in* $B(0, r_0)$.

We shall also establish the following regularity result for distribution solutions of (1.1).

Theorem 3. *Let* Ω *be a bounded open set in* E_N, $N \geq 2$, *and assume that* $\mathbf{u}$ *is in* $L^{\beta}(\Omega)$, $\beta > N$, *and that* p *is in* $L^1(\Omega)$. *Suppose furthermore that* $(\mathbf{u}, p)$ *is a distribution solution of the nonlinear stationary Navier-Stokes equations*

(1.1) *in* Ω, *where* $\mathbf{f}$ *is in* $C^{\alpha}(\Omega)$, $0 < \alpha < 1$. *Then* $(\mathbf{u}, p)$ *is equal almost everywhere in* Ω *to a regular solution of* (1.1).

It is clear that Theorem 1 is an immediate corollary to Theorems 2 and 3. We establish Theorems 2 and 3 below in §4 and §3, respectively.

2. **Fundamental lemmas.** In order to establish Theorems 2 and 3 we shall need some results in multiple trigonometric series.

We shall use the following notation: $B(x, r)$ will designate the open N-ball with center x and radius r; T_N will designate the N-dimensional torus $\{x: -\pi < x_j \leq \pi, j = 1, \cdots, N\}$; (x, y) will designate the usual inner product $x_1y_1 + \cdots + x_Ny_N$; and m will designate an integral lattice point.

For q in $L^1(T_N)$, we shall set

$$q\hat{}(m) = (2\pi)^{-N} \int_{T_N} q(x)e^{-i(m,x)}\,dx \tag{2.1}$$

and designate the Abel partial means of q for $t > 0$ by

$$q(x, t) = \sum_{m} q\hat{}(m)e^{i(m,x)-|m|t}. \tag{2.2}$$

(Throughout the rest of this paper, we shall not use the summation convention.)

It follows from [5, p. 56, (17)] that

$$q(x, t) = (2\pi)^{-N} \int_{T_N} q(y)A(x - y, t)\,dy \tag{2.3}$$

where

$$A(x, t) = \lim_{R\to\infty} b_N t \sum_{|m|\leq R} [t^2 + |x + 2\pi m|^2]^{-(N+1)/2} \tag{2.4}$$

and b_N is a positive constant.

Observing that for $t > 0$, $A(x, t) > 0$ for x in T_N and that $(2\pi)^{-N}\int_{T_N} A(x, t)dx = 1$, we see from (2.3) and (2.4) that

(2.5) if q in $L^{\gamma}(T_N)$, $1 \leq \gamma < \infty$, then

$\lim_{t\to 0} \int_{T_N} |q(x, t) - q(x)|^{\gamma}\,dx = 0.$

Next, for $j = 1, \cdots, N$, we introduce the functions $H_j(x)$ defined in E_N as follows:

$$H_j(x) = \lim_{t\to 0} \sum_{m\neq 0} im_j e^{i(m,x)-|m|t}/|m|^2. \tag{2.6}$$

From [5, p. 72], we obtain that the following properties prevail:

$$\begin{aligned} &H_j(x) \text{ is in } L^1(T_N), \quad \text{and} \\ &H_j^\wedge(m) = im_j/|m|^2 \quad \text{for } m \neq 0, \\ &H_j^\wedge(0) = 0. \end{aligned} \tag{2.7}$$

In a similar manner, we introduce the function $H(x)$ defined in E_N as follows:

$$H(x) = \lim_{t\to 0} \sum_{m\neq 0} e^{i(m,x)-|m|t}/|m|^2. \tag{2.8}$$

From [5, p. 72], we also obtain that the following properties prevail:

$$\begin{aligned} &H(x) \text{ is in } L^1(T_N), \text{ and} \\ &H^\wedge(m) = |m|^{-2} \text{ for } m \neq 0, \\ &H^\wedge(0) = 0. \end{aligned} \tag{2.9}$$

Also from [5, p. 72], we obtain

(2.10) $H(x) - |x|^2/2N$ and $H_j(x)$ for $j = 1, \cdots, N$ are harmonic in $E_N - \bigcup_m \{2\pi m\}$,

and from (2.6, (2.7), (2.8), (2.9), (2.10), and [6, Lemma 2] that

$$\partial H(x)/\partial x_j = H_j(x) \quad \text{for } x \text{ in } E_n - \bigcup_m \{2\pi m\}. \tag{2.11}$$

From [5, p. 72] we also have that there are finite constants b_N', b_N'', c_N', and c_N'' such that

$$\begin{aligned} &|H(x) - b_N'|x|^{-(N-2)}| < c_N' \quad \text{for } N \geq 3 \\ &|H(x) - b_N' \log|x|^{-1}| < c_N' \quad \text{for } N = 2 \\ &|H_j(x) - b_N' x_j |x|^{-N}| < c_N'' \quad \text{for } N \geq 2 \\ &\qquad \text{for } j = 1, \cdots, N \text{ and } x \text{ in } T_N - \{0\}. \end{aligned} \tag{2.12}$$

Next, using (2.6) through (2.12), we see that the following lemma holds:

Lemma 1. *Let $q(x)$ be in $L^1(T_N)$, let $0 < r_1 < 1$, and let n be a nonnegative integer. Suppose furthermore $q(x)$ is in $C^{n+\alpha}[B(0, r_1)]$, $0 < \alpha < 1$. Set*

$$Q(x) = (2\pi)^{-N} \int_{T_N} q(y)H(x-y)\,dy$$

and

$$Q_j(x) = (2\pi)^{-N} \int_{T_N} q(y) H_j(x-y)\, dy \quad \textit{for } j = 1, \cdots, N.$$

Then $Q(x)$ *is in* $C^{n+2+\alpha}[B(0, r)]$ *and* $Q_j(x)$ *is in* $C^{n+1+\alpha}[B(0, r)]$ *for every* r *such that* $0 < r < r_1$.

Next, we intend to establish the following lemma:

Lemma 2. *Let* $q(x)$ *be in* $L^1(T_N)$ *and let* $0 < r_1 < 1$. *Suppose that* $q(x)$ *is in* $L^\gamma[B(0, r_1)]$ *where* $1 < \gamma < N$. *For* $j = 1, \cdots, N$, *set*

$$Q_j(x) = (2\pi)^{-N} \int_{T_N} q(y) H_j(x-y)\, dy. \tag{2.13}$$

Then $Q_j(x)$ *is in* $L^{N\gamma/(N-\gamma)}[B(0, r)]$ *for every* r *such that* $0 < r < r_1$

Let

$$0 < r_4 < r_3 < r_2 < r_1. \tag{2.14}$$

The lemma will be established if we can show

$$Q_j(x) \text{ is in } L^{N\gamma/(N-\gamma)}[B(0, r_4)]. \tag{2.15}$$

In order to do this, we choose a function $\lambda(x)$ such that

$$\begin{aligned} \lambda(x) &= 1 \quad \text{for } 0 \le |x| \le r_3 \\ &= 0 \quad \text{for } r_2 \le |x| < \infty \quad \text{and} \\ \lambda(x) &\quad \text{in } C^\infty(E_N). \end{aligned} \tag{2.16}$$

Next, we set

$$\begin{aligned} Q_j'(x) &= (2\pi)^{-N} \int_{T_N} \lambda(y) q(y) H_j(x-y)\, dy \quad \text{and} \\ Q_j''(x) &= (2\pi)^{-N} \int_{T_N} [1 - \lambda(y)] q(y) H_j(x-y)\, dy. \end{aligned} \tag{2.17}$$

From (2.12), we have there is a constant c_N such that

$$|H_j(y)| \le c_N |y|^{-(N-1)} \quad \text{for } y \text{ in } T_N - \{0\}. \tag{2.18}$$

From (2.16), (2.17), and (2.18), we have that

$$|Q_j'(x)| \le c_N (2\pi)^{-N} \int_{B(0, r_2)} |\lambda(y) q(y)|\, |x-y|^{-(N-1)}\, dy \tag{2.19}$$

almost everywhere in $B(0, r_2)$.

