EQUIVARIANT METHOD FOR PERIODIC MAPS

BY

WU-HSIUNG HUANG

ABSTRACT. The notion of coherency with submanifolds for a Morse function on a manifold is introduced and discussed in a general way. A Morse inequality for a given periodic transformation which compares the invariants called qth Euler numbers on fixed point set and the invariants called qth Lefschetz numbers of the transformations is thus obtained. This gives a fixed point theorem in terms of qth Lefschetz number for arbitrary q.

Let f be a periodic transformation of a closed m-dimensional manifold M with fixed point set N. We develop in this note an equivariant approach using Morse theory. We introduce in §2 the notion of coherency with a submanifold S of M for a Morse function and show that such S-coherent Morse functions are dense in $C^\infty(M)$. Furthermore, in this approximation f-invariance will be preserved (§3). The coherency with the fixed point set N of f makes it possible to compare the difference of qth Euler number of N and qth Lefschetz number of f. More precisely, let $\beta_q(N)$ and $\lambda_q(f)$ be respectively the qth Betti numbers of N and the trace of $f^*$ on the qth homology group $H_q(M)$ with real coefficients. Let $B_q(N)$ and $\Lambda_q(f)$ be their alternative sums respectively, i.e.,

\[ \beta_q(N) = \beta_q(N) - \beta_{q-1}(N) + \cdots + (-1)^q \beta_0(N), \]
\[ \Lambda_q(f) = \lambda_q(f) - \lambda_{q-1}(f) + \cdots + (-1)^q \lambda_0(f), \]

where $0 \leq q \leq m$. We establish in §5 an inequality for arbitrary q that $|\beta_q(N) - \Lambda_q(f)|$ is no greater than the qth Morse difference of an arbitrary f-invariant N-coherent Morse function. We obtain as corollaries a fixed point theorem in terms of arbitrary $\Lambda_q$ (when $q = m$, this is the Lefschetz fixed point theorem) and a more geometric proof of the fact that $\beta_q(N) = \Lambda_q(f)$, i.e., the Euler number of N is equal to the Lefschetz number of f.

The Lemma 1 (§1) which states that a smooth function can be approximated by a Morse function with prescribed "boundary value" is essential to the construction of the approximations.

1. A Morse extension. For a real-valued smooth function F on M, let $C(F)$ denote the set of all critical points of F. F is called a Morse function if for any $p \in C(F)$, the determinant of the Hessian at $p$ does not vanish.

We assume without loss of generality that M is a riemannian manifold with a metric g. Let $g_{ij}$ be the metric tensor of g with respect to a local coordinate $(x^i)$

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and let $g^{-1}$ be the inverse of $g_{ij}$ as matrices. Using the metric $g$, the differential $dF(x)$ of $F$ at $x$ has a natural way to be identified with a tangent vector at $x$ which is called the gradient $\nabla F(x)$ at $x$. Locally we have $\nabla F(x) = g^{ij}(\partial F/\partial x^i)(\partial / \partial x^j)$.

We define $\|dF(x)\|$ by

$$\|dF(x)\|^2 = g(\nabla F, \nabla F) \text{ at } x$$

and define $\|F\|_{L^1, \Omega}$ and $\|F\|_{L^2, \Omega}$ of $F$ on an open set $\Omega$ in $M$ by

$$\|F\|_{L^1, \Omega} = \sup\{|F(x)|; x \in \Omega\},$$

$$\|F\|_{L^2, \Omega} = \sup\{|F(x)| + \|dF(x)\|; x \in \Omega\}.$$ 

Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be a $C^{\infty}$-function with $0 \leq |\phi(r)| \leq 1$, $\phi(0) = 1$, $\phi''(0) < 0$ and $\phi(r) = 0$ for $|r| \geq 1$. We denote throughout the induced function of mollifier by $\phi_\epsilon$, for each positive number $\epsilon$, i.e. $\phi_{\epsilon}(r) = \phi(r/\epsilon)$.

