TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 189, 1974

WEAKLY ALMOST PERIODIC AND UNIFORMLY CONTINUOUS FUNCTIONALS ON THE FOURIER ALGEBRA OF ANY LOCALLY COMPACT GROUP(1)

BY
EDMOND E. GRANIRER

ABSTRACT. We define for any locally compact group G, the space of bounded uniformly continuous functionals on $\hat{G}$, $UCB(\hat{G})$, in the context of P. Eymard [Bull. Soc. Math. France **92** (1964), 181–236. MR **37** #4208] (for notations see next section). For $u \in A(G)$ let $u^{\perp} = \{\phi \in VN(G); \phi[A(G)u] = 0\}$.

Theorem. *If for some norm separable subspace $X \subset VN(G)$ and some positive definite $0 \neq u \in A(G)$, $UCB(\hat{G}) \subset$ norm closure $[W(\hat{G}) + X + u^{\perp}]$ then G is discrete. If G is discrete then $UCB(\hat{G}) \subset AP(\hat{G}) \subset W(\hat{G})$.*

Introduction. A result of R. B. Burckel [13, p. 68] implies that if G is any locally compact abelian group and $W(\hat{G})$, $UCB(\hat{G})$, $C(\hat{G})$ are the spaces of bounded weakly almost periodic, uniformly continuous, continuous, functions on the dual $\hat{G}$ of G, respecively, then $W(\hat{G}) = C(\hat{G})$ iff G is discrete. A different proof of this same result is given in Ramirez-Dunkl [4, Corollary 6.5]. Our results in [7, p. 62] imply an improved version of this statement, namely that if G is locally compact abelian nondiscrete, then the quotient Banach space $UC(\hat{G})/W(\hat{G})$ with the usual quotient norm is not norm separable.

The Fourier algebra of G, $A(G)$ and its dual $VN(G)$ (none other than $L^{\infty}(\hat{G})$ in the abelian case) have been defined and extensively studied by P. Eymard in [6].

It is proved by Dunkl and Ramirez in [4, Corollary 6.4], that if $G = \prod_1^{\infty} G_n$ is the complete direct product of countably many nontrivial compact groups then $W(\hat{G})$ is a proper closed subspace of $VN(G)$ (denoted $\mathcal{L}^{\infty}(\hat{G})$ in [4]. For the definition of $W(\hat{G})$ see next section.) This result is improved by Dunkl and Ramirez in [15] to hold for any compact group and further improved by P. F. Renaud in a recent paper [16], which was not available to us before sending the present paper for publication, to hold for any locally compact group.

We define in the next section the space of bounded uniformly continuous functionals on $\hat{G}$, $UCB(\hat{G}) \subset VN(G)$. It coincides with $UCB(\hat{G})$ if G is locally compact abelian and with $\mathcal{L}^{\infty}(\hat{G}) = VN(G)$ if G is compact (then $\hat{G}$ is discrete, see [5]), as it should be.

Our main result, which improves all the above is

Received by the editors January 18, 1973 and, in revised form, April 9, 1973.

AMS (MOS) subject classifications (1970). Primary 43A60, 43A30, 43A35, 22D05, 22D15; Secondary 43A07, 22D35, 22D25.

(1) Research done while the author held a Canada Council Award. The support of the Canada Council is gratefully acknowledged.

Theorem 12. *Let G be any locally compact group. For $u \in A(G)$ let* $u^{\perp} = \{\phi \in VN(G); \phi[A(G)u] = 0\}$.

If for some positive definite nonzero $u \in A(G)$ and some norm separable subspace $X \subset VN(G)$

$$UCB(\hat{G}) \subset \textit{norm } \mathrm{cl}[W(\hat{G}) + X + u^{\perp}]$$

then G is discrete. If G is discrete then $UCB(\hat{G}) \subset AP(\hat{G}) \subset W(\hat{G})$.

If G is an amenable locally compact group then $W(\hat{G}) \subset UCB(\hat{G})$. For $u \in A(G)$ let

$$W(\hat{G}, u) = \text{norm } \mathrm{cl}[W(\hat{G}) + UCB(\hat{G}) \cap u^{\perp}].$$

For such groups one has

Corollary 13. *If for some positive definite nonzero $u \in A(G)$ the Banach space $UCB(\hat{G})/W(\hat{G}, u)$ (a fortiori $UCB(\hat{G})/W(\hat{G})$) with quotient norm is separable, then G is discrete. If G is discrete then $AP(\hat{G}) = W(\hat{G}) = UCB(\hat{G})$.*

We prove in the end the following theorem (see definitions in the next section) which is analogous but not equivalent to Theorem 12.

Theorem 14. *Let G be any amenable locally compact group and $\alpha \in L_1(G)$, $\alpha \geq 0, \int \alpha\, dx = 1$. Let $\alpha^{\perp} = \{f \in L^{\infty}(G); (f, L_1(G) * \alpha) = 0\}$. If $UCB_r(G) \subset$ norm $\mathrm{cl}\{W(G) + \alpha^{\perp} + X\}$ for some norm separable $X \subset L^{\infty}(G)$ then G is compact.*

Notations, definitions and some remarks. We describe at first the essentials of the construction of the Fourier algebra $A(G)$ and its conjugate Banach space $VN(G)$ (which reduces to $L^{\infty}(\hat{G})$ in case G is abelian) for any locally compact group G. This construction is due to P. Eymard [6] and is given here in concise form.