Consequently, it follows from [3, Lemma 5, p. 13] and the hypothesis of the lemma that

(2.20) $$Q_j' \text{ is in } L^{N\gamma/(N-\gamma)}[B(0, r_4)].$$

Also from (2.10), (2.16), and (2.17), we have that

(2.21) $$Q_j'' \text{ is in } C^{\infty}[B(0, r_3)].$$

From (2.13) and (2.17), we have that $Q_j = Q_j' + Q_j''$ whenever both Q_j' and Q_j'' are finite. But then (2.15) follows immediately from (2.20) and (2.21), and the proof to the lemma is complete.

Lemma 3. *Let q, r_1, and Q_j be as in Lemma 2 except now suppose that $N < \gamma < \infty$. Then Q_j is in $C^{\delta}[B(0, r)]$ for every δ such that $0 < \delta < 1 - (N/\gamma)$ and for every r such that $0 < r < r_1$.*

Once again we suppose that (2.14) holds. Then the lemma will be established if we can show

(2.22) $$Q_j \text{ is in } C^{\delta}[B(0, r_4)] \quad \text{for } 0 < \delta < 1 - (N/\gamma).$$

With b_N'' as in (2.12), set

(2.23) $$H_{j1}(x) = H_j(x) - b_N'' x_j |x|^{-N} \quad \text{for } x \text{ in } B(0, 2) - \{0\}.$$

Then it follows from (2.10) and (2.12) that H_{j1} can be defined at 0 so that

(2.24) $$H_{j1} \text{ is harmonic in } B(0, 2).$$

Next, with λ defined as in (2.16), we set for x in $B(0, r_2)$

(2.25) $$Q_{j1}(x) = (2\pi)^{-N} \int_{B(0,r_2)} \lambda(y) q(y) H_{j1}(x - y)\, dy$$

and

(2.26) $$Q_{j2}(x) = (2\pi)^{-N} \int_{B(0,r_2)} \lambda(y) q(y)(x_j - y_j)|x - y|^{-N}\, dy.$$

With Q_j'' defined as in (2.17), we see from (2.13), (2.16), (2.25), and (2.26) that

(2.27) $$Q_j(x) = Q_{j1}(x) + Q_{j2}(x) + Q_j''(x) \quad \text{for } x \text{ in } B(0, r_3).$$

Now as before (2.21) holds. From (2.24) and (2.25) we see that

$$Q_{j1} \text{ is harmonic in } B(0, r_3). \tag{2.28}$$

On the other hand, from (2.16), (2.26), the hypothesis of the lemma [3, Lemma 5, p. 13] and the following inequality (see [8, (21), p. 221])

$$|(x_j + h_j)|x + h|^{-N} - x_j|x|^{-N}| \leq K|h|^{\delta}[|x + h|^{-(N-1+\delta)} + |x|^{-(N-1+\delta)}]$$

where K is a constant independent of x and h, we see that

$$Q_{j2} \text{ is in } C^{\delta}[B(0, r_4)] \quad \text{for } 0 < \delta < 1 - (N/\gamma). \tag{2.29}$$

But then (2.22) follows from (2.21), (2.27), (2.28) and (2.29), and the proof of the lemma is complete.

Next, we set

$$K_{jk}(x) = x_j x_k/|x|^{2+N} \quad \text{for } x \neq 0 \text{ and } j, k = 1, \cdots, N, \quad j \neq k, \tag{2.30}$$

and observe that $K_{jk}(x)$ is a spherical harmonic Calderón-Zygmund kernel (see [1, p. 261]). We then define for $j \neq k$ and for $x \neq 2\pi m$

$$\begin{aligned} K_{jk}^{**}(x) &= \lim_{R\to\infty} \sum_{1 \leq |m| \leq R} [K_{jk}(x + 2\pi m) - K_{jk}(2\pi m)] \quad \text{and} \\ K_{jk}^{*}(x) &= K_{jk}(x) + K_{jk}^{**}(x). \end{aligned} \tag{2.31}$$

Also, we observe from [1, p. 252] that the series defining $K_{jk}^{**}(x)$ is absolutely convergent for x in T_N^2 where

$$T_N^2 = \{z: z = x + y, \ |x| < 2 \text{ and } |y_j| < \pi \text{ for } j = 1, \cdots, N\}. \tag{2.31'}$$

In particular, we note that $K_j^{**}(0)$ is defined.

Next, we observe from [1, pp. 257–261] that

$$\begin{aligned} &\lim_{\epsilon\to 0} (2\pi)^{-N} \int_{T_N - B(0,\epsilon)} K_{jk}^{*}(x) e^{i(m,x)}\, dx \\ &= \xi_{jk} m_j m_k |m|^{-2} \quad \text{for } m \neq 0 \\ &= 0 \quad\quad\quad\quad\quad\;\; \text{for } m = 0 \end{aligned} \tag{2.32}$$

where ξ_{jk} is a nonzero constant.

We next establish the following lemma:

Lemma 4. *Let $q(x)$ be in $L^1(T_N)$ and let $0 < r_1 < 1$. Suppose that $q(x)$ is in $L^\gamma[B(0, r_1)]$ where $1 < \gamma < \infty$. Then for almost every x in $B(0, r_1)$ the following limit exists and is finite:*

$$q_{jk}(x) = \lim_{\epsilon \to 0} \xi_{jk}^{-1}(2\pi)^{-N} \int_{T_N - B(x,\epsilon)} q(y)K_{jk}^*(x-y)\,dy. \tag{2.33}$$

Furthermore, q_{jk} is in $L^\gamma[B(0, r)]$ for every r such that $0 < r < r_1$.

Once again we suppose that (2.14) holds. The lemma will be established if we show that

(2.34) the limit in (2.33) is finite valued almost everywhere in $B(0, r_4)$

and

$$q_{jk} \text{ is in } L^\gamma[B(0, r_4)]. \tag{2.35}$$

We let $\lambda(x)$ be as in (2.16) and we set

$$q'_{jk}(x) = \lim_{\epsilon \to 0} \xi_{jk}^{-1}(2\pi)^{-N} \int_{T_N - B(x,\epsilon)} \lambda(y)q(y)K_{jk}^*(x-y)\,dy. \tag{2.36}$$

Since $\lambda(y)q(y)$ is in $L^\gamma(T_N)$, we have from [1, Theorem 1, p. 253] that

$$q'_{jk}(x) \text{ exists and is finite almost everywhere in } T_N, \tag{2.37}$$

and from [1, Theorem 2, p. 253] that

$$q'_{jk}(x) \text{ is in } L^\gamma(T_N). \tag{2.38}$$

Next, we set

$$q''_{jk}(x) = \lim_{\epsilon \to 0} \xi_{jk}^{-1}(2\pi)^{-N} \int_{T_N - B(x,\epsilon)} [1 - \lambda(y)]q(y)K_{jk}^*(x-y)\,dy. \tag{2.39}$$

Since for $z \neq 0$ and z in T_N^2 [where T_N^2 is defined in (2.31′)], we have that $K_{jk}^*(z) = K_{jk}(z) + K_{jk}^{**}(z)$ and since furthermore $K_{jk}^{**}(z)$ is continuous in T_N^2, we conclude from (2.16) and (2.39) that

$$q''_{jk}(x) \text{ is continuous in } B(0, r_3). \tag{2.40}$$

But then from (2.33), (2.36), and (2.39) we have that $q_{jk}(x) = q'_{jk}(x) + q''_{jk}(x)$ whenever both $q'_{jk}(x)$ and $q''_{jk}(x)$ exist and are finite. Consequently (2.34) follows from (2.14), (2.37), and (2.40). Also (2.35) follows from (2.14), (2.38), and (2.40), and the proof of the lemma is complete.

Next, we state the following lemma.

Lemma 5. *Let $q(x)$ be in $L^1(T_N)$ and let $0 < r_1 < 1$. Suppose that $q(x)$ is continuous in $B(0, r_1)$. For $t > 0$ and $j \neq k$ where $j, k = 1, \cdots, N$, set*

$$q_{jk}(x, t) = \sum_{m \neq 0} q^\frown(m) m_j m_k |m|^{-2} e^{i(m,x)-|m|t}. \tag{2.41}$$

Then

$$\lim_{t \to 0} \left[q_{jk}(x, t) - (2\pi)^{-N} \xi_{jk}^{-1} \int_{T_N - B(x,t)} q(y) K_{jk}^*(x - y)\, dy \right]$$

$$= 0 \quad \text{for } x \text{ in } B(0, r_1).$$

The above lemma follows from the material in [4, pp. 44–46]. We leave the details to the reader.