There exists a constant $a > 1$ such that

$$|\phi_{\epsilon}'(r)| < a/\epsilon.$$ 

It is well known ([4] or [3]) that any given real-valued smooth function on a compact manifold $M$ can be approximated by a Morse function in the norm $\|\|_{L^1, M}$. The following lemma establishes this approximation theorem even when the “boundary value” of the desired Morse function has been given.

**Lemma 1.** Let $\Omega$ and $D$ be open sets of a smooth manifold $M$ such that $\Omega$ has a compact closure $\bar{\Omega}$ with smooth boundary $\partial \Omega$ and $\bar{D} \subset \Omega$. Let $F$ be a Morse function defined on $M - \bar{D}$. Then $F|_{M - \bar{D}}$ can be extended to a Morse function $\tilde{F}: M \rightarrow \mathbb{R}$. Moreover, if a smooth function $G$ on $M$ with $\|F - G\|_{L^1, M - D} < \epsilon$, is given, then the above Morse extension can be made so that $\|\tilde{F} - G\|_{L^1, M} < 2\epsilon$.

**Proof.** Choose a metric $g$ for $M$. For a point $x$ inside $\Omega$, we denote by $r(x)$ the distance with respect to $g$ from $x$ to $\partial \Omega$. Let $\bar{\Omega}$ be the set $\{x \in \Omega \mid r(x) > r\}$. Since $C(F)$ is discrete and $\Omega$ is compact, there exist positive numbers $\eta$, $R$ and $\delta$ such that

$$\delta < \min\{1/2(1 + a), \sqrt{\epsilon/\eta}\} \quad \text{and} \quad \|dF(x)\| > \eta$$

for all $x$ in the strip $\Omega_{R+\delta} - \Omega_{R+2\delta}$ contained in $\Omega - D$.

Define $H: M \rightarrow \mathbb{R}$ by patching together $F$ and $G$ in $\Omega_{R+\delta} - \Omega_{R+2\delta}$ as follows:

$$H(x) = F(x), \quad x \in M - \Omega_{R+\delta},$$

$$= G(x) + \phi_\epsilon(R + \delta - r(x))(F(x) - G(x)), \quad x \in \Omega_{R+\delta} - \Omega_{R+2\delta},$$

$$= G(x), \quad x \in \Omega_{R+2\delta}.$$
It follows that \( \|H - G\|_{H,M} < \varepsilon \).

Let \( E \) be a Morse function on \( \Omega_{R_\delta} \) approximating \( H | \Omega_{R_\delta} \) such that

\[
(4) \quad \|E - H\|_{\Omega_{R_\delta}} < \delta^2 \eta < \varepsilon.
\]

Finally we define \( \tilde{F} \) on \( M \) by patching together \( E \) and \( F \) in the strip \( \Omega_{R_\delta} - \Omega_R \) as above. In order to see that \( \tilde{F} \) is a Morse function on \( M \), it suffices to show that \( \tilde{F} \) has no critical point in \( \Omega_{R_\delta} - \Omega_R \). In fact, for \( x \) in \( \Omega_{R_\delta} - \Omega_R \), we have

\[ H(x) = F(x) \]

and

\[
\|dF(x)\| > \|dF(x)\| - |\varphi_p(R - r(x))| \cdot \|dE(x) - dH(x)\| - |\varphi_p(R - r(x))| \cdot |E(x) - H(x)|
\]

\[ > \eta - \delta^2 \eta - (a/\delta) \delta^2 \eta > \eta(1 - \delta(1 + a)) > \eta/2 > 0, \]

since we have the estimates (1), (2) and (4). The approximation of \( \tilde{F} \) to \( G \) follows evidently from the construction.

2. Coherency with submanifold. Let \( S \) be a closed embedding submanifold of \( M \). In this section we define \( S \)-coherent Morse functions and show an approximation theorem of smooth functions by \( S \)-coherent Morse functions.

