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MINIMAL SEQUENCES IN SEMIGROUPS

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ABSTRACT. In this paper we generalize a result of Tamura on $\mathcal{S}$-indecomposable semigroups. Based on this, the concept of a minimal sequence between two points, and from a point to another, is introduced. The relationship between two minimal sequences between the same points is studied. The rank of a semigroup S is defined to be the supremum of the lengths of the minimal sequences between points in S. The semirank of a semigroup S is defined to be the supremum of the lengths of the minimal sequences from a point to another in S. Rank and semirank are further studied.

Introduction. Semilattice decompositions of semigroups were first defined and studied by Clifford [1]. Since then several people have worked on this topic, notably Tamura [5]–[9]. The author's work on the subject can be found in [3], [4]. In this paper, we start by generalizing a result of Tamura [8] (or [9]) on $\mathcal{S}$-indecomposable semigroups. Based on this, the concept of a minimal sequence between two points, and from a point to another, is introduced. The relationship between two minimal sequences between the same points is studied. The rank of a semigroup is defined to be the supremum of the lengths of the minimal sequences between points in the semigroup. The semirank of a semigroup is defined to be the supremum of the lengths of the minimal sequences from a point to another in the semigroup. Rank and semirank are further studied. To understand this paper, the reader need only be aware of the first few chapters of Clifford and Preston [2] and Tamura's decomposition theorem. (See any of [5], [6], [8] or [9]. It was rediscovered by Petrich [10].)

1 **Preliminaries.** Throughout, S will denote a semigroup and Z^+ the set of positive integers. A congruence σ on S is called a semilattice congruence if S/σ is a semilattice. $S \times S$ is the universal congruence on S. S is $\mathcal{S}$-indecomposable if $S \times S$ is the only semilattice congruence on S.

Definition. Let $a, b \in S$. Then

(1) $a \mid b$ if and only if $b \in S^1 a S^1$. $\mid$ is transitive and reflexive.

(2) $\rightarrow$ is defined as $a \rightarrow b$ iff $a \mid b^i$ for some $i \in Z^+$; let $\rightarrow^0$ denote $\rightarrow$, i.e., $\rightarrow^0 = \rightarrow$.

(3) $a \rightarrow^{n+1} b$ iff there exists $x \in S$ such that $a \rightarrow^n x \rightarrow b$.

(4) $a \rightarrow^\infty b$ iff $a \rightarrow^n b$ for some $n \in Z^+$.

(5) — is defined as $a — b$ iff $a \rightarrow \quad \rightarrow a$; let $—^0$ denote —, i.e., $—^0 = —$.

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(6) $a \text{---}^{n+1} b$ iff there exists $x \in S$ such that $a \text{---}^{n} x \text{---} b$.
(7) $a \text{---}^{\infty} b$ iff $a \text{---}^{n} b$ for some $n \in Z^+$. ---^{∞} is an equivalence relation.
The following theorem and corollary are due to Tamura [8] or [9].

Theorem [Tamura]. *Let S be a semigroup. Then $\to^{\infty} \cap (\to^{\infty})^{-1}$ is the finest semilattice congruence on S and each component is $\mathcal{S}$-indecomposable.*

Corollary [Tamura]. *Let S be an $\mathcal{S}$-indecomposable semigroup. Then $\to^{\infty}$ is the universal congruence on S.*

We generalize these results to:

Theorem 1.1. *Let S be a semigroup. Then ---^{∞} is the finest semilattice congruence on S. ---^{∞} is also the equivalence relation generated by the relations $ab \equiv aba \equiv ba$, for all $a, b \in S^1$ and $ab \in S$.*

Corollary 1.2. *Let S be an $\mathcal{S}$-indecomposable semigroup. Then ---^{∞} is the universal congruence on S.*

It is easy to deduce Tamura's result from ours.
To prove Theorem 1.1, we need the following

Lemma 1.3. *Let σ be an equivalence relation on a semigroup S satisfying $xy \,\sigma\, xyx \,\sigma\, yx$ for all $x, y \in S^1$. Then for all $a, b, c, d \in S^1$ (with the convention $1 \,\sigma\, 1$),*
(1) *$abc \,\sigma\, ab^ic$ for all $i \in Z^+$,*
(2) *$abcd \,\sigma\, acbd$,*
(3) *$a \text{---}^{\infty} b$ implies $xay \,\sigma\, xby$ for all $x, y \in S^1$.*
In particular $\text{---}^{\infty} \subseteq \sigma$.

Proof. (1) $abc \,\sigma\, cab \,\sigma\, b(ca)b = (bc)(ab) \,\sigma\, (ab)(bc) = ab^2c$. $ab^ic = (ab^{i-1})bc \,\sigma\, (ab^{i-1})b^2c = ab^{i+1}c$.
(2) Using (1), for any $A, B, C \in S^1$,

$$\begin{aligned}
ABC \,\sigma\, & A(BC)(BC) \,\sigma\, (ABCBC)(ABCBC) \\
&= (AB)(CBCA)(BC)^2 \,\sigma\, (AB)(CBCA)BC \\
&\quad \sigma\, (AB)(CBCA)(CBCA)BC \\
&= A(BC)^2(ACBCABC) \,\sigma\, A(BC)(ACBCABC) \\
&= (ABCACB)(CABC) \,\sigma\, (ABCACB)(ABCACB)(CABC) \\
&= (ABCACBA)(BC)(ACB)(CABC) \\
&\quad \sigma\, (ABCACBA)(BC)(BC)(ACB)(CABC) \\
&= (ABCACBAB)(CBCA)^2BC \,\sigma\, (ABCACBAB)(CBCA)BC \\
&= (ABCACBA)(BC)^2(ABC) \,\sigma\, (ABCACBA)(BC)(ABC) \\
&= (ABC)(ACB)(ABC)^2 \,\sigma\, (ABC)(ACB)(ABC).
\end{aligned}$$

In short $ABC\,\sigma\,(ABC)(ACB)(ABC)$. Interchanging B and C, we have $ACB\ \sigma\ (ACB)(ABC)(ACB)$. But $(ABC)(ACB)(ABC)\ \sigma\ (ABC)(ACB)\ \sigma\ (ACB)(ABC)(ACB)$. So $ABC\,\sigma\,ACB$.

Thus $abcd\ \sigma\ d(abc) = (da)bc\ \sigma\ (da)cb = d(acb)\ \sigma\ (acb)d$.