Next, we establish the following lemma.

Lemma 6. *Let $q(x)$ be in $L^1(T_N)$ and let $0 < r_1 < 1$. For $t > 0$ and $j \neq k$ where $j, k = 1, \cdots, N$, let $q_{jk}(x, t)$ be defined by (2.41). Suppose that $q(x)$ is in $L^\gamma[B(0, r_1)]$ where $1 < \gamma < \infty$. Then for almost every x in $B(0, r_1)$,*

$$\lim_{t \to 0} q_{jk}(x, t) = q_{jk}(x)$$

where $q_{jk}(x)$ is defined almost everywhere by the limit in (2.33).

Once again we suppose that (2.14) holds. The lemma will be established if we show

$$\text{for almost every } x \text{ in } B(0, r_4), \quad \lim_{t \to 0} q_{jk}(x, t) = q_{jk}(x). \tag{2.42}$$

To establish (2.42), we let λ be as in (2.16) and set

$$q''_{jk}(x, t) = \sum_{m \neq 0} [q^\frown(m) - (\lambda q)^\frown(m)] m_j m_k |m|^{-2} e^{i(m,x)-|m|t}. \tag{2.43}$$

Since $[1 - \lambda(x)]q(x) = 0$ in $B(0, r_3)$, it follows from (2.39), (2.40), and Lemma 5 that

$$\lim_{t \to 0} q''_{jk}(x, t) = q''_{jk}(x) \quad \text{for } x \text{ in } B(0, r_3). \tag{2.44}$$

Next, we set

$$q'_{jk}(x, t) = \sum_{m \neq 0} (\lambda q)^\frown(m) m_j m_k |m|^{-2} e^{i(m,x)-|m|t} \tag{2.45}$$

and obtain from [4, pp. 44–46] (or from [1] and [5, pp. 55–56]) that

$$\lim_{t \to 0} \left\{ q'_{jk}(x, t) - \xi_{jk}^{-1}(2\pi)^{-N} \int_{T_N - B(x,t)} \lambda(y) q(y) K^*_{jk}(x-y)\, dy \right\} \tag{2.46}$$
$$= 0 \quad \text{almost everywhere in } T_N.$$

But then it follows from (2.36), (2.37), and (2.46) that

$$\lim_{t \to 0} q'_{jk}(x, t) = q'_{jk}(x) \quad \text{almost everywhere in } T_N. \tag{2.47}$$

From (2.41), (2.43), and (2.45), we have that

$$q_{jk}(x, t) = q'_{jk}(x, t) + q''_{jk}(x, t). \tag{2.48}$$

On the other hand, from (2.33), (2.36), and (2.39) we have that $q_{jk}(x) = q'_{jk}(x) + q''_{jk}(x)$ whenever both $q'_{jk}(x)$ and $q''_{jk}(x)$ exist and are finite. But then (2.42) follows from this fact, (2.37), (2.40), (2.44) and (2.47). The proof of the lemma is consequently complete.

Next, we establish the analogue of Lemma 4 for Hölder continuous functions.

Lemma 7. *Let $q(x)$ be in $L^1(T_N)$ and let $0 < r_1 < 1$. Suppose that $q(x)$ is in $C^\gamma[B(0, r_1)]$, $0 < \gamma < 1$. Then the limit in* (2.33), *called $q_{jk}(x)$, exists and is finite for every x in $B(0, r_1)$. Furthermore $q_{jk}(x)$ is in $C^\gamma[B(0, r)]$ for every r such that $0 < r < r_1$.*

Once again we suppose that (2.14) holds. The lemma will be established if we show that

(2.49) the limit in (2.33) is finite valued in $B(0, r_4)$,

and

$$q_{jk} \text{ is in } C^\gamma[B(0, r_4)]. \tag{2.50}$$

We let $\lambda(x)$ be as in (2.16) and we define $q'_{jk}(x)$ by the limit in (2.36). From [1, Theorem 11, p. 262] we have

(2.51) $q'_{jk}(x)$ exists and is finite everywhere in $B(0, r_1)$

and furthermore

$$q'_{jk}(x) \text{ is in } C^\gamma[B(0, r_1)]. \tag{2.52}$$

Next, for x in $B(0, r_3)$ we define $q''_{jk}(x)$ by the limit in (2.39). Since $\lambda(y) = 1$ in $B(0, r_3)$, we have that

$$q''_{jk}(x) \text{ exists and is finite everywhere in } B(0, r_3). \tag{2.53}$$

Furthermore from (2.31), we have that for x in $B(0, r_3)$

$$\begin{aligned} q''_{jk}(x) &= \xi_{jk}^{-1}(2\pi)^{-N} \int_{T_N - B(0,r_3)} [1 - \lambda(y)]q(y)K_{jk}(x-y)\,dy \\ &\quad + \xi_{jk}^{-1}(2\pi)^{-N} \int_{T_N - B(0,r_3)} [1 - \lambda(y)]q(y)K^{**}_{jk}(x-y)\,dy. \end{aligned} \tag{2.54}$$

But an easy computation shows that $K^{**}_{jk}(z)$ is in $C^1(T^2_N)$ where T^2_N is defined by (2.31′). Consequently, it follows from (2.54) that

$$q''_{jk}(x) \text{ is in } C^1[B(0, r_4)]. \tag{2.55}$$

Since $q_{jk}(x) = q'_{jk}(x) + q''_{jk}(x)$ whenever both $q'_{jk}(x)$ and $q''_{jk}(x)$ exist and are finite, (2.49) follows from (2.51) and (2.53). Also (2.50) follows from (2.52) and (2.55). The proof of the lemma is consequently complete.

Next, we establish the analogue of Lemma 6 for Hölder continuous functions.

Lemma 8. *Let $q(x)$ be in $L^1(T_N)$ and let $0 < r_1 < 1$. For $t > 0$ and $j \neq k$ where $j, k = 1, \cdots, N$, let $q_{jk}(x, t)$ be defined by* (2.41). *Suppose that $q(x)$ is in $C^\gamma[B(0, r_1)]$ where $0 < \gamma < 1$. Then for every x in $B(0, r_1)$,*

$$\lim_{t\to 0} q_{jk}(x, t) = q_{jk}(x)$$

where $q_{jk}(x)$ is defined everywhere in $B(0, r_1)$ by the limit in (2.33).

To establish the above lemma, we observe from its hypothesis and from Lemma 5 that

$$\lim_{t\to 0}\left[q_{jk}(x, t) - \xi_{jk}^{-1}(2\pi)^{-N}\int_{T_N - B(x,t)} q(y)K^*_{jk}(x-y)\,dy\right] = 0 \quad \text{everywhere in } B(0, r_1).$$

Invoking Lemma 7, we conclude from this last fact that

$$\lim_{t\to 0}\,[q_{jk}(x, t) - q_{jk}(x)] = 0 \quad \text{everywhere in } B(0, r_1).$$

This completes the proof of the lemma.

Next, we set

$$K_j(x) = \left\{(N-1)x_j^2 - \sum_{k=1;j\neq k}^{N} x_k^2\right\} \Big/ |x|^{2+N} \quad \text{for } x \neq 0,\ j = 1, \cdots, N. \tag{2.56}$$

We then define, for $x \neq 2\pi m$,

$$\begin{aligned} K_j^{**}(x) &= \lim_{R\to\infty} \sum_{1\leq|m|\leq R} [K_j(x+2\pi m) - K_j(2\pi m)] \quad \text{and} \\ K_j^{*}(x) &= K_j(x) + K_j^{**}(x) \end{aligned} \tag{2.57}$$

and observe as before that the series defining $K_j^{**}(x)$ is absolutely convergent for x in T_N^2, and in particular that $K_j^{**}(0)$ is defined.

