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SPECTRA OF POLAR FACTORS OF HYPONORMAL OPERATORS(1)

BY

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ABSTRACT. An investigation is made of the interdependence and properties of the spectrum of a hyponormal operator T and of the spectra, and absolutely continuous spectra, of the factors in a polar factorization of T when the latter exists.

1. Introduction. Only bounded operators on a fixed separable Hilbert space H will be considered in this paper. An operator T will be said to have a polar factorization if $T = UP$ where U is unitary and P is a nonnegative selfadjoint operator. (Other factorizations in which U is not unitary but is only an isometry or a partial isometry, cf. Halmos [1, p. 68], or Kato [3, p. 334], will not be considered.) Thus, if T has a polar factorization $T = UP$, then $T^* = PU^*$ and $T^*T = P^2$, hence $P = (T^*T)^{1/2}$, so that

$$(1.1) \qquad T = UP, \qquad U \text{ unitary and } P = (T^*T)^{1/2}.$$

In general the unitary factor is not unique. In case T is nonsingular, that is, if 0 is not in its spectrum, the polar factorization exists, is unique, and was given by Wintner [12]; a generalization was obtained by von Neumann [4, p. 307].

As noted above, if $T = UP$ where U is unitary and P is nonnegative then necessarily $P = (T^*T)^{1/2}$. Also,

$$(1.2) \qquad TT^* = U(T^*T)U^* \text{ (equivalently, } (TT^*)^{1/2} = U(T^*T)^{1/2}U^*), \ U \text{ unitary.}$$

Conversely, it was shown by Hartman [2], using the above mentioned result of von Neumann, that if T is arbitrary then the nonzero spectra of T^*T and TT^* are identical, including multiplicities of both point and continuous spectra, while 0 may occur in the point spectra of T^*T and TT^* with different multiplicities. Further, (1.2) holds for some unitary U if and only if the multiplicities of 0 in the point spectra of T^*T and TT^* (equivalently, of T and of T^*) are equal, that is,

$$(1.3) \qquad \dim\{x: Tx = 0\} = \dim\{x: T^*x = 0\}.$$

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In addition (cf. [2, p. 234], T has a polar factorization (1.1), for some unitary U, if and only if (1.2) holds for some (not necessarily the same) unitary U, or, equivalently, if and only if (1.3) holds. In this case, the unitary operator U of (1.1) (but, of course, not that of (1.2)) is uniquely determined if 0 is not in the point spectrum of T (and/or T^*), that is, if the common dimension of (1.3) is 0.

Next, an operator is said to be hyponormal if

$$T^*T - TT^* = D \geq 0, \tag{1.4}$$

and completely hyponormal if, in addition, there is no nontrivial subspace reducing T on which T is normal. It was shown in Putnam [8] that if T is completely hyponormal then its spectrum, $\operatorname{sp}(T)$, has positive planar measure and, in fact,

(1.5) if T is completely hyponormal then $\operatorname{meas}_2(\operatorname{sp}(T) \cap \alpha) > 0$ whenever $\operatorname{sp}(T) \cap \alpha \neq$ empty set,

where α denotes any open disk of the complex plane.

Let $T_z = T - zI$ for any complex z. Then $T_z^* T_z - T_z T_z^* = T^*T - TT^*$ and hence

$$\{x\colon T_z x = 0\} \subset \{x\colon T_z^* x = 0\} \quad \text{if } T \text{ is hyponormal.} \tag{1.6}$$

Hence, if z is in the point spectrum of a hyponormal T the corresponding eigenspace is a reducing space of T on which it is normal. It is also clear that if T is hyponormal and if 0 is not in the point spectrum of T^* then T has a (unique) polar factorization (1.1). Of course, if T is normal, and whether or not 0 is in the point spectrum of T^*, equality holds in (1.6) for all z, in particular, for $z = 0$, and it follows that T must have a (that is, at least one) polar factorization. Such a factorization is easily constructed, for instance, from the spectral resolution of the operator. The unilateral shift (cf. Halmos [1, p. 40]) is an example of a completely hyponormal operator which fails to have a polar factorization (1.1).