(3) First suppose $a - b$. So we solve $sat = b^i$, $s'bt' = a^j$. Then using (1), (2), we have $xaby\ \sigma\ xab^iy = xasaty\ \sigma\ xsaaty\ \sigma\ xsaty = xb^iy\ \sigma\ xby$. Similarly, $xbay\ \sigma\ xay$. So by (2), $xay\ \sigma\ xby$. Now assume $a -^n b$, $n \geq 1$. So $a - a_1 - \cdots - a_n - b$. By the above $xay\ \sigma\ xa_1y$, $xa_iy\ \sigma\ xa_{i+1}y$ $(i = 1, \ldots, n-1)$, $xa_ny\ \sigma\ xby$. Thus, $xay\ \sigma\ xby$.

Thus $a -^\infty b$ implies $xay\ \sigma\ xby$ for all $x, y \in S^1$.

Proof of Theorem 1.1. Consider the following.

$$(*) \qquad xy \equiv xyx \equiv yx, \quad \text{for all } x, y \in S^1,\ xy \in S.$$

Let $a, b \in S^1$. Then $aba \mid (ab)^2$, $ab \mid aba$. So $aba - ab$. Now $ab \mid (ba)^2$, $ba \mid (ab)^2$. Thus $ab - ba$. So $ab -^\infty aba -^\infty ba$. Thus $-^\infty$ is an equivalence relation satisfying $(*)$. By Lemma 1.3, we conclude that $-^\infty$ is the smallest equivalence relation satisfying $(*)$. In the same lemma, replacing σ by $-^\infty$, we have $-^\infty$ is a semilattice congruence. Since any semilattice congruence satisfies $(*)$, we have that $-^\infty$ is the finest semilattice congruence on S.

Corollary 1.2 is now immediate. We will need the following lemmas later.

Lemma 1.4. *Let S be a semilattice of semigroups S_α $(\alpha \in \Omega)$, δ the corresponding semilattice congruence.*

(1) *Let $\alpha \in \Omega$, with $a, b \in S_\alpha$. If $a \to b$ in S, then $a \to b$ in S_α.*

(2) *Let $a \in S_\alpha$, $b \in S_\beta$, $a \to b$. Then $\alpha \geq \beta$.*

(3) *Let $a, b \in S$ with $a - b$. Then for some $\alpha \in \Omega$, $a, b \in S_\alpha$ and $a - b$ in S_α.*

Proof. (1) For some $x, y \in S^1$, $xay = b^i$. So $b^ixayb^i = b^{3i}$. Then $b^ix = xayx\ \delta\ xay = b^i\ \delta\ b$. So $b^ix \in S_\alpha$. Similarly $yb^i \in S_\alpha$. So $a \mid b^{3i}$ in S_α, whence $a \to b$ in S_α.

(2) $\alpha \to \beta$ in Ω. Since Ω is a semilattice we deduce $\alpha \mid \beta$ in Ω and then that $\alpha \geq \beta$.

(3) Let $a \in S_\alpha$, $b \in S_\beta$. By (2), $\alpha \geq \beta$, $\beta \geq \alpha$ and so $\alpha = \beta$. By (1), $a - b$ in S_α.

Lemma 1.5. *Let S be a semigroup and $a, b, c \in S$.*

(1) *Let $i \in Z^+$. Then $a \to b^i$ implies $a \to b$.*

(2) *$a \mid b \to c$ implies $a \to c$.*

(3) *Let $i, j \in Z^+$. Then $a^i - b^j$ implies $a - b$.*

Lemma 1.6. (1) *Let S be a semigroup with an ideal I and $a, b \in S$. Suppose b is not nilpotent in S/I, and $a \to b$ in S/I. Then $a \to b$ in S.*

(2) *Let S be a semigroup with zero, and suppose $a \in S$. Then $0 \to a$ if and only if a is nilpotent.*

Proof. (1) We can solve $xay = b^i$ in the semigroup S/I. Since b is not nilpotent in S/I, $b^i \in S\backslash I$. So $x, a, xa, y, xay \in S\backslash I$. Thus $xay = b^i$ in S.

(2) If $0 \to a$, then $0 \mid a^i$ for some $i \in Z^+$. Hence $a^i = 0$. Conversely if $a^i = 0$ for some $i \in Z^+$, then $0 \mid 0 = a^i$ whence $0 \to a$.

2. Minimal sequences.

Definition. Let S be a semigroup, $a, b \in S$.

(1) By a sequence *between* a and b, we mean a (possibly empty) finite sequence $\langle x_i \rangle_{i=1}^n$ in S such that $a — x_1, x_i — x_{i+1}$ $(i = 1, \ldots, n-1)$, $x_n — b$. We call n the length of $\langle x_i \rangle$. By $n = 0$, or $\langle x_i \rangle_{i=1}^n$ empty, we mean $a — b$. We say $\langle x_i \rangle$ is minimal if it is nonempty and there is no sequence of smaller length (including the empty sequence) between a and b.

(2) By a sequence *from a to b*, we mean a (possibly empty) finite sequence $\langle x_i \rangle_{i=1}^n$, such that $a \to x_1, x_i \to x_{i+1}$ $(i = 1, \ldots, n-1)$, $x_n \to b$. Again n is the length of the sequence, and by $n = 0$ (or $\langle x_i \rangle$ empty) we mean $a \to b$. $\langle x_i \rangle$ is minimal if it is nonempty and there is no sequence of smaller length (including the empty sequence) from a to b.

Lemma 2.1. *Let S be a semigroup with $a, b \in S$.*

(1) *Let $\langle x_i \rangle_{i=1}^n$, $\langle y_i \rangle_{i=1}^n$ be two sequences between a and b of the same length. If $\langle x_i \rangle$ is minimal, then so is $\langle y_i \rangle$.*

(2) *Let S be $\mathcal{S}$-indecomposable. Then either $a — b$ or there is a minimal sequence between a and b.*

(3) *Let S be $\mathcal{S}$-indecomposable. Then either $a \to b$ or there is a minimal sequence from a to b.*

Proof. (1) Obvious.

(2) and (3) are trivial using Corollary 1.2.