Now $K_j(x)$ is a spherical harmonic Calderón-Zygmund kernel (see [1, p. 261]). As a consequence we have from [1, pp. 257–261] that

$$\begin{aligned} &\lim_{\epsilon\to 0} (2\pi)^{-N} \int_{T_N - B(0,\epsilon)} K_j^*(x) e^{-i(m,x)}\, dx \\ &= \xi_j \left[(N-1)m_j^2 - \sum_{k=1;\alpha\neq k}^{N} m_k^2\right] |m|^{-2} \quad \text{for } m \neq 0, \\ &= 0 \qquad\qquad \text{for } m = 0, \end{aligned} \tag{2.58}$$

where ξ_j is a nonzero constant.

The following two lemmas follow in a manner similar to that in which Lemmas 4, 6, 7, and 8 were established. We leave the details to the reader.

Lemma 9. *Let q be in $L^1(T_N)$ and let $0 < r_1 < 1$. Suppose that $q(x)$ is in $L^\gamma[B(0, r_1)]$ where $1 < \gamma < \infty$. Then for almost every x in $B(0, r_1)$ the following limit exists and is finite.*

$$q_j(x) = \lim_{\epsilon\to 0} \xi_j^{-1} (2\pi)^{-N} \int_{T_N - B(x,\epsilon)} q(y) K_j^*(x-y)\, dy. \tag{2.59}$$

Also, q_j is in $L^\gamma[B(0, r)]$ for every r such that $0 < r < r_1$. Furthermore for $t > 0$, set

$$q_j(x, t) = \sum_{m\neq 0} q^\wedge(m) \left[(N-1)m_j^2 - \sum_{k=1, j\neq k}^{N} m_k^2\right] |m|^{-2} e^{i(m,x) - |m|t}. \tag{2.60}$$

Then for almost every x in $B(0, r_1)$, $\lim_{t\to 0} q_j(x, t) = q_j(x)$.

Lemma 10. *Let $q(x)$ be in $L^1(T_N)$ and let $0 < r_1 < 1$. Suppose that $q(x)$ is in $C^\gamma[B(0, r_1)]$, $0 < \gamma < 1$. Then the limit in (2.59), called $q_j(x)$, exists and is finite for every x in $B(0, r_1)$. Also, $q_j(x)$ is in $C^\gamma[B(0, r)]$ for every r such that $0 < r < r_1$. Furthermore with $q_j(x, t)$ defined by (2.60), $\lim_{t\to 0} q_j(x, t) = q_j(x)$ for every x in $B(0, r_1)$.*

Next, we establish the following lemma.

Lemma 11. *Let $q(x)$ be in $L^1(T_N)$ and let $0 < r_1 < 1$. For $j = 1, \cdots, N$, and $t > 0$, set*

(2.61)
$$q_{jj}(x, t) = \sum_{m \neq 0} q^\frown(m) m_j^2 |m|^{-2} e^{i(m,x) - |m| t}.$$

(a) *Suppose q is in $L^\gamma[B(0, r_1)]$, $1 < \gamma < \infty$. Then there exists a function $q_{jj}(x)$ in $L^\gamma[B(0, r)]$ for every r such that $0 < r < r_1$ with the property that $\lim_{t \to 0} q_{jj}(x, t) = q_{jj}(x)$ almost everywhere in $B(0, r_1)$. Furthermore, $q_{jj}(x) = [q(x) - q^\frown(0) + q_j(x)]/N$ almost everywhere in $B(0, r_1)$ where $q_j(x)$ is defined by the limit in* (2.59).

(b) *Suppose q is in $C^\gamma[B(0, r_1)]$, $0 < \gamma < 1$. Then there exists a function $q_{jj}(x)$ in $C^\gamma[B(0, r)]$ for every r such that $0 < r < r_1$ with the property that $\lim_{t \to 0} q_{jj}(x, t) = q_{jj}(x)$ everywhere in $B(0, r_1)$. Furthermore, $q_{jj}(x) = [q(x) - q^\frown(0) + q_j(x)]/N$ everywhere in $B(0, r_1)$.*

We first establish (a) of Lemma 11. Since this part of the lemma is true for the special case $q \equiv$ constant in T_N, we see from the start that with no loss in generality, we can assume that

(2.62)
$$q^\frown(0) = 0.$$

Next, we observe that

(2.63)
$$m_j^2 = \left\{ |m|^2 + \left[(N-1) m_j^2 - \sum_{k=1; j \neq k}^{N} m_k \right] \right\} \Big/ N.$$

We consequently conclude from (2.2), (2.60), (2.61), (2.62), and (2.63) that

(2.64)
$$q_{jj}(x, t) = [q(x, t) + q_j(x, t)]/N.$$

From [5, Theorem 2, pp. 55–56], we have that

(2.65)
$$\lim_{t \to 0} q(x, t) = q(x) \quad \text{almost everywhere in } B(0, r_1).$$

From Lemma 9, we have that

(2.66)
$$\lim_{t \to 0} q_j(x, t) = q_j(x) \quad \text{almost everywhere in } B(0, r_1).$$

We conclude from (2.64), (2.65), and (2.66) that

(2.67) $$\lim_{t\to 0} q_{jj}(x, t) = q(x) + q_j(x) \quad \text{almost everywhere in } B(0, r_1).$$

The conclusion to (a) of Lemma 11 then follows immediately from the hypothesis, (2.62), (2.67), and Lemma 9.

The proof of (b) of Lemma 11 follows in a similar manner. In (2.65) and (2.66), we replace almost everywhere by everywhere, and we use Lemma 10 instead of Lemma 9. We leave the details to the reader.

Next, we state the following lemma.

Lemma 12. *Let $q(x)$ be in $L^1(T_N)$ and let $0 < r_1 < 1$. For $j, k = 1, \cdots, N$ and $t > 0$, set*

$$q_{jk}(x, t) = \sum_{m\neq 0} q^{\wedge}(m) m_j m_k |m|^{-2} e^{i(m,x) - |m|t}.$$

Suppose that $q(x)$ is in $L^\gamma[B(0, r_1)]$, $1 < \gamma < N/(N-1)$, and that

$$\left\{ r^{-N} \int_{B(0,r)} |q(x)|^\gamma \, dx \right\}^{1/\gamma} = o(r^{-(N-1)}) \quad \text{as } r \to 0.$$

Then there exists a function $q_{jk}(x)$ which is in $L^\gamma[B(0, r)]$ for every r such that $0 < r < r_1$ with the property that $\lim_{t\to 0} q_{jk}(x, t) = q_{jk}(x)$ almost everywhere in $B(0, r_1)$. Furthermore

$$\left\{ r^{-N} \int_{B(0,r)} |q_{jk}(x)|^\gamma \, dx \right\}^{1/\gamma} = o(r^{-(N-1)}) \quad \text{as } r \to 0.$$

Lemma 12 follows from the techniques used to establish Lemmas 3, 4, and 11(a) and [9, Lemma 5.1, p. 206]. We leave the details to the reader.

Next, we state the following lemma.

Lemma 13. *Let $q(x)$ be in $L^1(T_N)$ and define $Q(x)$ and $Q_j(x)$ for $j = 1, \cdots, N$ almost everywhere in T_N by setting*

$$Q_j(x) = (2\pi)^{-N} \int_{T_N} q(y) H_j(x - y) \, dy$$

and

$$Q(x) = (2\pi)^{-N} \int_{T_N} q(y) H(x - y) \, dy.$$

Then $Q(x)$ and $Q_j(x)$ are in $L^1(T_N)$ and as $r \to 0$,

(2.68) $$r^{-N} \int_{B(0,r)} |Q_j(x)| \, dx = o(r^{-(N-1)})$$

and

$$r^{-N}\int_{B(0,r)}|Q(x)|\,dx = o(r^{-(N-2)}) \quad \textit{for } N \geq 3, \tag{2.69}$$
$$= o(\log r^{-1}) \quad \textit{for } N = 2.$$

The proof of (2.68) is very similar to that given on [7, p. 90]. (2.69) follows in an analogous manner.