Recall that A is a selfadjoint operator with the spectral resolution $A = \int t\,dE_t$ then the set $H_a(A)$ of elements x in H for which $\|E_t x\|^2$ is an absolutely continuous function of t is a subspace of H reducing A. The operator A is said to be absolutely continuous if $H_a(A) = H$. Similar concepts can be defined for a unitary operator $U = \int_0^{2\pi} e^{it}\,dE_t$; cf. [6, p. 19].

If T is hyponormal with the rectangular representation $T = A + iB$ (A, B selfadjoint) it was shown in Putnam [5] (cf. also [6, p. 46]) that, exactly as in the case when T is normal, the spectra of A and B are precisely the projections, as real sets, of the spectrum of T onto the real and imaginary axes. Further (cf. [6, pp. 42–43]), both $H_a(A)$ and $H_a(B)$ contain the least subspace of H reducing T and containing the range of $T^*T - TT^*$. In particular, if T is completely hyponormal, A and B are absolutely continuous. This paper will deal with an analogous investigation of the spectrum of T and of the spectra, and absolutely continuous spectra, of the components of a polar factorization of T, when the latter exists.

For use below, recall that a number t is said to be in the essential spectrum of a selfadjoint operator A, $\operatorname{essp}(A)$, if t is either a limit point of $\operatorname{sp}(A)$ or is an eigenvalue of infinite multiplicity. The point spectrum of any operator T will be denoted by $\operatorname{ptsp}(T)$.

2. **Theorem 1.** *Let T be hyponormal and let*

$$z \in \textit{boundary of } \operatorname{sp}(T). \tag{2.1}$$

Then

$$|z| \in \operatorname{sp}(T^*T)^{1/2} \cap \operatorname{sp}(TT^*)^{1/2}. \tag{2.2}$$

Further, if T is completely hyponormal, then

$$|z| \in \operatorname{essp}(T^*T)^{1/2} \cap \operatorname{essp}(TT^*)^{1/2}. \tag{2.3}$$

Proof. The hypothesis (2.1) implies that there exists a sequence of unit vectors, $\{x_n\}$, for which $(T - zI)x_n \to 0$. Since T is hyponormal, also $(T^* - \bar{z}I)x_n \to 0$ and so $(T^*T - |z|^2I)x_n \to 0$ and $(TT^* - |z|^2I)x_n \to 0$, hence also $((T^*T)^{1/2} - |z|I)x_n \to 0$ and $((TT^*)^{1/2} - |z|I)x_n \to 0$, and so (2.2) follows. Further, if z is an isolated point of $\operatorname{sp}(T)$, T has a normal part with eigenvalue z (cf. Stampfli [11, p. 473] or Putnam [8]). Hence, if T is completely hyponormal, it follows from [7, Theorem 2 of p. 506], that the above sequence $\{x_n\}$ can be chosen so as to converge weakly to 0, and hence (2.3) holds.

Remarks. The above argument shows that if T is normal, then (2.2) holds, if, instead of (2.1), it is supposed only that

$$z \in \operatorname{sp}(T). \tag{2.4}$$

In general, however, if T is only hyponormal, condition (2.4) does not imply (2.2). One need only let T denote the unilateral shift operator, so that on the l^2 sequence space $x = (x_1, x_2, \cdots)$, $Tx = (0, x_1, x_2, \cdots)$. Then $\operatorname{sp}(T)$ is the closed unit disk but $T^*T = I$ and $TT^* = \operatorname{diag}(0, 1, 1, \cdots)$.

As noted earlier, the unilateral shift fails to have a polar factorization (1.1). However, even if T is hyponormal and nonsingular, in which case a polar factorization (1.1) is assured, still (2.4) does not imply (2.2). To see this, consider the doubly infinite nonnegative diagonal matrices

$$A = \operatorname{diag}(\cdots, a_{-1}, a_0, a_1, \cdots) \quad \text{and} \quad B = \operatorname{diag}(\cdots, b_{-1}, b_0, b_1, \cdots)$$

with $a_i = 4$ for $i \geq 1$, $a_i = 1$ for $i \leq 0$, $b_i = 4$ for $i \geq 0$, $b_i = 1$ for $i \leq -1$. Let P denote the nonnegative square root of B and put $T = UP$, where U is the unitary bilateral shift on the sequence space of vectors $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$, $\Sigma |x_i|^2 < \infty$, defined by $(Ux)_n = x_{n-1}$ $(n = 0, \pm 1, \pm 2, \cdots)$. Since $A = UBU^*$ then

$$T^*T - TT^* = B - A = \operatorname{diag}(\cdots, d_{-1}, d_0, d_1, \cdots)$$

with $d_0 = 3$ and $d_i = 0$ for $i \neq 0$. Thus T is hyponormal but not normal. Also, $\operatorname{sp}(T) = \{z: 1 \leq |z| \leq 2\}$, as can be deduced, for instance, from the results of this paper (cf. Theorems 8, 9 below). However, $\operatorname{sp}(T^*T)^{1/2} = \operatorname{sp}(TT^*)^{1/2} = \{1, 2\}$.