**Definition 1.** A Morse function \( F \) on \( M \) is called \( S \)-coherent if for each \( p \) in \( C(F | S) \), there is a coordinate neighborhood \( (U_p, (x_i)) \) with origin at \( p \), \( U \cap S = \{x_{i+1} = \cdots = x_m = 0\} \), and

\[
F(x_1 \cdots x_m) = F(0) - x_1^2 - \cdots - x_i^2 + \cdots + x_m^2
\]

where \( s \) is the dimension of \( S \) at \( p \) with \( s \geq \lambda \).

Such a \( (U_p, (x_i)) \) is called an \( S \)-coherent coordinate neighborhood of \( p \) for \( F \). Evidently, if \( F \) is an \( S \)-coherent Morse function on \( M \), then \( F | S \) is a Morse function on \( S \) with \( C(F | S) \subseteq C(F) \) and at each \( p \) of \( C(F | S) \), the index of \( F | S \) is equal to the index of \( F \).

For the convenience of later use, we fix the following notation:

**Definition 2.** Given a smooth function \( \psi \) defined on a closed embedding submanifold \( S \) of \( M \), we denote by \( \psi^* \) an extension of \( \psi \) on a tubular neighborhood \( T_p \) of \( S \) with radius \( \rho \) defined as follows. Let \( \rho \) be so small that for any \( x \) in \( T_p \), there is a unique geodesic joining \( x \) to a point \( x' \) of \( S \) and having the length equal to the distance \( r(x) \) from \( x \) to \( S \). Let

\[
\psi^*(x) = \psi(x') \cdot (2 - \varphi_p(r(x)))
\]

where \( \varphi_p \) is the mollifier relative to \( \rho \) (see §1).

If \( \psi \) is a Morse function, so is \( \psi^* \). In fact,

\[
C(\psi) = C(\psi^*) \quad \text{and} \quad \psi''(0) < 0.
\]
Note that at \( p \in C(\psi) \), the index of \( \psi \) equals the index of \( \psi^* \).

**Theorem 1.** Given a closed submanifold \( S \) of \( M \), any smooth function \( G \) on \( M \) can be approximated uniformly by an \( S \)-coherent Morse function \( F \).

**Proof.** Let \( g \) be a Morse function on \( S \) approximating \( G \mid S \). By Lemma 1, the \( g^* \) on a tubular neighborhood of \( S \) can be extended to a Morse function \( F \) on \( M \). \( F \) is evidently \( S \)-coherent. If the tubular neighborhood of \( S \) is sufficiently small, \( F \) can be made to approximate \( G \). Q.E.D.

3. Review of isometric actions. In general, for a compact riemannian manifold \((M, g)\), let \( \text{ISO}(M, g) \) denote the full isometry group. Let \( G \) be a closed subgroup of \( \text{ISO}(M, g) \) and \( p \) a point in \( M \). By the isotropy group \( G_p \), we mean the subgroup of isometries which leave \( p \) fixed. The orbit \( G(p) \) of \( G \) at \( p \) is the set \( \{ \gamma(p); \gamma \in G \} \).

Each orbit is a closed submanifold embedded in \( M \). An orbit \( G(p) \) is called principal if

1. for any \( q \in M \), \( \dim G^p \leq \dim G^q \), and
2. the number of components of \( G^p \) is no greater than the number of components of \( G^q \) whenever \( \dim G^p = \dim G^q \).

We quote the following well-known result.

**Lemma 2** [5]. Let \( G \) be a closed subgroup of \( \text{ISO}(M, g) \) of a complete riemannian manifold \((M, g)\). Then the union of all the principal orbits of \( G \) is open and dense in \( M \).

We return to our given periodic map \( f \) of \( M \) with order \( v \). Without loss of generality, we may assume that \( f \) is an isometry of \((M, g)\) with some metric. In fact we can modify an arbitrarily given metric \( g \) by taking the mean of the induced metrics \((f^k)^*g\) for \( k = 1, 2, \ldots, v \).