3. **Proof of Theorem 3.** To prove Theorem 3, it is clear that we can assume with no loss in generality that $\Omega = B(0, r_0)$ where $0 < r_0 < 1$. We let $0 < r_1 < r_0$ and observe that the proof will be complete if we can show that there is a vector field $\mathbf{u}'$ and a function p' such that $\mathbf{u}'$ is in $C^{2+\alpha}[B(0, r)]$ and p' is in $C^{1+\alpha}[B(0, r)]$ for every r such that $0 < r < r_1$, suth that $(\mathbf{u}', p')$ satisfy (1.1) in $B(0, r_1)$ and such that $\mathbf{u}' = \mathbf{u}$ and $p' = p$ almost everywhere in $B(0, r_1)$. In order to accomplish this fact, we introduce

$$r_1 < r_1' < r_0' < r_0 < 1,$$

and choose a function $\eta(x)$ which is in $C^\infty(E_N)$, which equals one in $B(0, r_1')$ and equals zero in $E_N - B(0, r_0')$. We then define a periodic vector field $\mathbf{u}^*(x)$ as follows:

$$\begin{aligned} \mathbf{u}^*(x) &= \eta(x)\mathbf{u}(x) && \text{for } x \text{ in } B(0, r_0), \\ &= 0 && \text{for } x \text{ in } T_N - B(0, r_0), \\ \mathbf{u}^*(x) &= \mathbf{u}^*(x - 2\pi m) && \text{for } x \text{ in } T_N + 2\pi m. \end{aligned}$$

We define $p^*(x)$ and $\mathbf{f}^*(x)$ in a similar manner and observe that $(\mathbf{u}^*, p^*)$ is a distribution solution of (1.1) in $B(0, r_1)$ with $\mathbf{f}$ replaced by $\mathbf{f}^*$, i.e. $(\mathbf{u}^*, p^*)$ satisfies (1.3) with $\Omega = B(0, r_1)$ and $\mathbf{f}$ replaced by $\mathbf{f}^*$.

It then follows that Theorem 3 will be established if we establish the following lemma.

Lemma 14. *Let* $\mathbf{u}$, p *and* $\mathbf{f}$ *be defined in* E_N *and be periodic of period* 2π *in each variable. Assume also that* $\mathbf{u}$ *is in* $L^\beta(T_N)$, $\beta > N$, *that* p *is in* $L^1(T_N)$ *and that* $\mathbf{f}$ *is in* $C^\alpha(E_N)$, $0 < \alpha < 1$. *Suppose that* $(\mathbf{u}, p)$ *is a distribution solution of* (1.1) *in* $B(0, r_1)$, $0 < r_1 < 1$, *i.e.*, $(\mathbf{u}, p)$ *satisfies* (1.3) *with* $\Omega = B(0, r_1)$. *Then* $(\mathbf{u}, p)$ *is almost everywhere equal to a regular solution of* (1.1) *in* $B(0, r_1)$.

We see that in order to establish Lemma 14, we can assume from the start with no loss in generality that

$$\hat{f_j}(0) = 0, \qquad j = 1, \cdots, N. \tag{3.1}$$

We first intend to establish

(3.2) $$p(x) \text{ is in } L^{\beta/2}[B(0, r)] \quad \text{for every } r \text{ such that } 0 < r < r_1.$$

For ease of notation for $0 < r < 1$, we shall say

(3.3) $$\begin{aligned} &\phi \text{ is in } C_0^\infty[r, T_N] \text{ if } \phi \text{ is in } C_0^\infty[B(0, r)] \text{ and} \\ &\phi = 0 \text{ in } T_N - B(0, r). \end{aligned}$$

We next observe from (2.5) that if q is in $L^1(T_N)$, ϕ is in $C_0^\infty[r_1, T_N]$, and $q(x, t)$ is given by (2.2) that

(3.4) $$\begin{aligned} \int_{T_N} q(x) \frac{\partial\phi(x)}{\partial x j} dx &= \lim_{t\to 0} \int_{T_N} q(x, t) \frac{\partial\phi(x)}{\partial x j} dx \\ &= -(2\pi)^N \sum_m q\hat{\ }(m) i m_j \phi\hat{\ }(-m). \end{aligned}$$

As a consequence, we conclude from the hypothesis of Lemma 14, from (1.3), and from (3.1) that, for ϕ in $C_0^\infty[r_1, T_N]$,

(3.5) $$\sum_{m\neq 0} \left\{ \nu u_j\hat{\ }(m)|m|^2 + i \sum_{k=1}^N (u_j u_k)\hat{\ }(m) m_k + i m_j p\hat{\ }(m) - f_j\hat{\ }(m) \right\} \phi\hat{\ }(-m) = 0$$
$$\text{for } j = 1, \cdots, N,$$

and

(3.6) $$\sum_{m\neq 0} i \left(\sum_{j=1}^N m_j u_j\hat{\ }(m) \right) \phi\hat{\ }(-m) = 0.$$

(We shall not use the summation convention in establishing Lemma 14.)

Next, with $H_j(x)$ defined by (2.6), we set

(3.7) $$P_j(x) = (2\pi)^{-N} \int_{T_N} p(y) H_j(x - y)\, dy$$

and

(3.8) $$U_{jk}(x) = (2\pi)^{-N} \int_{T_N} u_j(y) u_k(y) H_k(x - y)\, dy.$$

With $H(x)$ defined by (2.8), we set

(3.9) $$\mathcal{F}_j(x) = (2\pi)^{-N} \int_{T_N} f_j(y) H(x - y)\, dy$$

and then we define

$$v_j(x) = P_j(x) + \sum_{k=1}^{N} U_{jk}(x) - \mathcal{F}_j(x). \tag{3.10}$$

Now it follows from (3.7), (3.8), (3.9), (3.10) that

$$\begin{aligned} v_j^\wedge(m) &= \left[im_j p^\wedge(m) + i \sum_{k=1}^{N} m_k (u_j u_k)^\wedge(m) - f_j^\wedge(m) \right] |m|^{-2} \quad \text{for } m \neq 0, \\ &= 0 \quad \text{for } m = 0. \end{aligned} \tag{3.11}$$

As a consequence, we obtain from (3.5) and (3.11) that

$$\sum_{m \neq 0} [\nu u_j^\wedge(m) + v_j^\wedge(m)] |m|^2 \phi^\wedge(-m) = 0 \quad \text{for } j = 1, \cdots, N. \tag{3.12}$$

Now (3.12) implies that

$$\int_{B(0,r_1)} [\nu u_j(x) + v_j(x)] \Delta\phi(x)\, dx = 0, \qquad j = 1, \cdots, N, \tag{3.13}$$

for all ϕ in $C_0^\infty[r_1, T_N]$.

We conclude from Weyl's lemma [2, p. 199] that for $j = 1, \cdots, N$ there is a function $w_j(x)$ such that

(3.14) $w_j(x)$ is harmonic in $B(0, r_1)$ and equal

almost everywhere in T_N to $\nu u_j(x) + v_j(x)$.