Theorem 2. *Let T be hyponormal and suppose that $z \in \operatorname{sp}(T)$ and $\bar{z} \notin \operatorname{ptsp}(T^*)$. Then $|z| \in \operatorname{essp}(T^*T)^{1/2} \cap \operatorname{essp}(TT^*)^{1/2}$.*

Proof. Since T is hyponormal, $T_z^* T_z \geq T_z T_z^*$, where $T_z = T - zI$, and so, since $z \in \operatorname{sp}(T)$, $0 \in \operatorname{sp}(T_z T_z^*)$. Also, since $\bar{z} \notin \operatorname{ptsp}(T^*)$, then $z \notin \operatorname{ptsp}(T)$. Consequently, 0 is in the essential spectra of both $T_z^* T_z$ and $T_z T_z^*$. In view of the inequality $T_z^* T_z \geq T_z T_z^*$, there exists a sequence of unit vectors, $\{x_n\}$, converging weakly to 0 for which both $(T - zI)x_n \to 0$ and $(T^* - \bar{z}I)x_n \to 0$ and hence $(T^*T - |z|^2 I)x_n \to 0$ and $(TT^* - |z|^2 I)x_n \to 0$. Thus, $|z|^2$ is in the essential spectra of both T^*T and TT^*, and the assertion of the theorem follows.

3. **Theorem 3.** *Let T be hyponormal with a polar factorization* (1.1). *Suppose that $z \neq 0$ and satisfies* (2.4) *and that $z = |z|e^{i\theta}$. Then, for any U of* (1.1),

$$e^{i\theta} \in \operatorname{sp}(U). \tag{3.1}$$

Proof. Let $z_1 = re^{i\theta}$ where $r = \max\{|z|: z = |z|e^{i\theta}$ and $z \in \operatorname{sp}(T)\}$ (hence $r > 0$). Clearly, z_1 is a boundary point of $\operatorname{sp}(T)$ and, as in Theorem 1, there exists a sequence of unit vectors, $\{x_n\}$, such that $(T - z_1 I)x_n \to 0$ and $(T^* - \bar{z}_1 I)x_n \to 0$ and hence also $((T^*T)^{1/2} - rI)x_n \to 0$. But $(T - z_1 I)x_n = U(T^*T)^{1/2}x_n - z_1 x_n \to 0$. Since $r > 0$, this implies that $(Ux_n - e^{i\theta}x_n) \to 0$ and, hence, that (3.1) holds.

4. **Theorem 4.** *Let T be hyponormal and nonsingular, so that T has a (unique) polar factorization* (1.1). *Then if $e^{i\theta} \in \operatorname{sp}(U)$, there exists a $z = |z|e^{i\theta} \neq 0$ satisfying* (2.4).

Proof. We have $T = UP$ and

$$P^2 - UP^2U^* = T^*T - TT^* = D \geq 0. \tag{4.1}$$

Since $e^{i\theta} \in \operatorname{sp}(U)$ there exists a sequence of unit vectors, $\{x_n\}$, satisfying $(U - e^{i\theta}I)x_n \to 0$, hence $(U^* - e^{-i\theta}I)x_n \to 0$. Clearly,

$$\|D^{1/2}x_n\|^2 = (Dx_n, x_n) = (P^2x_n, x_n) - (UP^2U^*x_n, x_n)$$
$$= (P^2x_n, x_n) - (P^2U^*x_n, U^*x_n) \to 0,$$