Lemma 2.2. *Let S be a semilattice of semigroups S_α $(\alpha \in \Omega)$.*

(1) *Let $a, b \in S$, with a sequence $\langle x_i \rangle_{i=1}^n$ between a and b. Then a, b and all the x_i's lie in some S_α. Moreover $\langle x_i \rangle$ is a sequence between a and b in S_α. The minimal sequences between a and b in S are exactly those in S_α.*

(2) *Let $\alpha \in \Omega$, with $a, b \in S_\alpha$. Let $\langle x_i \rangle_{i=1}^n$ be a sequence from a to b in S. Then all the x_i's lie in S_α and $\langle x_i \rangle$ is a sequence from a to b in S_α. The minimal sequences from a to b in S are exactly those in S_α.*

(3) *Let $a \in S_\alpha$, $b \in S_\beta$. Suppose there exists a sequence from a to b in S. Then $\alpha \geq \beta$.*

Proof. (1) That x_i's, a, b lie in some S_α and that $\langle x_i \rangle$ is a sequence between a and b within S_α follow from Lemma 1.4. So a minimal sequence between a and b in S is a sequence between a and b in S_α and obviously minimal in S_α. Let $\langle y_i \rangle$ be a minimal sequence between a and b in S_α. Let $\langle z_i \rangle$ be a sequence between a and b in S. By the above, $\langle z_i \rangle$ is a sequence between a and b in S_α. So $\langle z_i \rangle$ has length at least that of $\langle y_i \rangle$. So $\langle y_i \rangle$ is minimal in S.

(2) If $\langle x_i \rangle$ is empty, $a \to b$ in S and so in S_α, by Lemma 1.4. Otherwise let $x_i \in S_{\alpha_i}$, $i = 1, \ldots, n$. Then $a \to x_1 \to \cdots \to x_n \to b$. By Lemma 1.4, $\alpha \geq \alpha_1 \geq \cdots \geq \alpha_n \geq \alpha$. Consequently, $\alpha = \alpha_1 = \cdots = \alpha_n$. Now $\langle x_i \rangle$ is a sequence from a to b within S_α, by Lemma 1.4. The rest follows as in (1).

(3) If the sequence is empty, $a \to b$ and so by Lemma 1.4, $\alpha \geq \beta$. Otherwise, $a \to x_1 \cdots x_n \to b$, $x_i \in S_{\alpha_i}$. By Lemma 1.4, $\alpha \geq \alpha_1 \geq \cdots \geq \alpha_n \geq \beta$. So $\alpha \geq \beta$.

Definition. (1) A semigroup S is a Γ-semigroup iff for any $a, b \in S$, either $a \to b$ or $b \to a$. Clearly any semigroup S with $\mathcal{J}$-classes linearly ordered (equivalently the ideals are linearly ordered or still equivalently for any $a, b \in S$, $a \mid b$ or $b \mid a$) is a Γ-semigroup. Such an example is the full transformation semigroup. The null semigroup with more than one element is a Γ-semigroup, but its $\mathcal{J}$-classes are not linearly ordered.

(2) S is a Γ^*-semigroup iff S is a semilattice of Γ-semigroups.

Lemma 2.3. *Let S be a semigroup. Then the following are equivalent.*

(1) *S is a Γ^*-semigroup.*

(2) *S is a semilattice of Γ^*-semigroups.*

(3) *The $\mathcal{S}$-indecomposable components of S are Γ-semigroups.*

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Let S be a semilattice of Γ^*-semigroups $S_\alpha\,(\alpha \in \Omega)$. Let T be an $\mathcal{S}$-indecomposable component of S. Then $T \subseteq S_\alpha$ for some $\alpha \in \Omega$. S_α is a semilattice of Γ-semigroups U_β. So $T \subseteq U_\beta$ for some β. Let $a, b \in T$. Then $a, b \in U_\beta$. So $a \to b$ or $b \to a$ in U_β and hence in S. By Lemma 1.4, $a \to b$ or $b \to a$ in T. Consequently T is a Γ-semigroup.

(3) $\Rightarrow$ (1). Obvious.

Definition. Let $a, b \in S$. Then $a \rightsquigarrow b$ iff $a^i \to b$ for all $i \in Z^+$.

Lemma 2.4. *Let S be a semigroup with $a, b, c \in S$.*

(1) *If $a \to b \rightsquigarrow c$, then $a \to c$.*

(2) *If S is a Γ-semigroup, then either $a \rightsquigarrow b$ or $b \rightsquigarrow a$.*

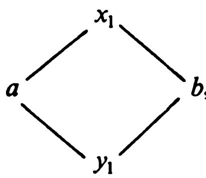
Proof. (1) $a \mid b^i \to c$ for some $i \in Z^+$. So $a \to c$.

(2) Suppose $a \not\rightsquigarrow b$. Then $a^i \not\to b$ for some $i \in Z^+$. So for any $k \in Z^+$, $a^i \not\to b^k$. Hence $b^k \to a^i$ and so $b^k \to a$. Since k is arbitrary, $b \not\rightsquigarrow a$.

Theorem 2.5. *Let S be a Γ^*-semigroup with $a, b \in S$. Let $\langle x_i \rangle_{i=1}^n$, $\langle y_i \rangle_{i=1}^n$ be two minimal sequences between a and b. Then $x_i - y_i$ for $i = 1, \ldots, n$. We can further conclude (if $n > 1$) that for $i = 1, \ldots, n-1$, either $x_i - y_{i+1}$ or $y_i - x_{i+1}$.*

Proof. S is a semilattice of Γ-semigroups $S_\alpha\,(\alpha \in \Omega)$. Using Lemma 2.2, we deduce that if the theorem is true for each S_α, it is true for S. So we can assume that S is a Γ-semigroup. We use Lemma 2.4 without further remark.

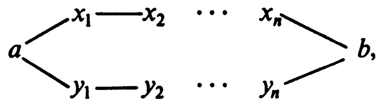
First we prove the theorem for $n = 1$. We have



$a \not\rightarrow b.$

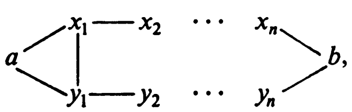
Now $a \rightsquigarrow b$ or $b \rightsquigarrow a$. By symmetry, we assume $a \rightsquigarrow b$. Since $a \not\rightarrow b$, we conclude that $b \nrightarrow a$. Now either $x_1 \rightsquigarrow a$ or $a \rightsquigarrow x_1$. If $x_1 \rightsquigarrow a$, we have (since $b \rightarrow x_1$) that $b \rightarrow a$, a contradiction. So, $a \rightsquigarrow x_1$. Since $y_1 \rightarrow a$, we have $y_1 \rightarrow x_1$. Similarly $x_1 \rightarrow y_1$. Thus $x_1 — y_1$.