Next, we set

$$\mathfrak{U}(x) = \sum_{j=1}^{N} (2\pi)^{-N} \int_{T_N} u_j(y) H_j(x-y)\, dy \tag{3.15}$$

and observe that

$$\begin{aligned} \mathfrak{U}^\wedge(m) &= \sum_{j=1}^{N} im_j u_j^\wedge(m) |m|^{-2} \quad && \text{for } m \neq 0, \\ &= 0 && \text{for } m = 0. \end{aligned} \tag{3.16}$$

From (3.6) and (3.16), it follows that

$$\int_{B(0,r_1)} \mathfrak{U}(x) \Delta\phi(x)\, dx = 0$$

for all ϕ in $C_0^\infty[r_1, T_N]$, and we conclude once again from Weyl's lemma that

$$\begin{aligned} &\mathfrak{U}(x) = \mathfrak{U}'(x) \quad \text{almost everywhere in } B(0, r_1) \\ &\qquad \text{where } \mathfrak{U}'(x) \text{ is harmonic in } B(0, r_1). \end{aligned} \tag{3.17}$$

From (3.17) and [6, Lemma 2], it follows that

$$\lim_{t\to 0}\sum_{m\neq 0}|m|^2\mathfrak{U}^\wedge(m)e^{i(m,x)-|m|t}=0 \quad \text{for } x \text{ in } B(0,r_1). \tag{3.18}$$

Also from (3.14) and [6, Lemma 2], it follows that

$$\lim_{t\to 0}\sum_{m\neq 0}\left(\sum_{j=1}^{N} im_jw_j^\wedge(m)\right)e^{i(m,x)-|m|t}=\sum_{j=1}^{N}\frac{\partial w_j(x)}{\partial x_j} \quad \text{for } x \text{ in } B(0,r_1). \tag{3.19}$$

But from (3.14) and (3.16), we have

$$\sum_{j=1}^{N} im_jw_j^\wedge(m)-\nu|m|^2\mathfrak{U}^\wedge(m)=\sum_{j=1}^{N} im_jv_j^\wedge(m) \quad \text{for } m\neq 0, \tag{3.20}$$

and we conclude from (3.18), (3.19), and (3.20) that

$$\lim_{t\to 0}\sum_{m\neq 0}\left(\sum_{j=1}^{N} im_jv_j^\wedge(m)\right)e^{i(m,x)-|m|t}=\sum_{j=1}^{N}\frac{\partial w_j(x)}{\partial x_j} \quad \text{for } x \text{ in } B(0,r_1). \tag{3.21}$$

Next, we set

$$F_j(x)=(2\pi)^{-N}\int_{T_N} f_j(y)H_j(x-y)\,dy \tag{3.22}$$

and note from the hypothesis of the lemma and from Lemma 1 that

(3.23) $F_j(x)$ is a continuous periodic function which in $C^{1+\alpha}[B(0,r_1)]$ for $j=1,\cdots,N$.

Also we note from (3.22) that

$$\begin{aligned} F_j^\wedge(m) &= im_jf_j^\wedge(m)|m|^{-2} && \text{for } m\neq 0,\\ &=0 && \text{for } m=0. \end{aligned} \tag{3.24}$$

Next, we observe from (3.11) and (3.24) that

$$\begin{aligned}\sum_{j=1}^{N} im_jv_j^\wedge(m) = &-p^\wedge(m)-\sum_{j=1}^{N}\sum_{k=1}^{N} m_jm_k(u_ju_k)^\wedge(m)|m|^{-2}\\ &-\sum_{j=1}^{N}F_j^\wedge(m) \quad \text{for } m\neq 0.\end{aligned} \tag{3.25}$$

Also, we observe that u_ju_k is in $L^{\beta/2}[B(0,r_1)]$, and conclude from Lemmas 4, 6 and 11(a) that there is a function

(3.26) $W_{jk}(x)$ in $L^{\beta/2}[B(0, r)]$ for every r such that $0 < r < r_1$

with the property that for $j, k = 1, \cdots, N$

$$\lim_{t\to 0} \sum_{m\neq 0} m_j m_k |m|^{-2}(u_j u_k)\hat{}(m)e^{i(m,x)-|m|t} = W_{jk}(x) \quad \text{almost everywhere in } B(0, r_1). \tag{3.27}$$

Also from [5, Theorem 2, pp. 55–56], we have that

$$\lim_{t\to 0} \sum_{m\neq 0} p\hat{}(m)e^{i(m,x)-|m|t} = p(x) - p\hat{}(0) \quad \text{almost everywhere in } B(0, r_1), \tag{3.28}$$

and from [5, Theorem 2] in conjunction with (3.23) and (3.24) that

$$\lim_{t\to 0} \sum_{m\neq 0} F_j\hat{}(m)e^{i(m,x)-|m|t} = F_j(x) \quad \text{for } x \text{ in } B(0, r_1). \tag{3.29}$$

We consequently conclude from (3.21), (3.25), (3.27), and (3.29) that

$$-p(x) + p\hat{}(0) - \sum_{j=1}^{N}\sum_{k=1}^{N} W_{jk}(x) - \sum_{j=1}^{N} F_j(x) = \sum_{j=1}^{N} \frac{\partial w_j(x)}{\partial x_j} \quad \text{almost everywhere in } B(0, r_1).$$

We rewrite this last fact as follows:

$$p(x) = p\hat{}(0) - \sum_{j=1}^{N}\sum_{k=1}^{N} W_{jk}(x) - \sum_{j=1}^{N} F_j(x) - \sum_{j=1}^{N} \frac{\partial w_j(x)}{\partial x_j} \quad \text{almost everywhere in } B(0, r_1). \tag{3.30}$$

But then our desired result, namely (3.2), follows immediately from (3.14), (3.23), (3.26), and (3.30).

Next, we intend to show the following:

(3.31) If $N < \beta < 2N$, u_j is in $L^{N\beta/(2N-\beta)}[B(0, r)]$ for every r such that $0 < r < r_1$, $j = 1, \cdots, N$.

First, we observe from (3.2), (3.7), and Lemma 2 that

(3.32) if $N < \beta < 2N$, P_j is in $L^{N\beta/(2N-\beta)}[B(0, r)]$

for every r such that $0 < r < r_1$.

Likewise, since $u_j u_k$ is in $L^{\beta/2}$ on compact subsets of $B(0, r_1)$, we have from Lemma 2 and (3.8) that

(3.33) if $N < \beta < 2N$, U_{jk} is in $L^{N\beta/(2N-\beta)}[B(0, r)]$

for every r such that $0 < r < r_1$.

Also, we have from the hypothesis of Lemma 12 and from (3.9) and Lemma 1 that

(3.34) $\mathcal{F}_j(x)$ is a continuous periodic function which is in $C^{2+\alpha}[B(0, r_1)]$.

Consequently, it follows from (3.32), (3.33), (3.34), and (3.10) that

(3.35) if $N < \beta < 2N$, v_j is in $L^{N\beta/(2N-\beta)}[B(0, r)]$ for every r such that $0 < r < r_1$.

But then (3.31) follows immediately from (3.14) and (3.35).

Next, we observe the following:

(3.36) if $\beta = N + \epsilon$ where $0 < \epsilon < N$, then $N\beta/(2N - \beta) \geq N + 2\epsilon$.

As a consequence of (3.31) and (3.36) and working iteratively, we can assume that

(3.37) u_j is in $L^{3N/2}[B(0, r)]$ for every r

such that $0 < r < r_1$, $j = 1, \cdots, N$.

But then $N(3N/2)/[2N - 3N/2] = 3N$ and we conclude from (3.31) and (3.37) that

(3.38) u_j is in $L^{3N}[B(0, r)]$ for every r

such that $0 < r < r_1$, $j = 1, \cdots, N$.

From (3.38), we obtain that

(3.39) $u_j u_k$ is in $L^{3N/2}[B(0, r)]$ for every r

such that $0 < r < r_1$.

As a consequence we see from Lemmas 4, 6, and 11(a) and (3.39) that the function W_{jk} introduced in (3.26) and used in (3.27) actually is better than stated.

In particular we now have that

(3.40) $W_{jk}(x)$ is in $L^{3N/2}[B(0, r)]$ for every r such that $0 < r < r_1$

and that (3.27) still holds. Also (3.30) still holds, and we conclude from (3.30), (3.23), (3.14), and (3.40) that

(3.41) $p(x)$ is in $L^{3N/2}[B(0, r)]$ for every r such that $0 < r < r_1$.

But then it follows from (3.7), (3.41), and Lemma 3 that

(3.42) P_j is in $C^{1/6}[B(0, r)]$ for every r such that $0 < r < r_1$.

Likewise it follows from (3.8) and (3.39) that

(3.43) U_{jk} is in $C^{1/6}[B(0, r)]$ for every r such that $0 < r < r_1$.

As a consequence, we obtain from (3.10), (3.34), (3.42) and (3.43) that there is a function v_j' with the following properties:

(3.44) v_j' is in $C^{1/6}[B(0, r)]$ for every r such that $0 < r < r_1$ and equals v_j almost everywhere in T_N.

But then from (3.14) and (3.44), we obtain that there is a function u_j' with the following properties:

(3.45) u_j' is in $C^{1/6}[B(0, r)]$ for every r such that $0 < r < r_1$ and equals u_j almost everywhere in T_N.