and so $Dx_n \to 0$. Hence, by (4.1), $P^2x_n - UP^2U^*x_n \to 0$, that is, $(U^* - e^{-i\theta}I)P^2x_n \to 0$. A similar argument shows that $(U^* - e^{-i\theta}I)f(P^2)x_n \to 0$, where $f(t)$ is a polynomial, or, via the functional calculus, a continuous function on $(-\infty, \infty)$. It then follows (cf. a similar argument in [6, p. 46]) that there exists a number $s > 0$ and a sequence of unit vectors $\{y_n\}$ such that $(P^2 - sI)y_n \to 0$ and $(U^* - e^{-i\theta}I)\, y_n \to 0$, hence also $(P - s^{1/2}I)y_n \to 0$ and $(U - e^{i\theta}I)y_n \to 0$. Consequently, if $z = s^{1/2}e^{i\theta}$, then $(T - zI)y_n \to 0$ (also $(T^* - \bar{z}I)y_n \to 0$) and so (2.4) holds.

5. **Theorem 5.** *Let T be hyponormal with a polar factorization* (1.1) *and suppose that*

$$0 \notin \operatorname{ptsp}(T). \tag{5.1}$$

If $e^{i\theta} \in \operatorname{sp}(U)$ then there exist $z_n = |z_n|e^{i\theta_n} \neq 0$, $z \in \operatorname{sp}(T)$, for which $\theta_n \to \theta$.

Remark. Note that if T is completely hyponormal then the hypotheses (1.1) and (5.1) are certainly fulfilled.

Proof. In case T is nonsingular the above theorem follows from Theorem 4. The theorem also is clear if T is singular and if there does not exist some open wedge

$$W = \{z\colon z = re^{it},\ r > 0,\ a < t < b\}, \qquad a < \theta < b, \tag{5.2}$$

for which

$$\operatorname{sp}(T) \cap W \text{ is empty.} \tag{5.3}$$

Consequently, it is sufficient to show that if T is singular then the assumption that there exists a wedge W of (5.2) satisfying (5.3) leads to a contradiction.

Suppose then the existence of such a wedge. Consider the bisector of W, that is, the half-line $\{z\colon z = re^{i\frac{1}{2}(a+b)},\ r > 0\}$ and choose complex numbers $s_n = |s_n|e^{i\frac{1}{2}(a+b)} \neq 0$ on this half-line satisfying $s_n \to 0$. It is clear that each (hyponormal) operator $T_n = T - s_nI$ is nonsingular and that, by (5.3), $\operatorname{sp}(T_n) \cap W$ is empty. If $T_n = U_nP_n$ is the (unique) polar factorization of T_n then, by Theorem 4,

$$e^{it} \in \operatorname{sp}(U_n) \quad (n = 1, 2, \cdots) \text{ whenever } a < t < b. \tag{5.4}$$

But $\|T - T_n\| \to 0$ and hence $\|P - P_n\| = \|(T^*T)^{1/2} - (T_n^*T_n)^{1/2}\| \to 0$. Also, $U_nP - UP = T_n - T + U_n(P - P_n)$, so that $\|U_nP - UP\| \to 0$ and, in particular, $U_nPx \to UPx$ (strongly) for all x in H. In view of (5.1), $0 \notin$ point spectrum of $P = (T^*T)^{1/2}$, hence the range of P is dense, and consequently

$$U_n \to U \quad \text{(strongly).} \tag{5.5}$$

By (5.4), $\|(U_n - e^{i\theta}I)x\| \geq c\|x\|$ for all x where c is some positive constant, and hence, by (5.5), $\|(U - e^{i\theta}I)x\| \geq c\|x\|$. This implies that $e^{i\theta} \notin \mathrm{sp}(U)$, a contradiction, and the proof of Theorem 5 is complete.

6. **Theorem 6.** *Let T be completely hyponormal and have a polar factorization $T = UP$ of* (1.1). *In addition, suppose that there exists some open wedge W of* (5.2) *satisfying* (5.3). *Then*

(6.1) $P = (T^*T)^{1/2}$ *(hence also $(TT^*)^{1/2}$) and U are absolutely continuous.*

Proof. It follows from Theorem 4 that no $e^{i\theta}$, $a < \theta < b$, can belong to $\mathrm{sp}(U)$. It now follows from (4.1) and the theorem of [6, p. 21], that both $H_a(P^2)$ $(= H_a(T^*T) = H_a((T^*T)^{1/2}))$ and $H_a(U)$ contain the least subspace, M, of H reducing P^2 and U (equivalently, reducing P and U) and containing the range of $D = T^*T - TT^*$. Since T is completely hyponormal, $M = H$, and, in particular, (6.1) follows.