Let G by *any* locally compact group with unit e and with left Haar measure dx. If $p \geq 1$ let $\|f\|_p = (\int |f(x)|^p dx)^{1/p}$, $\tilde{f}(x) = \overline{f(x^{-1})}$, $f^{\vee}(x) = f(x^{-1})$ for $x \in G$, and $\|f\|_{\infty} = \text{ess sup}|f(x)|$. Let ρ denote the left regular representation of $L^1(G)$ on $L^2(G)$, i.e. $(\rho f)(g) = f * g$ for $g \in L^2(G)$ (as in [8, p. 293, (20.14)]) and let $\|f\|_{\rho}$ denote the operator norm of ρf: $L^2(G) \to L^2(G)$. Then $\|f\|_{\rho}$ is a C^* norm on the convolution algebra $L^1(G)$.

Let $A(G) = \{g * \tilde{h}; g, h \in L^2(G)\}$ with the norm

$$\|u\|_A = \sup\left\{\left|\int u(x) f(x)\, dx\right|; f \in L^1(G); \|f\|_{\rho} \leq 1\right\}.$$

Then $(A(G), \| \ \|_A)$ under pointwise multiplication is a commutative Banach algebra whose maximal ideal space is G. This is one of the important achievements of Eymard in [6] (see (2.6), 1°, p. 192; p. 218; p. 222, Theorem 3.34).

In all that follows we will write $\|u\|$ instead of $\|u\|_A$ for $u \in A(G)$. For abelian G, $A(G)$ is none other than the algebra $\{Ff; f \in L^1(\hat{G})\}$, where F denotes Fourier transform, with norm $\|Ff\| = \|f\|_1$, i.e. the usual Fourier algebra of G.

We describe now $VN(G)$, (denoted $\mathcal{L}^\infty(\hat{G})$ for compact G in [5]). The W^*-algebra of all bounded linear operators T: $L^2(G) \to L^2(G)$ which commute with convolution from the right by continuous functions with compact support, equipped with the usual operator norm on $L^2(G)$, is denoted by $VN(G)$ [6, p. 210].

$VN(G)$ is identified with the conjugate of the Banach space $A(G)$ by: $T \in VN(G)$ is identified with the linear functional ϕ_T on $A(G)$ given by: If $f, g \in L^2(G)$ then: $\phi_T[(f * \tilde{g})] = \phi_T(\bar{g} * f^\vee) = (Tf, g)$ (inner product in $L^2(G)$). Then $\|\phi_T\| = \|T\|$ (operator norm) and $A(G)^* = \{\phi_T; T \in VN(G)\}$ (see [6, p. 210]). (If X is a normed space X^* will always denote its conjugate.) The space $A(G)$ is in fact the predual of the W^*-algebra $VN(G)$ [6, p. 212]. We will identify ϕ_T with $T \in VN(G)$ and write $T(u)$ or Tu instead of $\phi_T(u)$ (we never use the notation Tu as in [6, p. 213]).

We note now the module action of $A(G)$ on $VN(G)$ given by: For $u \in A(G)$, $\phi \in VN(G)$, $(u \cdot T)(v) = T(uv)$ for all $v \in A(G)$. One has $\|u \cdot T\| \leq \|u\|\,\|T\|$ [6, p. 224].

The set of all $\phi \in VN(G)$ for which the operator from $A(G)$ to $VN(G)$ given by $u \to u \cdot \phi$ is weakly compact [compact] is denoted by $W(\hat{G})$ [$AP(\hat{G})$], the weakly almost periodic [almost periodic] functionals in $VN(G)$ (see Dunkl and Ramirez [4, Definition 2.1, Proposition 2.2, and Chapters 7,8]).

Definition. Let $A(G) \cdot VN(G) = \{u \cdot \phi; u \in A(G), \phi \in VN(G)\}$ and $UCB(\hat{G})$ = norm closure $\{A(G) \cdot VN(G)\}$. The space $UCB(\hat{G})$ is said to be the set of bounded uniformly continuous functionals of $VN(G)$.(2)

The justification for this definition is the following: If G is abelian and $\phi \in L^\infty(\hat{G})$ and $u \in A(G)$ then $u \cdot \phi = (Fu)^\vee * \phi$ (Dunkl and Ramirez [4, Chapter 1]). Thus $A(G) \cdot VN(G) = L^1(\hat{G}) * L^\infty(\hat{G}) = UCB(\hat{G})$, the algebra of bounded uniformly continuous functions on $\hat{G}$ (see Hewitt and Ross [9, p. 283]).

If G is compact, not necessarily abelian, then $A(G)$ has an identity ($1 \in L^2(G)$ and $1 * \tilde{1} = 1$); hence $A(G) \cdot VN(G) = VN(G)$ as it should be since $\hat{G}$ is discrete (see [5]).

If G is any amenable locally compact group (as in [12, p. 29]) then we prove that $A(G) \cdot VN(G)$ is necessarily norm closed. We do not have an example of a group, for which this is not the case.

Let $P(G)$ denote all positive definite continuous functions on G and $P_1(G) = P(G) \cap \{u \in A(G); u(e) = 1\}$. We make extensive use of the set $P_1(G)$.

(2) Thanks are due to C. Herz for the following observation: $UCB(\hat{G})$ is always a linear space. In fact $A_c(G) = \{u \in A(G)$ with compact support$\}$ is dense in $A(G)$; hence $A_c(G) \cdot VN(G)$ is norm dense in $UCB(\hat{G})$. If $T_1, T_2 \in VN(G)$ and $u_1, u_2 \in A_c(G)$ let $F =$ supp $u_1 \cup$ supp u_2 and $F \subset O$ be open with compact closure. Let $u \in A(G)$ satisfy $0 \leq u \leq 1$, $u(F) = 1$, $u = 0$ off O [6, p. 208]. Then $u_1 T_1 + u_2 T_2 = u(u_1 T_1 + u_2 T_2) \in A_c(G) \cdot VN(G)$. Thus $UCB(\hat{G}) =$ norm cl $A_c(G) \cdot VN(G)$ is a linear space.