Let \( \Gamma \) be the subgroup generated by \( f \) in \( \text{ISO}(M, g) \). \( \Gamma \) is finite and cyclic with order \( v \). By the order of an orbit of \( \Gamma \), we mean the cardinal number of the orbit. For the integer \( k \) such that there exists an orbit \( \Gamma \) with order \( k \), let \( M_k \) be the union of the orbits of order \( l \) where \( l \) is a divisor of \( k \). Thus we have a lattice consisting of these \( M_k \)'s with inclusion as the partial ordering. The lower bound of the lattice is evidently the fixed point set \( N = M_1 \).

We now consider some geometries about \( N \) and more generally about \( M_k \)'s.

**Lemma 3.** The fixed point set \( N \) of an isometry \( f \) is a closed totally geodesic submanifold embedded in \( M \) [2]. If the isometry \( f \) is periodic, then each \( M_k \), defined in the above, is a closed totally geodesic submanifold embedded in \( M \) as well as in each \( M_j \) with \( j \) being a multiple of \( k \).

**Proof.** For the first statement, one can refer to [2]. An elementary proof with clearer geometric insight can be obtained by using the following two facts as the basis of induction to construct, in an obvious way, local coordinates of \( N \) for proving that \( N \) is a submanifold of \( M \).
(1) For two points $p$ and $q$ of $N$ which are sufficiently close to each other, the unique geodesic connecting $p$ and $q$ is contained in $N$.

(2) Let $\gamma_1$ and $\gamma_2$ be two geodesics of $M$ which are contained in $N$ and intersect with each other at a point $p$ of $N$. Then the parallel transportation of $\gamma_1$ along $\gamma_2$ generates a 1-parametered family of geodesics whose union is entirely contained in $N$.

For the second statement of the lemma, we need only to notice that $M_k$ is exactly the fixed point set of $f^k$ acting on $M$ as well as on $M_j$ with $j$ being a multiple of $k$. This completes the proof.

For any two $M_k$ and $M_p$, the intersection $M_k \cap M_p$ is evidently the $M_{(k,p)}$ where $(k, l)$ is the greatest common divisor of $k$ and $l$. On the other hand, $M = M_p$. In fact, for each $M_p$ and each $x$ in $M_p$, choose a convex neighborhood $U$ of $x$ such that for any $y$ in $U$, the geodesic joining $y$ to $x$ in $U$ is the only curve joining $y$ to $\Gamma(x)$ and having the length equal to the distance from $y$ to $\Gamma(x)$. It follows that $\Gamma^p \subset \Gamma^k$ and therefore the order of $\Gamma(x)$ is a divisor of that of $\Gamma(y)$. By Lemma 2, we see that the order of $\Gamma(x)$ is a divisor of $\nu$.

4. The approximation.

**Theorem 2.** Given a periodic transformation $f$ of $M$ with fixed point set $N$, an $f$-invariant smooth function $G: M \rightarrow \mathbb{R}$ can be uniformly approximated by an $f$-invariant $N$-coherent Morse function $F$.

**Proof.** We construct $F$ inductively in the following steps.

**Step 1.** Let $h_t$ be a Morse function on $N$ approximating $G \mid N$ uniformly. Recalling the Definition 2, we extend $h_t$ to $h_\Gamma$ on a tubular neighborhood $T_{2p}$ of $N$.

**Step 2.** For each prime number $p$ which is a divisor of $\nu$, we shall extend $h_\Gamma \mid T_p \cap M_p$ to an $f$-invariant Morse function $h_p: M_p \rightarrow \mathbb{R}$ which approximates $G \mid M_p$.