7. **Theorem 7.** *Let T be hyponormal and suppose that*

$$r \in \mathrm{sp}(T^*T) \quad (\textit{hence } r \geq 0). \tag{7.1}$$

Then there exists a $z \in \mathrm{sp}(T)$ for which $|z| = r^{1/2}$.

Remark. The hypothesis (7.1) of Theorem 7 can be replaced by

$$r \in \mathrm{sp}(TT^*) \quad (\text{hence } r \geq 0). \tag{7.1$'$}$$

In fact, if $r = 0$ then T is singular and $0 \in \mathrm{sp}(T)$. If $r > 0$, then (7.1)$'$ implies (7.1); cf. §1 above.

Proof. We first establish the theorem under the added hypothesis that T has a polar factorization (1.1) and that $\mathrm{sp}(U)$ is not the entire circle $|z| = 1$. Thus,

$$T = UP \quad \text{and} \quad \mathrm{meas}_1(\mathrm{sp}(U)) < 2\pi. \tag{7.2}$$

Let $e^{i\theta} \notin \mathrm{sp}(U)$ and define the unitary operator $U_\theta = e^{-i\theta}U$. Then $1 \notin \mathrm{sp}(U_\theta)$ and relation (4.1) becomes $P^2 - U_\theta P^2 U_\theta^* = D$. Now, U_θ is the Cayley transform of a selfadjoint operator A, where

$$U_\theta = (A - iI)(A + iI)^{-1} \qquad (U_\theta = e^{-i\theta}U). \tag{7.3}$$

If $C = \frac{1}{2}(A + iI)D(A + iI)^*$, it is seen that

$$AP^2 - P^2A = iC, \qquad C \geq 0. \tag{7.4}$$

(For a similar argument, see [6, pp. 16, 21].)

Next, by (7.1), $r \in \mathrm{sp}(P^2)$, so that $(P^2 - rI)x_n \to 0$ for some sequence of unit vectors $\{x_n\}$. But, by (7.4), $i\|C^{1/2}x_n\|^2 = (AP^2x_n, x_n) - (Ax_n, P^2x_n) \to 0$ and so $Cx_n \to 0$, and hence by (7.4) again, $(P^2 - rI)Ax_n \to 0$. Similarly, one obtains

$$(P^2 - rI)A^k x_n \to 0 \quad \text{as } n \to \infty \ (k = 0, 1, 2, \cdots). \tag{7.5}$$

In view of (7.3) and the relation $U_\theta^* = (A - iI)^{-1}(A + iI)$, it follows from (7.5) that

$$(P^2 - rI)U^k x_n \to 0, \qquad n \to \infty \ (k = 0, \pm 1, \pm 2, \cdots). \tag{7.6}$$

Hence, by an argument similar to that of [6, p. 46] (see also §4 above) there exists some $e^{i\phi} \in \operatorname{sp}(U)$ and a sequence of unit vectors, $\{y_n\}$, such that $(P^2 - rI)y_n \to 0$ (hence $(P - r^{1/2}I)y_n \to 0$) and $(U - e^{i\phi}I)y_n \to 0$. Thus, if $z = r^{1/2}e^{i\phi}$ then $(T - zI)y_n \to 0$, thus $z \in \operatorname{sp}(T)$, and so Theorem 7 is proved in the special case in which (7.2) is assumed.

Next, we consider the general case of Theorem 7. Let T have the rectangular representation $T = A + iB$ where A has the spectral resolution $A = \int t\, dE_t$. Let $S_n = (-\infty, \infty) - (-1/n, 1/n)$ for $n = 1, 2, \cdots$ and consider the hyponormal operator $T_n = E(S_n)TE(S_n)$ defined on the Hilbert space $E(S_n)H$. Then $\operatorname{sp}(T_n)$ lies outside the strip $|\operatorname{Re}(z)| < 1/n$ (see [6, p. 46]) and also (see [8], [9])

$$\operatorname{sp}(T_n) \subset \operatorname{sp}(T). \tag{7.7}$$

In particular, each T_n is nonsingular and hence has a polar factorization $T_n = U_n P_n$ and, by Theorem 4, $\operatorname{meas}_1(\operatorname{sp}(U_n)) < 2\pi$ $(n = 1, 2, \cdots)$, so that (7.2) holds with the role of T played by T_n. In addition, it is clear that

$$T_n \to T, \qquad T_n^* \to T^* \quad \text{(strongly)} \ [T_n \text{ here as an operator on } H]. \tag{7.8}$$