If (X,τ) is a locally convex space and $K \subset X$ then τ cl K denotes the τ-closure of K in X. If (X, X^1) are locally convex spaces in duality then $\sigma(X,X^1)$ is the weakest locally convex topology on X which makes all linear functionals in X^1 continuous.

If X is a normed space X^* always denotes its conjugate. $\sigma(X,X^*)$ $[\sigma(X^*,X)]$ are the weak (w) [weak* (w^*)] topology on X $[X^*]$.

We follow the notations of [6] unless specified otherwise.

G is said to be an amenable locally compact group if there is some $\psi \in L^\infty(G)^*$ such that $\psi \geq 0$, $\psi(1) = 1$ and $\psi(\phi * f) = \psi(f)$ for all $f \in L^\infty(G)$ and all $\phi \in L^1(G)$ with $\phi \geq 0$, $\int \phi\, dx = 1$. Denote by TLIM (G) the set of all such $\psi \in L^\infty(G)^*$. Any compact G or any locally compact abelian G is amenable (see [12] for these and more).

Denote by $UCB_r(G)$, $W(G)$ the right uniformly continuous, weakly almost periodic continuous, bounded complex valued functions on G, respectively, with sup norm. (Thus $f \in UCB_r(G)$ iff for any $\varepsilon > 0$ there is some neighborhood V of the unit e such that $|f(vx) - f(x)| < \varepsilon$ for each $v \in V$ and $x \in G$.)

Proposition 1. *Let G be an amenable locally compact group. Then* $W(\hat{G}) \subset UCB(\hat{G}) =$ *norm* $\mathrm{cl}(A(G) \cdot VN(G)) = A(G) \cdot VN(G)$.

Proof. By H. Leptin's theorem [10], $A(G)$ has an approximate unit e_α with $\|e_\alpha\| \leq 1$. Let $\phi \in W(\hat{G})$. Then $e_\alpha \cdot \phi \to \phi$ in the w^* topology of $VN(G)$ (i.e. $\sigma[VN(G), A(G)]$). But by [4, Proposition 2.2 and Chapter 8] there is some $\phi' \in VN(G)$ and a subnet such that $e_{\alpha_\beta} \cdot \phi \to \phi'$ weakly (i.e. in weak topology of $VN(G)$ as a Banach space). Thus $\phi = \phi'$ and since the weak and norm closure of $A(G) \cdot VN(G)$ are the same one has $W(\hat{G}) \subset UCB(\hat{G})$.

As for the equality $UCB(\hat{G}) = A(G) \cdot VN(G)$ we note the following: $(A(G), \|\ \|)$ is a Banach algebra with a bounded left approximate unit and $VN(G)$ is a left Banach $A(G)$-module (as in [9, p. 263, (32.14)], see also [6, p. 225]). By the factorisation theorem [9, (32.22), p. 268], $A(G) \cdot VN(G)$ is norm closed, which finishes this proof.

Proposition 2. *Let G be a discrete group. Then $UCB(\hat{G}) \subset AP(\hat{G}) \subset W(\hat{G})$. If G is amenable and discrete then $UCB(\hat{G}) = AP(\hat{G}) = W(\hat{G})$.*

Proof. For $a \in G$ let 1_a be the function whose value is one on a and zero otherwise. By Eymard [6, (3.2), p. 208], $1_a \in A(G)$.

We prove at first that for any $\phi \in VN(G)$, $1_a \cdot \phi \in AP(\hat{G})$.

Let $f, g \in A(G)$ and define $l_x\colon A(G) \to A(G)$ by $(l_x u)(y) = u(xy)$ and by $I \in VN(G)$ the identity operator. When I is regarded as a linear functional on $A(G)$ [6, (3.10), p. 210] one has, for $u \in A(G)$, $I(u) = u(e)$ where $e \in G$ is the identity of G. (By [6, (3.11), p. 210] if $u = f * \tilde{g}$ then $\phi_I(u^\vee) = (If, g) = (f, g)$ and $u(e) = (f * \tilde{g})^\vee(e) = f * \tilde{g}(e) = \int \overline{g(y)} f(y)\, dy = (f, g)$. Thus $\phi_I(u) = u(e)$ for a norm dense subspace of $A(G)$ and since both sides are continuous in the norm of $A(G)$ [6, p. 210], $\phi_I(u) = u(e) = I(u)$ for all $u \in A(G)$.) One has

$$[g \cdot (1_a \cdot \phi)](f) = \phi(g(a)f(a)1_a) = g(a)\phi(1_a)f(a) = g(a)\phi(1_a)[(l_a^* I)f],$$

i.e.

$$g \cdot (1_a \cdot \phi) = [g(a)\phi(1_a)]l_a^* I,$$

and if $\|g\| \leq 1$ then $g \cdot (1_a \cdot \phi) \in \{\alpha l_a^* I; |\alpha| \leq |\phi(1_a)|\}$ the latter being a bounded one dimensional subset of $VN(G)$. Thus the map from $A(G)$ to $VN(G)$ given by $g \to g \cdot (1_a \cdot \phi)$ is a compact linear operator so $1_a \cdot \phi \in AP(\hat{G})$.