But then it follows from (3.45) and Lemmas 7, 8, and 11(b) that the W_{jk} in (3.26), (3.27), and (3.40) is much better than stated. In particular, we now have that

(3.46) W_{jk} is in $C^{1/6}[B(0, r)]$ for every r such that $0 < r < r_1$.

Furthermore, we observe that (3.27) still holds and also that (3.30) still holds. We consequently conclude from (3.30), (3.23), (3.14), and (3.46) that there is a function $p'(x)$ with the following properties:

(3.47) $p'(x)$ is in $C^{1/6}[B(0, r)]$ for every r such that $0 < r < r_1$ and equals $p(x)$ almost everywhere in T_N.

But then it follows from (3.7), (3.47), and Lemma 1 that

(3.48) P_j is in $C^{1+1/6}[B(0, r)]$ for every r such that $0 < r < r_1$.

Likewise it follows from (3.8) and (3.45) that

(3.49) U_{jk} is in $C^{1+1/6}[B(0, r)]$ for every r such that $0 < r < r_1$.

As a consequence, we obtain from (3.10), (3.34), (3.48), and (3.49) that v_j' in (3.46) is such that

(3.50) v_j' is in $C^{1+1/6}[B(0, r]$ for every r such that $0 < r < r_1$.

But then from (3.14) and (3.50) we have that u_j' in (3.45) is such that

(3.51) u_j' is in $C^{1+1/6}[B(0, r)]$ for every r such that $0 < r < r_1$.

Next, it follows from (3.8), (3.51), and Lemma 1 that

(3.52) U_{jk} is in $C^{2+1/6}[B(0, r)]$ for every r such that $0 < r < r_1$.

Also, we note from (3.8) that

$$\begin{aligned} \hat{U_{jk}}(m) &= (u_j u_k)\hat{}(m) i m_k |m|^{-2} && \text{for } m \neq 0, \\ &= 0 && \text{for } m = 0. \end{aligned}$$

As a consequence, for $t > 0$,

$$\frac{\partial U_{jk}(x, t)}{\partial x_j} = -\sum_{m \neq 0} m_j m_k |m|^{-2} (u_j u_k)\hat{}(m) e^{i(m,x) - |m| t}. \tag{3.53}$$

We conclude from (3.52), (3.53), and [6, Lemma 2] that the function $W_{jk}(x)$ introduced in (3.26) and used in (3.27) is such that

(3.54) $W_{jk}(x)$ is in $C^{1+1/6}[B(0, r)]$ for every r such that $0 < r < r_1$

and furthermore

$$\lim_{t \to 0} \frac{\partial U_{jk}(x, t)}{\partial x_j} = W_{jk}(x) \quad \text{everywhere in } B(0, r_1). \tag{3.55}$$

Since (3.30) still holds, and we conclude from (3.54), (3.14), and (3.23) that $p'(x)$ in (3.47) is such that

(3.56) $p'(x)$ is in $C^{1+\zeta}[B(0, r)]$ for every r such that $0 < r < r_1$ where $\zeta = \min[\alpha, 1/6]$.

But then it follows from (3.7), (3.56), and Lemma 1 that

(3.57) P_j is in $C^{2+\zeta}[B(0, r)]$ for every r such that $0 < r < r_1$.

As a consequence, we obtain from (3.10), (3.44), (3.34), (3.52), and (3.57) that

(3.58) v_j' is in $C^{2+\zeta}[B(0, r)]$ for every r such that $0 < r < r_1$.

From (3.14), (3.45) and (3.58), we obtain

(3.59) u_j' is in $C^{2+\zeta}[B(0, r)]$ for every r such that $0 < r < r_1$.

But then it follows from (3.8), (3.59), and Lemma 1 that

(3.60) U_{jk} is in $C^{3+\zeta}[B(0, r)]$ for every r such that $0 < r < r_1$.

As a consequence, we see that we can replace (3.54) with the statement

(3.61) $W_{jk}(x)$ is in $C^{2+\zeta}[B(0, r)]$ for every r such that $0 < r < r_1$.

But (3.30) still holds, and we conclude from (3.14), (3.23), and (3.61) that

(3.62) $p'(x)$ is in $C^{1+\alpha}[B(0, r)]$ for every r such that $0 < r < r_1$.

Next, it follows from (3.7), (3.62), and Lemma 1 that

(3.63) P_j is in $C^{2+\alpha}[B(0, r)]$ for every r such that $0 < r < r_1$.

As a consequence, we obtain from (3.10), (3.34), (3.60), and (3.63) that

(3.64) v_j' is in $C^{2+\alpha}[B(0, r)]$ for every r
such that $0 < r < r_1$.

But then from (3.14) and (3.64), we obtain that

(3.65) u_j' is in $C^{2+\alpha}[B(0, r)]$ for every r
such that $0 < r < r_1$.

The conclusion to the lemma follows immediately from (3.45), (3.47), (3.62), and (3.65), and the proof of the lemma and consequently the theorem is complete.

4. **Proof of Theorem 2.** To prove Theorem 2 we can assume, with no loss in generality, that

(4.1) $$0 < r_1 < r_0 < 1,$$

and that

(4.2) $$\mathbf{u},\ p,\ \text{and}\ \mathbf{f} = 0 \quad \text{in}\ T_N - B(0, r_0).$$

Also, we assume that

(4.3) $(\mathbf{u}, p)$ is a distribution solution of (1.1) in $B(0, r_1) - \{0\}$

and observe from (4.1), (4.2), and (4.3) that the theorem will be established once we establish the following fact:

(4.4) For ϕ in $C_0^\infty[r_1, T_N]$, both (3.5) and (3.6) are valid,

where with no loss in generality we have assumed that (3.1) holds.

In order to establish (4.4), we say

(4.5) ϕ is in $C_0^\infty[r_1, T_N - \{0\}]$ if ϕ is in $C_0^\infty[B(0, r_1) - \{0\}]$ and
$\phi = 0$ in $T_N - B(0, r_1)$.

We observe from (4.3) that the following fact does hold:

(4.6) For ϕ in $C_0^\infty[r_1, T_N - \{0\}]$, both (3.5) and (3.6) are valid.

Next, we let P_j, U_{jk}, $\mathcal{F}_{jk}$, and v_j be defined almost everywhere in T_N by

(3.7), (3.8), (3.9), and (3.10) respectively and observe from (3.11) and (4.6) that

(4.7) For ϕ in $C_0^\infty[r_1, T_N - \{0\}]$, (3.12) holds.

From (4.7), we conclude that (3.13) holds for all ϕ in $C_0^\infty[r_1, T_N - \{0\}]$. As a consequence we conclude from Weyl's lemma that for $j = 1, \cdots, N$ there is a function $w_j(x)$ such that

(4.8) $w_j(x)$ is harmonic in $B(0, r_1) - \{0\}$ and

equal almost everywhere in T_N to $\nu u_j(x) + v_j(x)$.

From condition (ii) in the hypothesis of the theorem and Schwarz's inequality, we have for $j, k = 1, \cdots, N$ that

(4.9) $$r^{-N} \int_{B(0,r)} |u_j u_k| \, dx = o(r^{-(N-1)}) \text{ as } r \to 0 \text{ for } N \geq 3.$$

[We note for future reference that (4.9) is also true for $N = 2$.]

As a consequence, we conclude from condition (i) in the hypothesis of the theorem, from (4.9), (3.7), and (3.8), and from [7, Lemma 4] that for $j, k = 1, \cdots, N$ both

(4.10) $$\begin{aligned} &r^{-N} \int_{B(0,r)} |P_j| \, dx = o(r^{-(N-2)}), \quad \text{as } r \to 0 \quad \text{and} \\ &r^{-N} \int_{B(0,r)} |U_{jk}| \, dx = o(r^{-(N-2)}), \quad \text{as } r \to 0 \quad \text{for } N \geq 3. \end{aligned}$$

Likewise, we obtain from (3.9) and Lemma 13 that

(4.11) $$r^{-N} \int_{B(0,r)} |\mathcal{F}_j| \, dx = o(r^{-(N-2)}) \text{ as } r \to 0 \text{ for } N \geq 3.$$

We consequently conclude from condition (ii) in the hypothesis of the theorem, from (4.10), (4.11) and (3.10) and from (4.8) that for $j = 1, \cdots, N$

(4.12) $$r^{-N} \int_{B(0,r)} |w_j| \, dx = o(r^{-(N-2)}) \text{ as } r \to 0 \text{ for } N \geq 3.$$

Using the theory of spherical harmonics as in [7, (3.61) and (3.62), p. 94], we conclude from (4.8) and (4.12) that the following holds:

(4.13) For $N \geq 3$, w_j can be defined at 0

so that w_j is harmonic in $B(0, r_1)$.