For a general $k$ with $1 \leq k \leq \nu$, let $U_k$ denote the union of all orbits of order $k$. By Lemma 2, $U_k$ is open and dense in $M_k$. Now $h_\Gamma \mid T_p \cap U_p$ induces a Morse function

$$h_\Gamma^* : (T_p \cap U_p) / \Gamma \rightarrow \mathbb{R}$$

where the quotient by $\Gamma$ means the orbit space of $T_p \cap U_p$ under $\Gamma$. By Lemma 1, $h_\Gamma^*$ can be extended to a Morse function

$$h_{\Gamma} : U_p / \Gamma \rightarrow \mathbb{R}$$

approximating $G / \Gamma$ restricted on $U_p / \Gamma$. This $h_{\Gamma}$ induces an $f$-invariant $N$-coherent Morse extension $h_{\Gamma} : M_p \rightarrow \mathbb{R}$ of $h_\Gamma^* \mid T_p \cap M_p$. $h_{\Gamma}$ evidently still approximates $G \mid M_p$.

**Step 3.** If $\nu \neq p$, we extend $h_p$ to an $f$-invariant Morse function $H_p$ defined on a tubular neighborhood $T_p(M_p)$ of $M_p$ by considering $h_\Gamma^* : T_p(M_p) \rightarrow \mathbb{R}$, and then patching $h_\Gamma^*$ and $h_\Gamma$ together near $N$ as follows.
\[ H_p(x) = h_{\parallel}(x), \quad x \in T_\eta \cap T_{\eta}(M_p), \]
\[ = h_{\parallel}(x) + \varphi_{\parallel}(r(x) - \eta)(h_{\parallel}(x) - h_{\parallel}(x)), \quad x \in (T_{2\eta} - T_\eta) \cap T_{\eta}(M_p), \]
\[ = h_{\parallel}(x), \quad x \in T_{\eta}(M) - T_{2\eta}, \]

where \( \eta = \rho/3 \) and \( r(x) \) denotes the distance from \( x \) to \( N \).

By taking \( \rho_p \) sufficiently small, \( h_{\parallel} \) and \( h_{\parallel}^* \) as well as their derivatives will differ from each other only by a small amount in the patching strip. This guarantees that no critical point of \( H_p \) will appear in the strip. Clearly \( H_p \) approximates \( G \). \( H_p \) is also \( f \)-invariant, since \( h_{\parallel} \) and \( h_{\parallel}^* \) are \( f \)-invariant and \( \varphi_{\parallel} \) is symmetric with respect to 0.

**Step 4.** For \( M_k \), we assume according to the induction hypothesis that for each divisor \( l \) of \( k \), \( H_l \) has been constructed. By the Lemma 1, we extend the function \( UH,_{\parallel} M_k \cap (\bigcup_{l} T_{\eta}(M_l)) \)

to an \( f \)-invariant \( N \)-coherent Morse function \( h_k : M_k \to \mathbb{R} \) in the way similar to that described in Step 2. \( h_k \) approximates \( G \) again. If \( k < r \), we construct again \( h_{\parallel} \) and patch together \( h_{\parallel}^* \) and \( h_{\parallel} \), for all divisors \( l \) of \( k \), as in Step 2 to obtain \( H_k \). If \( k = r \), we take \( F = h_r \). This completes the construction of \( F \).

**Remark.** Such \( F \) is indeed \( M_l \)-coherent for all \( l \).

5. The Inequality and its applications. In general, for \( Y \subset X \subset M \), let

\[ \beta_q(X, Y) = \text{the Betti number of the pair } (X, Y), \]
\[ \lambda_q(X, Y) = \text{the trace of } f_\ast \text{ on } H_q(X, Y), \]

and let

\[ \beta_q(X, Y) = \beta_q(X, Y) - \beta_{q-1}(X, Y) + \cdots + (-1)^q \beta_0(X, Y), \]
\[ \lambda_q(X, Y) = \lambda_q(X, Y) - \lambda_{q-1}(X, Y) + \cdots + (-1)^q \lambda_0(X, Y). \]

We fix an \( f \)-invariant \( N \)-coherent Morse function \( F \) chosen arbitrarily. For a real number \( a \), let \( M^a \) be the set \( \{ x \in M \mid F(x) \leq a \} \).