If $g_0 = \sum_1^n \alpha_i 1_{a_i}$ then $g_0 \cdot \phi = \sum_1^n \alpha_i(1_{a_i}\phi)$ is a linear combination of compact linear operators, hence is a compact operator [2, p. 486]. Thus, for any $g_0 \in A(G)$, with compact (in fact finite) support and any $\phi \in VN(G)$, $g_0 \cdot \phi \in AP(\hat{G})$. Let now $g_0 \in A(G)$ be arbitrary and $g_n \in A(G)$ with compact support be such that $\|g_0 - g_n\| \to 0$ (see [6, p. 208, (3.4)]). If $\phi \in VN(G)$ and $v, u \in A(G)$, $\|u\| \leq 1$ and $\|v\| \leq 1$,

$$|(v \cdot [(g_n - g_0) \cdot \phi])u| = |\phi((g_n - g_0)uv)| \leq \|\phi\| \, \|g_n - g_0\| \to 0.$$

Thus $g_n \cdot \phi \to g_0 \cdot \phi$ in the operator (uniform) norm (as operators from $A(G)$ to $VN(G)$). Since norm limits of compact operators are compact [2, p. 486], $g_0 \cdot \phi \in AP(\hat{G})$. But $AP(\hat{G})$ is a norm closed subspace of $VN(G)$ [4, Theorem 7.2 and Chapter 8]. Thus for any discrete G,

$$UCB(\hat{G}) = \text{norm cl}[A(G)VN(G)] \subset AP(\hat{G}) \subset W(\hat{G}).$$

If G is addition amenable then apply Proposition 1.

The following proposition is in the folklore and follows from I. Namioka [17, Theorem 4.3].

Proposition 3. *Let B be a W^*-algebra and S $[S_N]$ the set of [ultraweakly continuous = normal] states on B. Then S_N is w^* dense in S.*

Corollary 4. $P_1(G) = P(G) \cap \{u \in A(G); u(e) = 1\}$ *is w^* dense in the set of all states on $VN(G)$.*

Proof. The set $P_1(G)$ coincides with the set of ultraweakly (i.e. normal) states on $VN(G)$ (see Eymard [6, p. 212 in the proof of (3.15) and (2.6), 2°, p. 193]).

Definition. Let $S = \{0 \leq \psi \in VN(G)^*; \psi(I) = 1\}$ be the set of states on the W^*-algebra $VN(G)$.

The element $\psi \in S$ is said to be a *topological invariant mean* (TIM) if $\psi(u \cdot \phi) = \psi(\phi)$ for all $u \in P_1(G)$ and $\phi \in VN(G)$. The set of all TIM's on $VN(G)$ is denoted by TIM $(\hat{G})$. The justification for this definition is as follows:

If G is a locally compact abelian group with dual group $\hat{G}$ and $f \in A(G)$ and $\phi \in L^\infty(\hat{G}) = VN(G)$ then $f \cdot \phi = (Ff)^\vee * \phi$ where F denotes Fourier transform (see Dunkl and Ramirez [4, Chapter 1]). Furthermore if $f \in P_1(G)$ then

$(Ff) \in L^1(\hat{G})$ is nonnegative of $L^1(\hat{G})$ norm one. In this case, for ψ to be TIM on $VN(G)$ just means that $\psi(g * \phi) = \psi(\phi)$ for all $0 \leq g \in L^1(\hat{G})$ of norm one and all $\phi \in L^\infty(\hat{G})$, which agrees with the usual definition of a topological invariant mean as in Greenleaf [12].

If $u \in A(G)$ let t_u: $VN(G) \to VN(G)$ be defined by $t_u(\phi) = u \cdot \phi$ (i.e. the module action of $A(G)$ on $VN(G)$ [6, p. 224]) and T_u: $VN(G)^* \to VN(G)^*$ be the adjoint of t_u, i.e. $(T_u\psi)\phi = \psi(t_u\phi)$. Note that t_u is the adjoint of the operator t_u^1: $A(G) \to A(G)$, $t_u^1(v) = uv$. Also, $\psi \in$ TIM $(\hat{G})$ iff $\psi \in S$ and $T_u\psi = \psi$ for all $u \in P_1(G)$.

We note that $\{t_u^1; u \in P_1(G)\}$ is a commutative semigroup of operators since $t_u^1 t_v^1 = t_{uv}^1$ for $u, v \in P_1(G)$. ($P_1(G)$ is closed under pointwise multiplication. This follows from [6, proof of (2.16), p. 197] and the definition of $P_1(G)$.) Therefore $\{t_u; u \in P_1(G)\}$ and $\{T_u; u \in P_1(G)\}$ will be commutative semigroups of operators.

Part (a) of the following proposition is due to Renaud [16, p. 287] while part (b) is due to Dunkl and Ramirez [4, Chapter 2, Theorem 2.11, and Chapter 8]. The proofs of these results given in [4] and [16] are pretty difficult in that they use the concept of ergodicity of a semigroup, an Eberlein ergodic theorem, the fact that each primary ideal in $A(G)$ is maximal, the notion of support $\phi \in VN(G)$ and some of its properties, in adition to a Beurling type theorem, Corollary 2, (4.11) of Eymard [6, p. 229]. Our proof of (a) is a trivial application of the Markov-Kakutani fixed point theorem while that of (b) is much simpler and shorter than the one in [4] besides, requiring only the basic tools of functional analysis.

The nonnegative linear functional ψ on $W(\hat{G})$ with $\psi(I) = 1$ is a TIM *on* $W(\hat{G})$ if $\psi(u \cdot \phi) = \psi(\phi)$ for all $u \in P_1(G)$ and $\phi \in W(\hat{G})$. That $u \cdot W(\hat{G}) \subset W(\hat{G})$ for all $u \in P_1(G)$ (i.e. that $W(\hat{G})$ is a submodule of $VN(G)$) is proved in [4, Theorem 2.6], or directly.

Proposition 5. *Let G be any locally compact group. Then*:

(a) TIM $(\hat{G}) \neq \varnothing$.

(b) *There exists a unique* $\psi \in W(\hat{G})^*$ *such that* $\psi(I) = 1$ *and* $\psi(t_u\phi) = \psi(\phi)$ *for all* $u \in P_1(G)$ *and* $\phi \in W(\hat{G})$.