Next, for $N \geq 2$, we define $\mathfrak{U}(x)$ almost everywhere in T_N by (3.15) and observe from (4.6), (3.6), and (3.16) that

$$\int_{B(0,r_1)} \mathfrak{U}(x)\Delta\phi(x)\,dx = 0 \quad \text{for } \phi \text{ in } C_0^\infty[r_1, T_N - \{0\}]. \tag{4.14}$$

From (4.14) and Weyl's lemma, we conclude that

(4.15) $\mathfrak{U}(x) = \mathfrak{U}'(x)$ almost everywhere in $B(0, r_1)$
where $\mathfrak{U}'(x)$ is harmonic in $B(0, r_1) - \{0\}$.

From condition (ii) in the hypothesis of the theorem, we obtain

$$r^{-N} \int_{B(0,r)} |u_j|\,dx = o(r^{-(N-1)/2}) \quad \text{as } r \to 0 \quad \text{for } N \geq 2 \text{ and } j = 1, \cdots, N. \tag{4.16}$$

But then we obtain from [7, Lemma 4] and (3.15) that

$$\begin{aligned} r^{-N} \int_{B(0,r)} |\mathfrak{U}|\,dx &= o(r^{-(N-2)}) \quad \text{for } N \geq 3, \\ &= o(\log r^{-1}) \quad \text{for } N = 2, \quad \text{as } r \to 0. \end{aligned} \tag{4.17}$$

We conclude from the theory of spherical harmonics that

(4.18) From $N \geq 2$, $\mathfrak{U}'$ can be defined at the origin so that it is harmonic in $B(0, r_1)$.

Next, we show that (4.13) holds also for $N = 2$. To see this we observe from [7, Lemma 4] and from Lemma 13 that

(4.19) For $N = 2$, the expressions on the right-hand side of the equality sign in (4.10) and (4.11) are to be replaced by $o(\log r^{-1})$.

From (4.8), (3.10), (4.16), and (4.19), we next obtain that

$$r^{-2} \int_{B(0,r)} |w_j|\,dx = o(r^{-1/2}) \quad \text{as } r \to 0 \quad \text{for } j = 1, 2. \tag{4.20}$$

Using the theory of spherical harmonics as before we conclude from (4.8) and (4.20) that for $j = 1, 2$,

(4.21) $w_j(x) = c_j \log|x| + G_j(x)$ for x in $B(0, r_1) - \{0\}$
where c_j is a constant and $G_j(x)$ is harmonic in $B(0, r_1)$.

From (4.15) and (4.21), we have that (3.18) and (3.19) hold for x in $B(0, r_1) - \{0\}$. We therefore obtain from (3.20) that (3.21) holds for x in $B(0, r_1) - \{0\}$. We write this out as follows:

$$(4.22)\qquad \sum_{j=1}^{N} c_j x_j |x|^{-2} + \sum_{j=1}^{N} \frac{\partial G_j(x)}{\partial x_j} = \lim_{t\to 0} \sum_{m\neq 0} \left(\sum_{j=1}^{N} i m_j v_j^{\wedge}(m) \right) e^{i(m,x)-|m|t}$$

for x in $B(0, r_1) - \{0\}$.

Next, we define $F_j(x)$ almost everywhere in T_N by (3.22) and observe that (3.24) and (3.25) hold. Also from condition (ii) in the hypothesis of the theorem, we have that

(4.23) there is an r_2 with $0 < r_2 < r_1$ such that $u_j u_k$ is in $L^{\gamma/2}[B(0, r_2)]$ for $j, k = 1, 2$.

Since $\gamma > 2$, we conclude from (4.23), (3.25), Lemmas 6 and 11 that

(4.24) there is an r_3 with $0 < r_3 < r_2$ and a function $M(x)$ in $L^1[B(0, r_3)]$ such that the expression on the right-hand side of the equality sign in (4.22) is equal to $M(x)$ almost everywhere in $B(0, r_3)$.

From (3.22) and Lemma 13, we have that

$$(4.25)\qquad r^{-2} \int_{B(0,r)} \sum_{j=1}^{N} |F_j|\, dx = o(r^{-1}).$$

From condition (ii) in the hypothesis of the theorem, we have that

$$(4.26)\qquad \left\{ r^{-2} \int_{B(0,r)} |u_j u_k|^{\gamma/2}\, dx \right\}^{2/\gamma} = o(r^{-1}),$$

where with no loss in generality we can also assume that

$$(4.27)\qquad 2 < \gamma < 4.$$

Now it follows from (3.25) and (4.24) that

$$(4.28)\qquad \begin{aligned} &M(x) + p(x) - p^{\wedge}(0) + \sum_{j=1}^{N} F_j(x) \\ &= - \lim_{t\to 0} \sum_{m\neq 0} \left[\sum_{j=1}^{N} \sum_{k=1}^{N} m_j m_k (u_j u_k)^{\wedge}(m) \right] e^{i(m,x)-|m|t} \end{aligned}$$

almost everywhere in $B(0, r_3)$.

We conclude from condition (i) in the hypothesis of the theorem, from (4.25), (4.23), (4.26), (4.27), Lemma 12, and (4.28) that

$$r^{-2}\int_{B(0,r)}|M(x)|\,dx = o(r^{-1}) \quad \text{as } r \to 0. \tag{4.29}$$

From (4.21), we have that $G_j(x)$ is harmonic in $B(0, r_1)$ for $j = 1, \cdots, N$. Therefore $\partial G_j(x)/\partial x_j$ is harmonic in $B(0, r_1)$. Since $r_3 < r_1$, we conclude from (4.22) and (4.24) that

(4.30) there is a function $G(x)$ harmonic in $B(0, r_3)$ such that

$\Sigma_{j=1}^2 c_j x_j |x|^{-2} = M(x) + G(x)$ almost everywhere in $B(0, r_3)$.

From (4.30) we obtain that

$$c_k \int_{B(0,r)} x_k^2 |x|^{-3}\,dx = \int_{B(0,r)} (M+G)x_k|x|^{-1}\,dx \quad \text{for } 0 < r < r_3 \text{ and } k = 1, 2. \tag{4.31}$$

From (4.29), (4.30), and (4.31) we further obtain that

$$c_k \pi r = o(r) \quad \text{for } k = 1, 2. \tag{4.32}$$

Therefore $c_k = 0$ for $k = 1, 2$, and we conclude from (4.21) that

(4.33) for $N = 2$, w_j can be defined at 0

so that it is harmonic in $B(0, r_1)$.

From (4.13) and (4.33), we obtain that

$$\int_{T_N} w_j(x)\Delta\phi(x)\,dx = 0 \text{ for } \phi \text{ in } C_0^\infty[r_1, T_N] \text{ and } j = 1, \cdots, N. \tag{4.34}$$

Likewise from (4.15) and (4.18), we have

$$\int_{T_N} \mathfrak{U}(x)\Delta\phi(x)\,dx = 0 \quad \text{for } \phi \text{ in } C_0^\infty[r_1, T_N]. \tag{4.35}$$

From (4.8), we have that $w_j(x) = \nu u_j(x) + v_j(x)$ almost everywhere in T_N. From this fact, (4.34), and (3.11) we see that (3.5) holds for ϕ in $C_0^\infty[r_1, T_N]$. Likewise from (3.16) and (4.35), we see that (3.6) holds for ϕ in $C_0^\infty[r_1, T_N]$. Consequently (4.4) is established and the proof of Theorem 2 is complete.

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