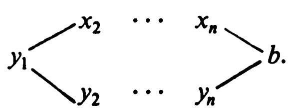
We now proceed by induction on n. We have,



$n > 1, a \not\rightarrow b.$

Now either $a \rightsquigarrow b$ or $b \rightsquigarrow a$. By symmetry, we assume $a \rightsquigarrow b$. Since $a \not\rightarrow b$, $b \nrightarrow a$. Again, either $x_n \rightsquigarrow a$ or $a \rightsquigarrow x_n$. If $x_n \rightsquigarrow a$, we obtain (since $b \rightarrow x_n$), $b \rightarrow a$, a contradiction. So $a \rightsquigarrow x_n$. Now assume $a \rightsquigarrow x_{j+1}, j \geq 1$. Then $x_{j+1} \nrightarrow a$, for otherwise, $a — x_{j+1}$ and so $\langle x_i \rangle_{i=j+1}^n$ is a sequence between a and b, contradicting the minimality of $\langle x_i \rangle_{i=1}^n$. Now either $x_j \rightsquigarrow a$ or $a \rightsquigarrow x_j$. If $x_j \rightsquigarrow a$, then since $x_{j+1} \rightarrow x_j$ we have $x_{j+1} \rightarrow a$, a contradiction. So $a \rightsquigarrow x_j$. Thus $a \rightsquigarrow x_i$ for all $i = 1, \ldots, n$. Similarly $a \rightsquigarrow y_i$ for all $i = 1, \ldots, n$. In particular $a \rightsquigarrow x_1$, $a \rightsquigarrow y_1$. Since $x_1 \rightarrow a$, we have $x_1 \rightarrow y_1$. Similarly, $y_1 \rightarrow x_1$. Thus $x_1 — y_1$. We further have $y_1 \rightarrow a \rightsquigarrow x_2$, and so $y_1 \rightarrow x_2$. Similarly, $x_1 \rightarrow y_2$. Now $x_1 \rightsquigarrow y_1$ or $y_1 \rightsquigarrow x_1$. We assume $x_1 \rightsquigarrow y_1$, the other case being taken care of similarly. So $x_2 \rightarrow x_1 \rightsquigarrow y_1$ and hence $x_2 \rightarrow y_1$. Since we already established $y_1 \rightarrow x_2$, we have $x_2 — y_1$. Thus we obtain:





In the figure on the right, the sequence $\langle y_i \rangle_{i=2}^n$ is a minimal sequence between y_1 and b. This is because a sequence between y_1 and b of length less than $n - 1$ would produce a sequence between a and b of length less than n, contradicting the minimality of $\langle y_i \rangle_{i=1}^n$. By Lemma 2.1, $\langle x_i \rangle_{i=2}^n$ is a minimal sequence between y_1 and b. By our induction hypothesis, we have $x_i — y_i$, for $i \geq 2$. Also if $n > 2$, $x_i — y_{i+1}$, or $y_i — x_{i+1}$, $i = 2, \ldots, n - 1$. Since we already know $x_1 — y_1$, $y_1 — x_2$, the theorem is proved.

We will see later that Theorem 2.5 is not true for arbitrary semigroups, even for $n = 1$.

Problem 2.6. In Theorem 2.5 can we conclude that $x_i — y_{i+1}$ *and* $y_i — x_{i+1}$, $i = 1, \ldots, n - 1$?

Problem 2.7. Call a sequence $\langle x_i \rangle$ between a and b indecomposable if $\langle x_i \rangle$ is nonempty and no proper subsequence of $\langle x_i \rangle$ is a sequence between a and b. Clearly a minimal sequence is indecomposable but not conversely. An indecomposable sequence of length 1 is minimal. Is Theorem 2.5 true for indecomposable sequences of the same length? For $n \leq 2$, the proof goes through.

Lemma 2.8. *Let S be a Γ-semigroup and $a, b \in S$. Let $\langle x_i \rangle_{i=1}^n$ be a minimal sequence from a to b. Then $\langle x_i \rangle_{i=1}^n$ is a minimal sequence between a and b.*

Proof. We have $a \to x_1 \cdots x_n \to b$, $n \geq 1$. Set $x_0 = a$, $x_{n+1} = b$. For $n \geq i \geq 1$, $x_i \rightsquigarrow x_{i+1}$ implies $x_{i-1} \to x_{i+1}$ (since $x_{i-1} \to x_i$) contradicting the minimality of $\langle x_i \rangle_{i=1}^n$. So $x_{i+1} \rightsquigarrow x_i$. Thus $x_i — x_{i+1}$, $n \geq i \geq 1$. Now $a \not\rightsquigarrow x_2$ by minimality of $\langle x_i \rangle_{i=1}^n$. So $x_2 \rightsquigarrow a$. Since $x_1 \to x_2$, we have $x_1 \to a$. So $a — x_1$. Consequently, $\langle x_i \rangle_{i=1}^n$ is a sequence *between* a and b. Since any sequence between a and b is a sequence from a to b, we have that $\langle x_i \rangle$ is a minimal sequence between a and b.

Corollary 2.9. *Let S be a Γ-semigroup with $a, b, c \in S$. Let $\langle x_i \rangle_{i=1}^n$ and $\langle y_i \rangle_{i=1}^n$ be minimal sequences of the same length from b to a and c to a respectively. Then $x_i — y_i$ for $i = 1, \ldots, n$. For $n > 1$, we can further conclude that for each $i = 1, \ldots, n-1$ either $x_i — y_{i+1}$ or $y_i — x_{i+1}$.*

Proof. Now either $x_1 \rightsquigarrow y_1$ or $y_1 \rightsquigarrow x_1$. By symmetry, we assume $x_1 \rightsquigarrow y_1$. Since $b \to x_1$, we have $b \to y_1$. Thus $\langle y_i \rangle_{i=1}^n$ is a sequence from b to a. Since $\langle x_i \rangle_{i=1}^n$ is minimal, we obtain that $\langle y_i \rangle_{i=1}^n$ is also a minimal sequence from b to a. By Lemma 2.8, $\langle x_i \rangle$ and $\langle y_i \rangle$ are minimal sequences *between* b and a. By Theorem 2.5, we are done.

Problem 2.10. Is Corollary 2.9 true for Γ^*-semigroups?

3. Rank and semirank.

Definition. Let S be a semigroup.

(1) The rank $\rho_1(S)$ of a semigroup S is zero if there is no minimal sequence between any two points. Otherwise $\rho_1(S)$ is the supremum of the lengths of the minimal sequences between points in S.

(2) The semirank $\rho_2(S)$ of S is zero if there is no minimal sequence from a point to another in S. Otherwise $\rho_2(S)$ is the supremum of the lengths of the minimal sequences from one point to another in S.

The following is an easy consequence of Lemma 2.1.