Proof. The set S of states on $VN(G)$ is $w*$ compact and convex. If $u \in P_1(G)$ then $T_u(S) \subset S$, since $uP_1(G) \subset P_1(G)$ and by Corollary 4. By the Markov-Kakutani fixed point theorem [2, p. 456] there exists some $\psi_0 \in S$ such that $T_u\psi_0 = \psi_0$ for all $u \in P_1(G)$. Thus $\psi_0 \in$ TIM $(\hat{G}) \neq \varnothing$, which proves (a).

The restriction of ψ_0 to $W(\hat{G})$ is a TIM *on* $W(\hat{G})$. Keep ψ_0 fixed and let u_α be a net in $P_1(G)$ such that $\phi(u_\alpha) \to \psi_0(\phi)$ for all $\phi \in VN(G)$. Let ψ_1 be a continuous linear functional on $W(\hat{G})$ such that $\psi_1(I) = 1$ and $\psi_1(u \cdot \phi) = \psi_1(\phi)$ for all $u \in P_1(G)$, $\phi \in W(\hat{G})$ and let $c = \psi_0(\phi_0)$. Then

$$(t_{u_\alpha}\phi_0)(u) = (t_u\phi_0)(u_\alpha) \to \psi_0(t_u\phi_0) = c = cu(e) = (cI)(u)$$

for $u \in P_1(G)$. Thus $t_{u_\alpha}\phi \to cI$ in the w^* topology of $VN(G)$. But $\{t_u\phi_0, u \in P_1(G)\}$ is relatively weakly compact; hence $t_{u_\alpha}\phi \to cI$ weakly (i.e. in $\sigma(VN(G), VN(G)^*)$) which implies that cI is in the norm closure of the convex set $\{t_u\phi_0; u \in P_1(G)\}$. Therefore $\psi_1(\phi_0) = \psi_1(t_u\phi_0) = c\psi_1(I) = \psi_0(\phi_0)$. Thus $\psi_1 = \psi_0$ on $W(\hat{G})$ which finishes this proof.

The following proposition is known and is a trivial application of [6, p. 207, (3.1) and p. 218].

Proposition 6. *Let G be a separable metric locally compact group. Then $A(G)$ is norm separable.*

Definition. For $u \in A(G)$ let $u^\perp = \{\phi \in VN(G); \phi(vu) = 0$ for all $v \in A(G)\}$.

Theorem 7. *Let G be a separable metric locally compact group. If, for some $u \in P_1(G)$ and some norm separable subspace $X \subset VN(G)$,*

$$UCB(\hat{G}) \subset \textit{norm}\ \mathrm{cl}[W(\hat{G}) + X + u^\perp]$$

then G is discrete.

Remark. $W(\hat{G})$ can be replaced by any submodule $Y \subset VN(G)$ which admits a unique TIM.

Proof. Let $\{u_n\}$ be a norm dense sequence in $P_1(G)$. If $\psi \in VN(G)^*$ satisfies $T_{u_n}\psi = \psi$ for all n then $T_v\psi = \psi$ for all $v \in P_1(G)$, as readily seen.

Denote now $K = \{vu; v \in P_1(G)\} \subset P_1(G)$ [6, p. 197] and identify $P_1(G)$ with its image in $VN(G)^*$. If $\phi \in u^\perp$ then $\phi(vu) = 0$ for all $v \in P_1(G)$. Therefore, for all $\psi \in w^*\ \mathrm{cl}\ K$, $\psi(\phi) = 0$ for all $\phi \in u^\perp$.

Choose and fix now some $\psi_0 \in \{w^*\ \mathrm{cl}\ K\} \cap \mathrm{TIM}\ (\hat{G})$. (Such ψ_0 exists since $w^*\ \mathrm{cl}\ K$ is a w^* compact convex set of states on $VN(G)$ which is invariant with respect to the commutative semigroup of w^* continuous operators $\{T_v; v \in P_1(G)\}$. Apply the Markov-Kakutani theorem [2, p. 456].)

Let ϕ_n be a norm dense sequence in X and let $\psi_0(\phi_n) = \alpha_n$. Consider

$$A = \{w^*\ \mathrm{cl}\ K\} \cap \{\psi \in VN(G)^*; (t^1_{u_n} - I^1)^{**}\psi = 0, \psi(\phi_n) = \alpha_n\}$$

where $I^1: A(G) \to A(G)$ is the identity. Note that $(t^1_{u_n} - I^1)^{**} = T_{u_n} - I^*$ where I^* is the identity on $VN(G)^*$. By the above observations it follows that

$$A = \{w^*\ \mathrm{cl}\ K\} \cap \{\psi \in \mathrm{TIM}\ (\hat{G}); \psi(\phi_n) = \alpha_n, n \geq 1\}.$$

Hence $\psi_0 \in A$. Moreover $A = \{\psi_0\}$. In fact, if $\psi_1 \in A$ then $\psi_1(u^\perp) = 0$, $\psi_1 = \psi_0$ on X since they agree on a dense subspace and $\psi_1 = \psi_0$ on $W(\hat{G})$, by Proposition 5(b). Our assumption implies that $\psi_1 = \psi_0$ on $UCB(\hat{G}) = \mathrm{norm\ cl}\ A(G) \cdot VN(G)$. If $\phi \in VN(G)$ is arbitrary and $v \in P_1(G)$ then $\psi_1(\phi) = \psi_1(v \cdot \phi) = \psi_0(v \cdot \phi) = \psi_0(\phi)$, i.e. $\psi_0 = \psi_1$ and $A = \{\psi_0\}$.