Consequently, $T_n^*T_n \to T^*T$ (strongly) and, by (7.1), there exist $r_n \in \operatorname{sp}(T_n^*T_n)$ for which $r_n \to r$. (The argument is similar to that following formula line (5.5) above.) Since the assertion of Theorem 7 has already been proved for the T_n, there exist $z_n \in \operatorname{sp}(T_n)$ such that $|z_n| = r_n^{1/2} \to r^{1/2}$. Since, by (7.7), $z_n \in \operatorname{sp}(T)$, and since $\{z_n\}$ is a bounded sequence, there exists a convergent subsequence $\{z_{n_k}\}$, say $z_{n_k} \to z$. Clearly, this z satisfies the conditions of Theorem 7 and the proof is complete.

8. **Theorem 8.** *Let T be completely hyponormal and suppose that*

$$r \ (\geq 0) \textit{ is an isolated point of } \operatorname{sp}(TT^*). \tag{8.1}$$

Let $a = \inf\{s\colon s \in \operatorname{sp}(TT^*),\ s \leq r$ and, if $s < r$, (s, r) contains no points of the essential spectrum of $TT^*\}$; and $b = \sup\{s\colon s \in \operatorname{sp}(TT^*),\ s \geq r$ and if $s > r$, (r, s) contains no points of the essential spectrum of $TT^*\}$. Then $a < b$ and either

$$a < r \quad \textit{and} \quad \{z\colon a^{1/2} < |z| < r^{1/2}\} \subset \operatorname{ptsp}(T^*) \tag{8.2}$$

or

$$(8.3)\qquad b > r \quad \textit{and} \quad \{z: r^{1/2} < |z| < b^{1/2}\} \subset \text{ptsp}(T^*).$$

Remark. It should be noted that even if both inequalities $a < r < b$ hold, still, as simple examples show, only one of the relations (8.2) and (8.3) need hold.

Proof. First we show that $a < b$. Otherwise, $a = b = r$ and $\text{sp}(TT^*)$ is the singleton $\{r\}$. If $r = 0$ then $T = 0$, hence T is normal, a contradiction. If $r > 0$ then, since $T^*T \geq TT^*$, $\text{sp}(T^*T) = \text{sp}(TT^*) = \{r\}$, that is, $T^*T = TT^* = rI$ and again T must be normal, a contradiction.

It follows from Theorem 7 and the remark following it that there exists a number $z_0 \in \text{sp}(T)$ for which $|z_0| = r^{1/2}$. Also, there exist $z_n \in \text{sp}(T)$ with $|z_n| \neq |z_0|$ satisfying $z_n \to z_0$ as $n \to \infty$. Otherwise, there exists an open disk α centered at z_0 and such that $\alpha \cap \text{sp}(T)$ is not empty and has zero planar measure. This is impossible by (1.5). It follows from Theorem 1 and the definitions of a and b in Theorem 8 that no boundary points of $\text{sp}(T)$ can lie in the difference set $\{z: a^{1/2} < |z| < b^{1/2}\} - \{z: |z| = r^{1/2}\}$.

Since $|z_n| \neq |z_0| = r^{1/2}$ then, for any n, either $|z_n| < r^{1/2}$ or $|z_n| > r^{1/2}$. Suppose first that $|z_n| < r^{1/2}$ for some n. Then clearly $a < r$ and, since no boundary point of $\text{sp}(T)$ can lie in $\{z: a^{1/2} < |z| < r^{1/2}\}$, it follows that $\{z: a^{1/2} < |z| < r^{1/2}\} \subset \text{sp}(T)$. Relation (8.2) now follows from Theorem 2. Similarly, if $|z_n| > r^{1/2}$ for some n, relation (8.3) holds.