Let all the critical values \( c_a \)'s of \( F \) be ordered such that \( c_1 > c_2 > \cdots > c_r \). Let \( p^1, \ldots, p^l, \ldots, p^k \) be all the critical points of \( F \) with critical value \( c_a \) and of indices \( v^1, \ldots, v^l, \ldots, v^k \) respectively, where \( p^1, \ldots, p^k \) are precisely the ones contained in \( N \). (\( l \) and \( k \) depend on \( a \). The superscript \( a \) will be omitted everywhere when no confusion can occur.)

For each \( p_j, 1 \leq j \leq k \), there is an \( N \)-coherent coordinate neighborhood \( (x_j) \) of \( p_j \). Let \( e_j \) be the \( v_j \)-cell \( (x_{j+1} = x_{j+2} = \cdots = x_m = 0) \). Consider numbers \( a_0, a_1, \ldots, a_m \) such that
When $a_\alpha$ is chosen sufficiently close to $c_\alpha$, we can have

1. $e_j$'s are disjoint and $\partial e_j \subset M^{a_\alpha}$;
2. $\{(e_j, \partial e_j) \mid j = 1, \ldots, l \}$ and $\{(e_j, \partial e_j) \mid j = 1, \ldots, l, \ldots, k \}$ are respectively the generators of the homology groups $H(N^{a_\alpha-1}, N^{a_\alpha})$ and $H(M^{a_\alpha-1}, M^{a_\alpha})$; and
3. for $1 \leq j \leq l$, $f$ is the identity map on $e_j$ and for $l < j \leq k$, $f_\alpha(e_j, \partial e_j) = (e_j, \partial e_j)$ with $i \neq j$, where $f_\alpha$ is the induced map of $f$ on $H(M^{a_\alpha-1}, M^{a_\alpha})$.

It follows that for each $q$ and $\alpha$ both of $\beta_q(N^{a_\alpha-1}, N^{a_\alpha})$ and $\lambda_q(M^{a_\alpha-1}, M^{a_\alpha})$ are equal to the number of $e_j$'s with $\gamma_j = q$ and $1 \leq j \leq l$. Hence we have

$$\beta_q(N^{a_\alpha-1}, N^{a_\alpha}) = \lambda_q(M^{a_\alpha-1}, M^{a_\alpha}),$$

$$B_q(N^{a_\alpha-1}, N^{a_\alpha}) = A_q(M^{a_\alpha-1}, M^{a_\alpha}).$$

From the exactness of

$$0 \rightarrow \partial_q(H_q(N, N^{a_\alpha})) \rightarrow H_q(N^{a_\alpha}, N^{a_\alpha-1}) \rightarrow H_q(N, N^{a_\alpha-1}) \rightarrow \cdots,$$

we have

$$B_q(N, N^{a_\alpha}) = B_q(N^{a_\alpha-1}, N^{a_\alpha}) + B_q(N, N^{a_\alpha-1}) - \epsilon_{q, \alpha}$$

where $\epsilon_{q, \alpha}$ is the rank of $\partial_q(H_q+1(N, N^{a_\alpha-1}))$. Similarly, we have

$$\Lambda_q(M, M^{a_\alpha}) = \Lambda_q(M^{a_\alpha-1}, M^{a_\alpha}) + \Lambda_q(M, M^{a_\alpha-1}) - \eta_{q, \alpha}$$

where $\eta_{q, \alpha}$ is the trace of $f_\alpha$ on $\partial_q(H_q+1(M, M^{a_\alpha-1}))$. By induction we have

$$B_q(N) = \sum_a B_q(N^{a_\alpha}, N^{a_\alpha-1}) - \sum \epsilon_{q, \alpha}$$

and

$$\Lambda_q(f) = \sum_a \Lambda_q(M^{a_\alpha}, M^{a_\alpha-1}) - \sum \eta_{q, \alpha}.$$