Apply now our Corollary (1.3) on p. 21 of [7]. It follows that $\psi_0 \in A \cap w^*$ seq. cl K, i.e. that for some sequence $v_n \in K \subset P_1(G)$ and all $\phi \in VN(G)$, $\phi(v_n) \to \psi_0(\phi)$. Thus v_n is weak Cauchy sequence in $A(G)$. By a theorem of Sakai [14] (see also [1]) the predual of a W^*-algebra is weakly sequentially complete. There is hence some $v_0 \in A(G)$ such that $\phi(v_0) = \psi_0(\phi)$ for all $\phi \in VN(G)$. Since $\psi_0 \in$ TIM $(\hat{G})$ it follows that for all $v \in P_1(G)$ and $\phi \in VN(G)$; $\phi(vv_0) = \phi(v_0)$. Thus

$$(**) \qquad vv_0 = v_0 = v(e)v_0 \quad \text{for all } v \in P_1(G).$$

But $P_1(G)$ linearly spans $A(G)$. Thus $vv_0 = v(e)v_0$ for all $v \in A(G)$. Let $e \neq a \in G$ and v be a compact neighborhood of e such that $a \notin V^2$. Let $v \in A(G)$ satisfy $0 \leq v \leq 1$, $v(x) = 1$ if $x \in V$, $v(x) = 0$ if $x \notin V^2$ [6, p. 208]. For this v one has $vv_0 = v_0$. Thus $v_0(x) = 0$ if $x \notin V^2$, and since V was arbitrary $v_0 = v_0(e)1_e \in A(G)$. But $v_0 \neq 0$ since $I(v_0) = \psi_0(I) = 1$. Thus $1_e \in A(G)$ is a continuous function, i.e. G is discrete.

Remarks. Let $K \subset G$ be a normal compact subgroup of G, let $G_0 = G/K$ and σ: $G \to G_0$ the canonical homomorphism. Let h: $A(G_0) \to A(G)$, $(hf)(x) = f(\sigma x)$ for $x \in G$ and $H = h^*$: $VN(G) \to VN(G_0)$. By Eymard [6, p. 217], h is an isometric isomorphism of $A(G_0)$ onto the subalgebra A_K of $A(G)$ of all $u \in A(G)$ which are constant on the cosets of G mod K. Moreover $H[VN(G)] = VN(G_0)$ and H is an ultraweakly continuous homomorphism.

Proposition 8. (a) $H[A_K \cdot VN(G)] = A(G_0) \cdot VN(G_0) = \{u \cdot \phi; u \in A(G_0), \phi \in VN(G_0)\}$.

(b) $H[W(\hat{G})] \subset W(\hat{G}_0)$.

Proof. (a) Any $k \in A_K$ can be written as $k = h(u)$ for some $u \in A(G_0)$. If $u, v \in A(G_0)$ and $\phi \in VN(G)$ then $H[(hu) \cdot \phi](v) = \phi[(hu)(hv)] = (H\phi)(uv) = [u \cdot (H\phi)](v)$, i.e.

$$(*) \qquad H((hu) \cdot \phi) = u \cdot (H\phi) \in A(G_0) \cdot VN(G_0).$$

Conversely, if $\psi \in VN(G_0)$ then $\psi = H\phi$ for some $\phi \in VN(G)$. If $v, u \in A(G_0)$ then

$$[u \cdot \psi](v) = [u \cdot H\phi](v) = (H\phi)(uv) = \phi(hu)(hv) = [H((hu) \cdot \phi)](v)$$

so $u \cdot \psi \in H[A_K \cdot VN(G)]$ which completes (a).

(b) Let $\phi \in W(\hat{G})$, $B_K = \{u \in A_K(G); \|u\| \leq 1\}$, $B = \{u \in A(G), \|u\| \leq 1\}$, $B_0 = \{v \in A(G_0); \|v\| \leq 1\}$. Then, since h: $A(G_0) \to A(G)$ is an isometry $h(B_0) = B_K$. Now

$$B_0 \cdot H(\phi) = H[(hB_0) \cdot \phi] = H[B_K \cdot \phi].$$

Hence, in order to prove that $B_0 \cdot H(\phi)$ is relatively weakly compact it is enough (since H is norm, hence w-w continuous) to prove that $B_K \cdot \phi$ is relatively weakly compact in $VN(G)$. But more than this statement is true. Since $\phi \in W(\hat{G})$ even $B \cdot \phi$ relatively weakly compact, by the definition of $W(\hat{G})$, and $B_K \subset B$.

Proposition 9. *Let G be a compactly generated group* [8, *p.* 35]. *If for some norm separable subspace $X \subset VN(G)$ and some $u \in P_1(G)$, $UCB(\hat{G}) \subset$ norm* $\mathrm{cl}[W(G) + X + u^{\perp}]$, *then G is discrete.*

Proof. u is continuous and tends to 0 at ∞ [6, p. 210], hence is uniformly continuous. Let V_n be neighborhoods of e with compact closure, such that $|u(xy) - u(x)| < 1/n$ for all $y \in V_n$, $x \in G$ and in addition $\lambda(V_n) \to 0$ where λ is left Haar measure. Such V_n exist, if G is not discrete. Let $K \subset \bigcap_1^\infty V_n$ be a compact normal subgroup of G such that $G_0 = G/K$ is separable metric [8, p. 75]. Clearly $u \in A(G)$ is constant on cosets of G mod K, thus $u \in A_K$ and hence there is some $u_0 \in A(G_0)$ such that $hu_0 = u$ (in the notation of Proposition 8). Let $\{\phi_k\}$ be norm dense in X and let X_0 be the linear span of $\{H\phi_k; k \geq 1\}$ in $VN(G_0)$. We claim that

$$(*) \qquad UCB(\hat{G}_0) \subset \text{norm } \mathrm{cl}[W(\hat{G}_0) + X_0 + u_0^{\perp}].$$