Lemma 3.1. *Let S be an $\mathcal{S}$-indecomposable semigroup, and $a, b \in S$. Then there exists a sequence between a and b of length at most $\rho_1(S)$.*(2) *Also there exists a sequence from a to b of length at most $\rho_2(S)$.*

Lemma 3.2. *Let S be the semilattice of $\mathcal{S}$-indecomposable semigroups S_α ($\alpha \in \Omega$). Then*

(1) $\rho_1(S) = \sup_{\alpha \in \Omega} \rho_1(S_\alpha)$,

(2) $\rho_2(S) = \sup_{\alpha \in \Omega} \rho_2(S_\alpha)$.

(2) If $\rho_1(S) = \infty$, the length is less than $\rho_1(S)$. Similarly for $\rho_2(S)$.

Proof. (1) Immediate from Lemma 2.2.

(2) By Lemma 2.2, we have $\rho_2(S_\alpha) \leq \rho_2(S)$ for all $\alpha \in \Omega$. So $\sup_{\alpha\in\Omega}\rho_2(S_\alpha) \leq \rho_2(S)$. Now let $a, b \in S$, $\langle x_i\rangle_{i=1}^n$ a minimal sequence from a to b. We have to show that $n \leq \sup_{\alpha\in\Omega}\rho_2(S_\alpha)$. Let $a \in S_\gamma$, $b \in S_\beta$. By Lemma 2.2, $\gamma \geq \beta$. So $ab \in S_\beta$. By Lemma 3.1, there exists a sequence $\langle y_i\rangle_{i=1}^k$ from ab to b, $k \leq \rho_2(S_\beta)$. Since $a \mid ab$, by Lemma 1.5, $\langle y_i\rangle_{i=1}^k$ is a sequence from a to b. By minimality of $\langle x_i\rangle_{i=1}^n$ we have $n \leq k \leq \rho_2(S_\beta) \leq \sup_{\alpha\in\Omega}\rho_2(S_\alpha)$. Thus $\rho_2(S) \leq \sup_{\alpha\in\Omega}\rho_2(S_\alpha)$. Combined with the previous result, $\rho_2(S) = \sup_{\alpha\in\Omega}\rho_2(S_\alpha)$.

A semigroup S is archimedean if and only if for all $a, b \in S$, $a \to b$ (see [3], [7] and [8]).

Theorem 3.3. *Let S be a semigroup.*

(1) $\rho_1(S)$ *is the smallest* $n \leq \infty$ *for which* $-^n$ *is transitive (i.e.,* $-^n = -^\infty$ *or equivalently* $-^n = -^{n+1}$).

(2) $\rho_2(S)$ *is the smallest* $n \leq \infty$ *for which* $\to^n$ *is transitive (i.e.,* $\to^n = \to^\infty$ *or equivalently* $\to^n = \to^{n+1}$).

(3) *Let S be a semilattice of semigroups* $S_\alpha (\alpha \in \Omega)$. *Then* $\rho_i(S) = \sup_{\alpha\in\Omega}\rho_i(S_\alpha)$, $i = 1, 2$.

(4) $\rho_2(S) \leq \rho_1(S)$.

(5) $\rho_1(S) = 0$ *if and only if* $\rho_2(S) = 0$ *if and only if S is a semilattice of archimedean semigroups.*

(6) *If S is a* Γ^**-semigroup,* $\rho_1(S) = \rho_2(S)$.

(7) *A finite semigroup has finite semirank and rank.*

Proof. (1) and (2) are easy consequences of the definition.

(3) For each $\alpha \in \Omega$, S_α is the semilattice of the $\mathcal{S}$-indecomposable components of S, contained in S_α. So the $\mathcal{S}$-indecomposable components of S are just those of all of the S_α's. Now the result follows from Lemma 3.2.

(4) By (3), we can assume S is $\mathcal{S}$-indecomposable. Let $a, b \in S$ and $\langle x_i\rangle_{i=1}^n$ a minimal sequence from a to b. By Lemma 3.1, there exists a sequence $\langle y_i\rangle_{i=1}^m$ between a and b, such that $m \leq \rho_1(S)$. But $\langle y_i\rangle$ can be considered a sequence from a to b. By the minimality of $\langle x_i\rangle$, $n \leq m \leq \rho_1(S)$. So $\rho_2(S) \leq \rho_1(S)$.

(5) Again we can assume S is $\mathcal{S}$-indecomposable. Clearly if S is archimedean, it has no minimal sequences and so $\rho_1(S) = \rho_2(S) = 0$. If for $i = 1$ or 2, $\rho_i(S) = 0$, then by Lemma 3.1, S is archimedean.

(6) By Lemma 2.3 and by (3) and (4) above, we can assume that S is an $\mathcal{S}$-indecomposable Γ-semigroup and that $\rho_2(S) \leq \rho_1(S)$. We have to show $\rho_1(S) \leq \rho_2(S)$. Let $a, b \in S$ and let $\langle x_i\rangle_{i=1}^n$ be a minimal sequence between a and b. Then $a \not- b$. So either $a \not\to b$ or $b \not\to a$. By symmetry we assume $a \not\to b$. By Lemma 2.1, there exists a minimal sequence $\langle y_i\rangle_{i=1}^m$ from a to b. So $m \leq \rho_2(S)$. By Lemma 2.8, $\langle y_i\rangle_{i=1}^m$ is a minimal sequence *between* a and b. Thus $n = m \leq \rho_2(S)$. So $\rho_1(S) \leq \rho_2(S)$, whence $\rho_1(S) = \rho_2(S)$.

(7) Obvious.

Lemma 3.4. *Let S be an $\mathcal{S}$-indecomposable semigroup and T a homomorphic image of S. Then $\rho_i(T) \leq \rho_i(S)$, $i = 1, 2$.*

Proof. Let $a, b \in S$, $\varphi: S \to T$ an onto homomorphism. Let there be a minimal sequence $\langle y_i \rangle_{i=1}^n$ in T, between $\varphi(a)$ and $\varphi(b)$. So $\varphi(a) \not\to \varphi(b)$ and so $a \not\to b$. By Lemma 2.1, there exists a minimal sequence $\langle x_i \rangle_{i=1}^m$ between a and b. Thus $m \leq \rho_1(S)$. $\langle \varphi(x_i) \rangle_{i=1}^m$ is a sequence between $\varphi(a)$ and $\varphi(b)$. By minimality of $\langle y_i \rangle_{i=1}^n$, we have $n \leq m \leq \rho_1(S)$. Thus $\rho_1(T) \leq \rho_1(S)$. A similar argument shows that $\rho_2(T) \leq \rho_2(S)$.

Problem 3.5. Is Lemma 3.4 true for arbitrary semigroups?