9. **Theorem 9.** *Let T be completely hyponormal and suppose that*

$$(9.1)\qquad \text{meas}_1(\text{sp}(T^*T)) \quad (= \text{meas}_1(\text{sp}(TT^*))) = 0.$$

Then there exists a finite or denumerably infinite number of pairwise disjoint open annuli $A_n = \{z: a_n < |z| < b_n\}$ $(n = 1, 2, \cdots)$ such that

$$(9.2)\qquad \text{sp}(T) \textit{ is the closure of the set } \bigcup A_n$$

and

$$(9.3)\qquad \bigcup A_n \subset \text{ptsp}(T^*).$$

Proof. Let $z_0 \in \text{sp}(T)$. Then consider any open disk α containing z_0. Then necessarily α contains a closed disk β satisfying

$$(9.4)\qquad \beta = \{z: |z - z_1| \leq s,\ s > 0\} \subset \text{sp}(T).$$

In fact, otherwise, all points of $\alpha \cap \text{sp}(T)$ would be boundary points of $\text{sp}(T)$. Further, if the half-line L: $\theta = c$ (const.) intersects α, then, by Theorem 1, each $r \geq 0$ satisfying $re^{ic} \in L \cap (\alpha \cap \text{sp}(T))$ belongs to $\text{sp}(T^*T)^{1/2}$. Hence, by (9.1), the set of such numbers r has linear measure 0. It readily follows from Fubini's theorem that $\alpha \cap \text{sp}(T)$ (which contains z_0 and hence is not empty) has zero planar measure and hence, by (1.5), T is not completely hyponormal, a contradiction. This proves (9.4).

If $a = \inf\{|z|: z \in \beta\}$ and $b = \sup\{|z|: z \in \beta\}$ then $a < b$. By (9.1), $\mathrm{sp}(T^*T)^{1/2}$ cannot contain $[a, b]$ and it follows from Theorem 1 that

$$\{z: a \leq |z| \leq b\} \subset \mathrm{sp}(T). \tag{9.5}$$

It is clear then that $\mathrm{sp}(T)$ is the closure of a set consisting of a possibly uncountable number of closed annuli each of the form (9.5). By a standard procedure (of combining intersecting annuli), one easily shows that $\mathrm{sp}(T)$ can be taken as the closure of a countable union of disjoint closed annuli, each of the form $\{z: c \leq |z| \leq d\}$ with $c < d$.

For a fixed such annulus let $\bigcup (c_n, d_n)$ denote the canonical decomposition of the linear open set $[c, d] - \{[c, d] \cap \mathrm{sp}(TT^*)^{1/2}\}$. (Note that any z satisfying $|z| = d$ is a boundary point of $\mathrm{sp}(T)$ and that a similar statement holds for $|z| = c$ provided $c > 0$. In view of $T^*T \geq TT^*$ it is clear from Theorem 1 that both c and d (even if $c = 0$) belong to $\mathrm{sp}(TT^*)^{1/2}$.) It follows from (9.1) that $\{z: c \leq |z| \leq d\}$ is the closure of $\bigcup B_n$, where $B_n = \{z: c_n < |z| < d_n\}$, and, from Theorem 2, that each $B_n \subset \mathrm{ptsp}(T^*)$. This completes the proof of Theorem 9.

A final result is the following

Theorem 10. *Let T be hyponormal and suppose that*

$$\mathrm{sp}(TT^*) \neq \textit{interval.} \tag{9.6}$$

Then T has a nontrivial invariant subspace.

Proof. Clearly, it can be supposed that T is completely hyponormal. Further, by (9.6) (cf. the beginning of the proof of Theorem 8), $\mathrm{sp}(TT^*)$ contains at least two points r_1 and r_2 satisfying $r_1 < r_2$ and for which $(r_1, r_2) \cap \mathrm{sp}(TT^*)$ is empty. It follows from Theorem 7 and the remark following it that there exist $z_1, z_2 \in \mathrm{sp}(T)$ where $|z_1| = r_1^{1/2}$ and $|z_2| = r_2^{1/2}$.

Clearly, T has a nontrivial invariant subspace if $\mathrm{ptsp}(T^*)$ is not empty. Hence, it can be supposed that this set is empty, and so, by Theorem 2, $\mathrm{sp}(T) \cap \{z: r_1^{1/2} < |z| < r_2^{1/2}\}$ is empty. Thus, $\mathrm{sp}(T)$ is not connected and hence (cf. [10, p. 421]) T has a nontrivial invariant subspace.

It may be noted that Theorem 10 is applicable to the unilateral shift but that it would not be if (9.6) is replaced by the (stronger) condition $\mathrm{sp}(T^*T) \neq$ interval.

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