In fact let $\phi_0 \in A(G_0) \cdot VN(G_0) = H[A_K \cdot VN(G)]$ (Proposition 8). Let $\phi \in A_K \cdot VN(G)$ be such that $H\phi = \phi_0$. Then there exist sequences $w_k \in W(\hat{G})$, ϕ_{n_k} and $\eta_k \in u^{\perp}$ such that $w_k + \phi_{n_k} + \eta_k \to \phi$. Thus $Hw_k + H\phi_{n_k} + H\eta_k \to \phi_0$. Now $Hw_k \in W(\hat{G}_0)$ (Proposition 8) and $H\phi_{n_k} \in X_0$. We show that $H\eta_k \in u_0^{\perp}$; then $(*)$ will follow. In fact let $\eta \in u^{\perp}$ and $v_0 \in P_1(G_0)$. Then $(H\eta)(v_0 u_0) = \eta(hv_0 hu_0) = \eta((hv_0)u) = 0$ since $hv_0 \in A(G)$ and $\eta \in u^{\perp}$. Apply now Theorem 7 to the separable metric group G_0. Then $G_0 = G/K$ is discrete; thus K is open in G. But $\lambda(K) = 0$ since $K \subset \bigcap V_n$, which cannot be. Thus G is discrete.

Remarks. Let G_1 be an open subgroup of G. If f is any function on G_1 let $\mathring{f}$ be the function of G equal to f on G_1 and zero otherwise. Define t: $A(G_1) \to A(G)$ by $tu = \mathring{u}$ and $T = t^*$: $VN(G) \to VN(G_1)$. Then by Eymard [6, p. 215] t is an isometry and T is *onto* (and $\|T\| \leq 1$).

Proposition 10. $T[UCB(\hat{G})] = UCB(\hat{G}_1)$.

Proof. Let $u \in A(G)$, $\psi \in VN(G)$. Define $ru = u|_{G_1} = u_1$. Then r maps $A(G)$ onto $A(G_1)$ [6, p. 215, (3.21), 2°] and $\|r\| \leq 1$ [6, p. 199, (2.20), 1°].

If $v_1 \in A(G_1)$ then $T(u \cdot \psi)v_1 = \psi(u(tv_1)) = \psi(t(u_1 v_1)) = (u_1 \cdot T\psi)(v_1)$, i.e.

$$(*) \qquad T(u \cdot \psi) = u_1 \cdot (T\psi) \in A(G_1) \cdot VN(G_1).$$

Thus $T(A(G) \cdot VN(G)) = A(G_1) \cdot VN(G_1)$ since T is also onto. Since T is continuous $T[UCB(\hat{G})] \subset UCB(\hat{G}_1)$.

Let now $r^* = R$. It is readily shown that for all $v_1 \in A(G_1)$, $\phi_1 \in VN(G_1)$, $R(v_1 \phi_1) = (tv_1) \cdot R\phi_1$. In particular $R[UCB(\hat{G}_1)] \subset UCB(\hat{G})$ since R is continuous.

Now rt: $A(G_1) \to A(G_1)$ is the identity; hence TR: $VN(G_1) \to VN(G_1)$ is the identity. Consequently $UCB(\hat{G}_1) = T[R(UCB(\hat{G}_1))] \subset T[UCB(\hat{G})]$ which finishes this proof.

With the above notations we also have the

Proposition 11. *$T[W(\hat{G})] \subset W(\hat{G}_1)$ and $T[AP(\hat{G})] \subset AP(\hat{G}_1)$.*

Proof. Let $B_1 = \{u \in A(G_1); \|u\| \leq 1\}$, $B = \{v \in A(G); \|v\| \leq 1\}$. If $u \in B_1$, $tu = \mathring{u} \in B$ and if $\psi \in W(\hat{G})$ then, by (*) of Proposition 10,

$$u \cdot T\psi = T(\mathring{u} \cdot \psi) \in T(B \cdot \psi).$$

Since T is w-w continuous [2, p. 422], $T(B \cdot \psi)$ is w relatively compact (since so is $B \cdot \psi$) and since $B_1 \cdot T\psi \subset T(B \cdot \psi)$ it follows that $B_1 \cdot T\psi$ is w relatively compact, i.e. $T\psi \in W(\hat{G}_1)$. The same argument works to show the rest.

Theorem 12. *Let G be any locally compact group. If for some norm separable subspace $X \subset VN(G)$ and some $u \in P_1(G)$*

$$UCB(\hat{G}) \subset \textit{norm } \mathrm{cl}[W(\hat{G}) + X + u^{\perp}]$$

then G is discrete.

Proof. Let U be a neighborhood of e with compact closure and $G_1 = \bigcup_{n=-\infty}^{\infty} U^n$. Let ϕ_n be a dense subset of X and $\phi_n^1 = T\phi_n \in VN(G_1)$. Let X_1 be the linear span of $\{\phi_n^1\}$ and let $u_1 = u|_{G_1} \in A(G_1)$ [6, p. 215]. It is clear that $u_1(e) = 1$ and u_1 is positive definite; thus $u_1 \in P_1(G_1)$. We clam that

$$(*) \qquad UCB(\hat{G}_1) \subset \mathrm{norm\ cl}[W(\hat{G}_1) + X_1 + u_1^{\perp}].$$

Let $\phi^1 \in UCB(\hat{G}_1)$ and $\phi \in UCB(\hat{G})$ be such that $T\phi = \phi^1$. Let $w_k \in W(\hat{G})$, ϕ_{n_k}, $\eta_k \in u^{\perp}$ be sequences such that $w_k + \phi_{n_k} + \eta_k \to \phi$. Then $Tw_k + \phi_{n_k}^1 + T\eta_k \to \phi^1$ and $Tw_k \in W(\hat{G}_1)$ by Proposition 11. It is enough hence to show that $T\eta_k \in u_1^{\perp}$.

Let $v_1 \in P(G_1)$. Then

$$(T\eta_k)(v_1 u_1) = \eta_k(\mathring{v}_1 \mathring{u}_1) = \eta_k(\mathring{v}_1 u) = 0$$

since $\eta_k \in u^{\perp}$ and $u_1 = u|_{G_1}$ and $\mathring{v}_1^0 = tv_1 \in A(G)$.