Theorem 3.6. *Let S be an $\mathcal{S}$-indecomposable semigroup with an ideal I. Then*

$$\rho_2(S/I) \leq \rho_2(S) \leq \rho_2(I) + \rho_2(S/I).$$

Proof. That $\rho_2(S/I) \leq \rho_2(S)$ follows from Lemma 3.4. By [5] (or [10]), both I and S/I are $\mathcal{S}$-indecomposable. Let $a, b \in S$. We have to show the existence of a sequence from a to b of length at most $\rho_2(I) + \rho_2(S/I)$. (3)

Case 1. $a \in S$, $b \in I$. Then $ab \in I$. So by Lemma 3.1, there exists a sequence $\langle y_i \rangle_{i=1}^m$ from ab to b in I such that $m \leq \rho_2(I)$. By Lemma 1.5, $\langle y_i \rangle_{i=1}^m$ is a sequence from a to b.

Case 2. $a \in I$, $b \in S \backslash I$, b is nilpotent in S/I. Then $b^k \in I$ for some $k \in Z^+$. Then by Lemma 3.1, there exists a sequence $\langle y_i \rangle_{i=1}^m$ from a to b^k, $m \leq \rho_2(I)$. By Lemma 1.5, $\langle y_i \rangle_{i=1}^m$ is a sequence from a to b.

Case 3. $a \in I$, $b \in S \backslash I$, b is not nilpotent in S/I. So in S/I, $0 \not\leftrightarrow b$. By Lemma 2.1, there exists a minimal sequence $\langle y_i \rangle_{i=1}^n$ from 0 to b. So $n \leq \rho_2(S/I)$. So in S/I, $0 \to y_1 \to \cdots \to y_n \to b$. If y_j is nilpotent in S/I, for some $j > 1$, we would have, by Lemma 1.6, $0 \to y_j \to \cdots \to y_n \to b$ contradicting the minimality of $\langle y_i \rangle_{i=1}^n$. So y_j is not nilpotent for $j > 1$. By Lemma 1.6, $y_1 \to \cdots \to y_n \to b$ in S. Now since $0 \to y_1$, y_1 is nilpotent in S/I. So by Case 2, there exists a sequence $\langle x_i \rangle_{i=1}^m$ from a to y_1 in S such that $m \leq \rho_2(I)$. So in S,

$$a \to x_1 \to \cdots \to x_m \to y_1 \to \cdots \to y_n \to b, \qquad m + n \leq \rho_2(I) + \rho_2(S/I).$$

Case 4. $a \in S \backslash I$, $b \in S \backslash I$. If b is nilpotent in S/I, then $b^k \in I$ for some $k \in Z^+$. By Case 1, there exists in S a sequence $\langle y_i \rangle$ from a to b^k of length $\leq \rho_2(I)$. By Lemma 1.5, $\langle y_i \rangle$ is a sequence from a to b in S.

Thus we may assume b is not nilpotent in S/I. So if $a \to b$ in S/I, then by Lemma 1.6, $a \to b$ in S and we would be done. So we assume $a \not\leftrightarrow b$ in S/I. By Lemma 2.1, there exists a minimal sequence $\langle x_i \rangle_{i=1}^n$ in S/I from a to b. So $n \leq \rho_2(S/I)$. If none of the x_i's is nilpotent, then $\langle x_i \rangle_{i=1}^n$ is a sequence from a to b in S, by Lemma 1.6. So let some x_j be nilpotent in S/I. Then $a \to x_j \to$

(3) The theorem is trivial if $\rho_2(I) = \infty$ or $\rho_2(S/I) = \infty$. So we assume $\rho_2(I) < \infty$ and $\rho_2(S/I) < \infty$.

$\cdots \to x_n \to b$ in S/I. By minimality of $\langle x_i \rangle_{i=1}^n$, $j = 1$. Thus x_1 is nilpotent and x_j is not nilpotent for $j > 1$. Thus by Lemma 1.6, $x_1 \to \cdots \to x_n \to b$ in S. From what we proved above there exists a sequence $\langle y_i \rangle_{i=1}^m$ in S from a to x_1 such that $m \le \rho_2(I)$. Thus in S,

$$a \to y_1 \to \cdots \to y_m \to x_1 \to \cdots \to x_n \to b, \qquad m + n \le \rho_2(I) + \rho_2(S/I).$$

Problem 3.7. Is Theorem 3.6 true for arbitrary semigroups? In Theorem 3.6, can we replace ρ_2 by ρ_1?

Consider the following condition on semigroups:

(A) $a \in S$ implies there exists a fixed $n = n(a) \in Z^+$ such that for all $i \in Z^+$, $a^{in} \mid a^n$.

Clearly any semigroup with a power of each element lying in a subgroup (in particular a periodic semigroup) satisfies (A).

Lemma 3.8. *Let S be a semigroup satisfying* (A). *Let $a, b \in S$, $k \in Z^+$, such that $b \to a^k$. Then $b \mid a^{n(a)}$.*

Proof. For some $i \in Z^+$, $b \mid a^i \mid a^{in(a)} \mid a^{n(a)}$. So $b \mid a^{n(a)}$.

Theorem 3.9. *Let S be a semigroup satisfying* (A). *Suppose $\rho_2(S) \le 1$. Then $\rho_1(S) \le 4$.*