The well-known Morse inequality states that given an arbitrary Morse function on $M$, we have

$$B_q(M) \leq C_q \overset{\text{def}}{=} c_q - c_{q-1} + \cdots + (-1)^q c_0$$

where $c_q$ denotes the number of critical points of the Morse function with index $q$. The difference $C_q - B_q(M)$ is given by

$$\sum_a \text{rank}[\partial_q(H_q+1(M^{a_\alpha}, M^{a_\alpha-1}))]$$
if we adopt the subdivision of $M$ according to the Morse function as we did in the above.

**Definition 3.** We call the difference $C_q - B_q(M)$ the $q$th Morse difference. We denote the $q$th Morse difference of $F$ by $\delta_q(F)$. However,

$$|\eta_{q,a} - \epsilon_{q,a}| \leq \text{rank}[\partial_*(H_{q+1}(M^a, M^{a-1}))].$$

Therefore we obtain

**Theorem 3.** Given a periodic transformation $f$ of a compact smooth $m$-dimensional manifold $M$ with fixed point set $N$, we have the inequality

$$|\Lambda_q(f) - B_q(N)| \leq \delta_q(F)$$

for each $q = 0, \ldots, m$ and each $f$-invariant $N$-coherent Morse function $F$, where

$$\Lambda_q(f) = \sum_{r=0}^{q} (-1)^{q-r} \text{trace of } f^r \text{ on } H_r(M),$$

$$B_q(N) = \sum_{r=0}^{q} (-1)^{q-r} \text{rth Betti number of } N,$$

and $\delta_q(F)$ is the $q$th Morse difference of $F$.

As corollaries we obtain a fixed point set theorem.

**Theorem 4.** Given a periodic transformation $f$ of a compact smooth manifold $M$, if $|\Lambda_q(f)| > \delta_q(F)$ for some $q = 1, \ldots, m$ and some $f$-invariant Morse function $F$ on $M$, then $f$ has a fixed point.

**Proof.** Suppose $f$ is fixed point free. Then every Morse function is $N$-coherent. Also $B_q(N) = 0$. These lead to a contradiction.

**Remark 2.** In particular when $q = m$, $\Lambda_m$ is the usual Lefschetz number and $\delta_m(F) = 0$ for all $F$. Therefore this corollary is a generalization of the Lefschetz fixed point theorem for a periodic map.

**Remark 3.** Such a fixed point theorem based on $\Lambda_q$ and $\delta_q(F)$ for arbitrary $q$ and $F$ gives the best possible estimation. In fact, let $T^2 = S^1 \times S^1 = \{(e^\theta, e^{i\varphi}) \mid \theta, \varphi < 2\pi\}$ and consider $f : (e^\theta, e^{i\varphi}) \to (e^\theta, e^{-i\varphi})$ and $F(e^\theta + e^{i\varphi}) = \cos \theta + \cos 2\varphi$. Then $F$ is an $f$-invariant Morse function with $\Lambda_1 = 1 = \delta_1(F)$ but $f$ has no fixed point.

Since $\delta_m(F) = 0$, we obtain

**Corollary 1.** Given a periodic transformation $f$ on a compact smooth manifold $M^m$ with fixed point set $N$, we have the Lefschetz number of $f$ equal to the Euler number of the fixed point set $N$ and therefore equal to the integral over $N$ of the restricted “intrinsic curvature” in the sense of Chern [1].
This statement can be regarded as a generalization of the Gauss-Bonnet theorem. A stronger result for any isometry can be proven rather directly by Mayer-Vietoris sequence applying on a tubular neighborhood of $N$. However, the above approach using the viewpoint of Morse theory may help one to have better geometric insight.

REFERENCES


DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202

Current address: Department of Mathematics, National Taiwan University, Taipei, Taiwan

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