By Proposition 9, G_1 is discrete and open in G. Thus G is discrete.

Definition. If $u \in A(G)$ let

$$W(\hat{G}, u) = \mathrm{norm\ cl}[W(\hat{G}) + UCB(\hat{G}) \cap u^{\perp}].$$

Corollary 13. *Let G be an amenable locally compact group and $u \in P_1(G)$. If G is not discrete then $UCB(\hat{G})/W(\hat{G}, u)$ (a fortiori $UCB(\hat{G})/W(\hat{G})$) is not norm separable.*

Proof. $W(\hat{G}) \subset UCB(\hat{G})$ for amenable G, by Proposition 1. If $X \subset UCB(\hat{G})$ is a norm separable subspace then $UCB(\hat{G})$ is not included in norm $\text{cl}[W(\hat{G},u) + X]$, by Theorem 12. Thus the quotient Banach space $UCB(\hat{G})/W(\hat{G},u)$ is not norm separable (see, for example, [7, remark on p. 62] or prove directly).

Theorem 14. *Let G be any amenable locally compact group and $\alpha \in L_1(G)$, $\alpha \geq 0$, with $\|\alpha\| = 1$. Let $\alpha^\perp = \{f \in L^\infty(G); (f, L_1(G) * \alpha) = 0\}$. If $UCB_r(G) \subset$ norm $\text{cl}[W(G) + \alpha^\perp + X]$ for some norm separable $X \subset L^\infty(G)$ then G is compact.*

Proof. Our assumption implies that any real valued $f \in UCB_r(G)$ is a uniform limit of functions $w + g + x$ where w, g and x are real parts of functions in $W(G)$, $\alpha^\perp$, X. We can and hence shall assume that $L^\infty(G)$, $UCB_r(G)$, $W(G)$ denote the respective real valued function spaces while α, X denote the space of real parts of functions in the above $\alpha^\perp$ and X, respectively.

Assume at first that G is σ-compact and let $\{f_n\}$ be norm dense in X. Let $Q = \{\beta \in L_1(G); \beta \geq 0, \|\beta\| = 1\}$. Let $\psi_0 \in [w^* \text{cl}\{Q * \alpha\}] \cap \text{TLIM}(G)$ (where we identify $Q * \alpha$ with its image in $L^\infty(G)^*$). That such ψ_0 exists is an easy consequence of J. C. S. Wong [Proc. Amer. Math. Soc. **27** (1971), 572–578, Theorem 3.1] (or is proved directly using the fixed point property [12, p. 55] and an easy extension argument). Denote $\psi_0(f_n) = \alpha_n$. We claim that

$$(*) \qquad \{\psi_0\} = w^* \text{cl}\{Q * \alpha\} \cap \{\psi \in \text{TLIM}(G); \psi f_n = \alpha_n\}.$$

Let ψ belong to the right side. If $f \in \alpha^\perp$ then $(f, L_1(G) * \alpha) = 0$; thus $(w^* \text{cl}\{Q * \alpha\}, f) = 0$. Hence $\psi(\alpha^\perp) = 0 = \psi_0(\alpha^\perp)$. Since $\psi = \psi_0$ on $W(G)$ (due to uniqueness) and $\psi = \psi_0$ on X, it follows that $\psi = \psi_0$ on $UCB_r(G)$ and hence on $L^\infty(G)$ since both ψ and ψ_0 are TLIM's (as in [7, p. 63]). Thus $(*)$ holds. Apply now Theorm 5 of [7, p. 53]. Then G is compact.

Assume now that G is not σ-compact. Let H be a noncompact, σ-compact open subgroup of G containing the support of α. Then H is amenable (see [12]). We can and shall assume that the Haar measure on G is an extension of that on H. If $f \in \alpha^\perp$ then the restriction $f \mid H$ of f to H, satisfies $f \mid H \in \{g \in L^\infty(H); (g, L_1(H) * \alpha) = 0\} = (\alpha \mid H)^\perp \subset L^\infty(H)$ (where α is now considered in $L_1(H)$, $\perp$ is taken in $L^\infty(H)$ and $*$ is convolution in $L_1(H)$). We clam that

$$(**) \qquad UCB_r(H) \subset \text{norm cl}\{W(H) + X \mid H + (\alpha \mid H)\} \subset L^\infty(H).$$

In fact if $f_0 \in UCB_r(H)$, $\varepsilon > 0$ let $f \in UCB_r(G)$ be an extension of f_0, $w \in W(G)$, $h \in X$, $g \in \alpha^\perp \subset L^\infty(G)$ be such that $\|f - (w + h + g)\|_\infty < \varepsilon$. Then $\|f_0 - (w \mid H + h \mid H + g \mid H)\|_\infty < \varepsilon$ in $L^\infty(H)$. Now $w \mid H \in W(H)$ as in [7, p. 64], $h \mid H \in X \mid H$ and $g \mid H \in (\alpha \mid H)^\perp \subset L^\infty(H)$, so $(**)$ holds. By the first part H is compact, which cannot be.

Remarks. 1. The present theorem improves our theorem on p. 62 of [7]. While the result in [7] can also be obtained using Ching Chou's results in Trans. Amer. Math. Soc. **151** (1970), 443–456, the above Theorem 14 cannot.

2. For abelian G, Theorems 12 and 14 are equivalent. This is not the case for nonabelian G.

3. The reader will note that $UCB_r(G) = L_1(G) * L^\infty(G)$, as is well known, which is in analogy to our $UCB(\hat{G})$. We do not know whether the amenability of G can be dropped in Theorem 14.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BRITISH COLUMBIA, CANADA