Proof. Let δ be the finest semilattice congruence on S and S_α ($\alpha \in S/\delta$) the $\mathcal{S}$-indecomposable components of S. Let $a \in S_\alpha$. By (A), there exists $n = n(a) \in Z^+$ such that for all $i \in Z^+$, $a^{(i+2)n} \mid a^n$ in S. So there exists $x, y \in S^1$ such that $xa^n a^{in} a^n y = a^n$. But then $xa^n = x(xa^{(i+2)n}y) \,\delta\, xa^{(i+2)n}y = a^n \,\delta\, a$. So $xa^n \in S_\alpha$. Similarly $a^n y \in S_\alpha$. Thus $a^{in} \mid a^n$ in S_α. Consequently each S_α satisfies (A). By Theorem 3.3, it suffices to prove the theorem for each S_α. Consequently we may and do assume that S is an $\mathcal{S}$-indecomposable semigroup. Let $a, b \in S$. We have to show the existence of a sequence between a and b of length at most 4. We use Lemma 3.1 and Lemma 3.8 without further remark. Let $n_1 = n(a)$, $n_2 = n(b)$. There exists $c \in S$ such that $a^{n_1} \to c \to b^{n_2}$. Set $n_3 = n(c)$. So $a^{n_1} \mid c^{n_3}$. There exists $d_1 \in S$ such that $c^{n_3} \to d_1 \to a^{n_1}$. Set $m_1 = n(d_1)$. So $d_1 \mid a^{n_1} \mid c^{n_3} \mid d_1^{m_1}$. So $d_1 \mid c^{n_3}$ and $a^{n_1} \mid d_1^{m_1}$. Thus $c^{n_3} - d_1 - a^{n_1}$. By Lemma 1.5, $c - d_1 - a$. Now since $c \to b^{n_2}$, $c \mid b^{n_2}$. There exists $d_2 \in S$ such that $b^{n_2} \to d_2 \to c$. Let $m_2 = n(d_2)$. Then $c \mid b^{n_2} \mid d_2^{m_2}$. Thus $c \mid d_2^{m_2}$ and hence, $b^{n_2} \to d_2 - c$. There exists $d_3 \in S$ such that $d_2^{m_2} \to d_3 \to b^{n_2}$. Let $m_3 = n(d_3)$. So $d_3 \mid b^{n_2} \mid d_2^{m_2} \mid d_3^{m_3}$. Hence $d_3 \mid d_2^{m_2}$ and $b^{n_2} \mid d_3^{m_3}$. Thus $d_2^{m_2} - d_3 - b^{n_2}$. By Lemma 1.5, $d_2 - d_3 - b$. Hence $c - d_2 - d_3 - b$. Consequently $a - d_1 - c - d_2 - d_3 - b$, and the theorem is proved.

Problem 3.10. Can the bound on ρ_1 be improved in Theorem 3.9? Does a semigroup of finite semirank necessarily have finite rank?

4. **Examples.** It can be deduced from Corollary 1.2 or from Tamura's corollary that a 0-simple semigroup is $\mathcal{S}$-indecomposable if and only if it has a nonzero nilpotent element. Also notice that for an $\mathcal{S}$-indecomposable semigroup S with zero, $\rho_1(S) = 0$ if and only if $\rho_2(S) = 0$ if and only if S is nil.

Example 4.1. Let S be a 0-simple semigroup with a nonzero nilpotent element b. Then $0 — b$. If x is a nonnilpotent element in S, then $0 — b — x$, $0 \to b \to x$ are minimal sequences. So $\rho_1(S) = \rho_2(S) = 1$.

Example 4.2. Let S_1, S_2 be two 0-simple semigroups with $b_1 \in S_1$, $b_2 \in S_2$ being nonzero nilpotent elements of S_1 and S_2 respectively. Identify the zeros of S_1 and S_2. Let $S = S_1 \cup S_2$ and $S_1 \cap S_2 = S_1 S_2 = S_2 S_1 = \{0\}$. Let $x \in S_1$, $y \in S_2$ be nonnilpotent. Then $x — b_1 — b_2 — y$ is a minimal sequence between x and y. So $\rho_1(S) = 2$. But $x \to b_2 \to y$, $y \to b_1 \to x$, whence $\rho_2(S) = 1$. Thus the rank of a semigroup can be strictly larger than the semirank.

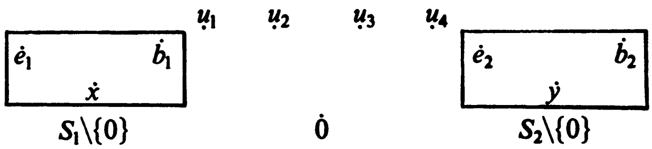
Example 4.3. Let S, S_1, S_2 be as in Example 4.2. This time choose $b_1 \in S_1$, $b_2 \in S_2$ such that $b_1, b_2 \neq 0$, $b_1^2 = b_2^2 = 0$. Let $T = S \cup \{u\}$, $u \notin S$. Define $u^2 = 0$, $xu = xb_1$, $ux = b_1 x$, $uy = b_2 y$, $yu = yb_2$ where $x \in S_1$, $y \in S_2$. It can be seen that T is a semigroup with ideal S. If $s_1 \in S_1$, $s_2 \in S_2$, then $u \mid b_1 \mid s_1$, $u \mid b_2 \mid s_2$, $s_1 \mid u^2 = 0$, $s_2 \mid u^2 = 0$. Hence $s_1 — u — s_2$. Thus $\rho_1(T) = 1$. But $\rho_1(S) = 2$ by Example 4.2. Thus the rank of an ideal of a semigroup can be greater than that of the semigroup. Can a similar thing happen with semirank? Can it happen for Γ-semigroups? Can the rank of a semigroup be less than that of an ideal by an arbitrary number?

Example 4.4. We are now going to construct $\mathcal{S}$-indecomposable Γ-semigroups of every rank (and hence semirank). A group is an example of an $\mathcal{S}$-indecomposable, Γ-semigroup of rank and semirank zero. Now let $\langle S_i \rangle_{i \in Z^+}$ be a sequence of 0-simple semigroups. Assume S_i has zero 0_i, a nonzero nilpotent element b_i, $b_i^2 = 0_i$, and a nonzero idempotent e_i. (For instance S_i could be a completely 0-simple semigroup which is not a Clifford semigroup.) Now identify e_i and 0_{i+1}. Thus $S_i \cap S_{i+1} = \{e_i\} = \{0_{i+1}\}$. Let $S = \bigcup_{i \in Z^+} S_i$. Set $I_i = \bigcup_{j \le i} S_j$. We define multiplication on S by defining multiplication on each I_i. $I_1 = S_1$ is a semigroup. Assume multiplication has been defined on I_i. Let $x \in I_i$ and $y \in S_{i+1}$. Then define $xy = xe_i$, $yx = e_i x$. Then it can be seen that the multiplication is consistent with the previous (i.e., when x or $y = e_i = 0_{i+1}$), and also that now I_{i+1} is a semigroup. Consequently, we obtain a semigroup S. Let $0_1 = 0$. Then 0 is the zero of S, and each I_i is an ideal of S. Now let x_i be a nonnilpotent element of S_i. For $k \in Z^+$, $b_k — e_k = b_{k+1}^2$ and so $b_k — b_{k+1}$. Thus $0 — b_1 — b_2 — \cdots — b_i — x_i$ is a minimal sequence between 0 and x_i of length i in both I_i and S. It now follows easily that $\rho_1(S) = \infty$, $\rho_1(I_i) = i$. Since S, I_i are Γ-semigroups, $\rho_2(S) = \infty$ and $\rho_2(I_i) = i$. S and each I_i are $\mathcal{S}$-indecomposable since there is a sequence between any two points. It is also clear that if we choose S_i's finite, we obtain finite semigroups of every finite semirank and rank.

Example 4.5. Now we are going to show that Theorem 2.5 need not be true for arbitrary (even finite) semigroups. Let S, S_1, S_2 be as in Example 4.2. Now we further assume that there exists a nonzero idempotent e_i in S_i, a nonzero nilpotent element $b_i \in S_i$ such that $b_i^2 = e_i b_i = b_i e_i = 0$, $i = 1, 2$. (For instance S_i could be the Rees-matrix semigroup over the trivial group $\{1\}$ with the 3×3 identity sandwich matrix and

$$e_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad b_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $T = S \cup \{u_1, u_2, u_3, u_4\}$, $u_k \notin S$, $k = 1, 2, 3, 4$, $u_i \neq u_j$ for $i \neq j$. We have



We complete the multiplication table as follows:

(1) $u_i u_j = 0$ for $i \neq j$.
(2) $u_1^2 = e_1$.
(3) $u_2^2 = 0$.
(4) $u_3^2 = e_2$.
(5) $u_4^2 = 0$.

(6)

$x \in S_1$	$y \in S_2$
$u_1 x = e_1 x$	$u_1 y = b_2 y$
$x u_1 = x e_1$	$y u_1 = y b_2$
$u_2 x = b_1 x$	$u_2 y = 0$
$x u_2 = x b_1$	$y u_2 = 0$
$u_3 x = b_1 x$	$u_3 y = e_2 y$
$x u_3 = x b_1$	$y u_3 = y e_2$
$u_4 x = 0$	$u_4 y = b_2 y$
$x u_4 = 0$	$y u_4 = y b_2$

The multiplication intersects when $x = y = 0$ but then the values are identically equal to 0. It can be shown with some effort that T is a semigroup with zero 0. Furthermore

(i) u_1 divides every element of S.
(ii) u_2 divides every element of S_1 but no element of $S_2\backslash\{0\}$.
(iii) u_3 divides every element of S.
(iv) u_4 divides every element of S_2 but no element of $S_1\backslash\{0\}$. Thus,

$$u_2 \mid e_1 = u_1^2, \qquad u_1 \mid 0 = u_2^2, \qquad \text{whence } u_1 - u_2.$$

$$u_3 \mid e_1 = u_1^2, \qquad u_1 \mid e_2 = u_3^2, \qquad \text{whence } u_1 - u_3.$$

$$u_4 \mid e_2 = u_3^2, \qquad u_3 \mid 0 = u_4^2, \qquad \text{whence } u_3 - u_4.$$

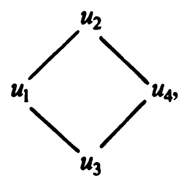
$$u_4 \mid 0 = u_2^2, \qquad u_2 \mid 0 = u_4^2, \qquad \text{whence } u_2 - u_4.$$

But,

$$u_4 \nmid e_1 = u_1^2 \qquad \text{and so} \quad u_1 \nrightarrow u_4.$$

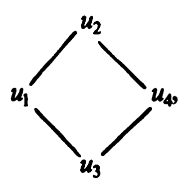
$$u_2 \nmid e_2 = u_3^2 \qquad \text{and so} \quad u_2 \nrightarrow u_3.$$

Thus,



$$u_1 \nrightarrow u_4, \; u_2 \nrightarrow u_3.$$

So Theorem 2.5 is not true for T. Note that $u_3 \mid 0 = u_2^2$ and so $u_3 \rightarrow u_2$. A slightly more complicated example can be given where



$$u_1 \not\twoheadrightarrow u_4, \; u_4 \not\twoheadrightarrow u_1, \; u_2 \not\twoheadrightarrow u_3, \; u_3 \not\twoheadrightarrow u_2.$$

Example 4.6. Let X be a finite set $|X| > 2$, $\mathcal{T}_X$ the full transformation semigroup on X. Let $\mathcal{V}_X = \mathcal{T}_X \backslash S_x$ where S_X is the group of permutations on X. Then $\mathcal{V}_X$ is a prime ideal in $\mathcal{T}_X$. $\mathcal{V}_X$ is $\mathcal{S}$-indecomposable. Moreover, there exists a fixed $a_0 \in \mathcal{V}_X$ such that for all $c \in \mathcal{V}_X$, $a_0 - c$. So $\rho_1(\mathcal{V}_X) = \rho_2(\mathcal{V}_X) = \rho_1(\mathcal{T}_X) = \rho_2(\mathcal{T}_X) = 1$. Since $\mathcal{T}_X$, $\mathcal{V}_X$ are Γ-semigroups, Theorem 2.5 is true for these semigroups. As a side remark we mention that $\mathcal{V}_X$ cannot even be decomposed into disjoint union of proper subsemigroups.

Example 4.7. Let X be an infinite set and $\mathcal{T}_X$ the full transformation semigroup on X. Then there exists a fixed $a_0 \in \mathcal{T}_X$ such that for all $c \in \mathcal{T}_X$, $a_0 - c$. Furthermore $\mathcal{T}_X$ is a Γ-semigroup. So $\mathcal{T}_X$ is an $\mathcal{S}$-indecomposable semigroup of rank and semirank 1. Furthermore Theorem 2.5 holds for $\mathcal{T}_X$. Can $\mathcal{T}_X$ be decomposed into a disjoint union of proper subsemigroups?

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REFERENCES

1. A. H. Clifford, *Semigroups admitting relative inverses*, Ann. of Math. (2) **42** (1941), 1037–1049. MR **3**, 199.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*. Vol. 1, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R.I., 1961. MR **24** #A2627.

3. M. S. Putcha, *Semilattice decompositions of semigroups*, Semigroup Forum **6** (1973), 12–34.

4. M. S. Putcha and J. Weissglass, *A semilattice decomposition into semigroups having at most one idempotent*, Pacific J. Math. **39** (1971), 225–228.

5. T. Tamura, *The theory of construction of finite semigroups*. I, Osaka Math. J. **8** (1956), 243–261. MR **18**, 717.

6.———, *Another proof of a theorem concerning the greatest semilattice decomposition of a semigroup*, Proc. Japan. Acad. **40** (1964), 777–780. MR **31** #3530.

7.———, *On Putcha's theorem concerning semilattice of archimedean semigroups*, Semigroup Forum **4** (1972), 83–86.

8.———, *Quasi-orders, generalized archimedeaness and semilattice decompositions* (to appear).

9.———, *Note on the greatest semilattice decomposition of semigroups*, Semigroup Forum **4** (1972), 255–261.

10. M. Petrich, *The maximal semilattice decomposition of a semigroup*, Math. Z. **85** (1964), 68–82. MR **29** #4824.

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