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SUBGROUPS OF GROUPS OF CENTRAL TYPE

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ABSTRACT. Let λ be a linear character on the center Z of a finite group Z of a finite group H, such that

(1) $\lambda^H = \sum_{i=1}^p \phi_i(1)\phi_i$ where the ϕ_i's are inequivalent irreducible characters on H of the same degree, and

(2) if $\sum_{i=1}^p m_i\phi_i(x) = 0$ for some $x \in H$ and nonnegative integers m_i, then either $\phi_i(x) = 0$ for all i or $m_i = m_j$ for all i, j.

The object of the paper is to describe finite groups which satisfy conditions (1) and (2) in terms of the multiplication of the group. If S is a p Sylow subgroup of the group H, and $R = S \cdot Z$, then H satisfies conditions (1) and (2) if and only if

(a) $\{x \in H: x^{-1}h^{-1}xh \in Z \Rightarrow \lambda(x^{-1}h^{-1}xh) = 1, h \in H\}/Z$ consists of elements of order a power of p in H/Z, and these elements form p conjugacy classes of H/Z, and

(b) the elements of $\{x \in R: x^{-1}r^{-1}xr \in Z \Rightarrow \lambda(x^{-1}r^{-1}xr) = 1, r \in R\}/Z$ form p conjugacy classes of R/Z.

Introduction. Let G be a finite group with center Z. In [3] F. R. DeMeyer and G. J. Janusz called G a group of central type if there is an irreducible (complex) character χ on G such that $\chi(1)^2 = [G: Z]$. Groups of central type arise in Schur's theory of projective representations [5, pp. 628–655] and the general Galois theory of rings [1].

We study groups which appear as normal subgroups of index p for some prime p in groups of central type. Let H be a finite group with center Z, and let p be a prime. Let λ be a linear character on the center Z of a finite group H, such that $\lambda^H = \sum_{i=1}^p \phi_i(1)\phi_i$ where the ϕ_i's are inequivalent irreducible characters on H of the same degree. Assume that if $\sum_{i=1}^p m_i\phi_i(x) = 0$ for nonnegative integers m_i, then either $\phi_i(x) = 0$ for all i or $m_i = m_j$ for all i, j. We call a group satisfying these conditions p-special with respect to λ.

We show that if H is a normal subgroup of index p in a group G of central type, then either H is of central type or H is p-special (Theorem 2.1). We next give necessary and sufficient conditions on a p-special group H that it be a normal subgroup of index p in a group of central type (Theorem 2.2).

We then examine some properties of p-special groups. For a group H, let $\mathrm{Cl}_H(x)$ be the conjugacy class in H containing x. Let Z be the center of H and let λ be a linear character on Z. Define

$$T(H,\lambda) = \{x \in H: x^{-1}\,\mathrm{Cl}_H(x) \cap Z \subseteq \mathrm{kernel}(\lambda)\}.$$

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If S is a p Sylow subgroup of H, let $R = S \cdot Z$ and define

$$T(R,\lambda) = \{x \in R: x^{-1}\,\mathrm{Cl}_R(x) \cap Z \subseteq \mathrm{kernel}(\lambda)\}.$$

We show (Corollary 2.26) that if H is a finite group with center Z, then H is p-special with respect to λ if and only if

(1) every element of $T(H,\lambda)/Z$ has order a power of p and $T(H,\lambda)/Z$ consists of p conjugacy classes of H/Z, and

(2) $T(R,\lambda)/Z$ consists of p conjugacy classes of R/Z.

Additional information is given concerning the set of elements $T(H,\lambda)$ and how it relates to the character on H.

Throughout this paper, all groups are finite and all characters are complex. If H is a group, $Z(H)$ denotes the center of H. If $x \in H$, $\langle x \rangle$ denotes the subgroup of H, generated by x. The conjugacy class of x is denoted by $\mathrm{Cl}_H(x)$ or simply by $\mathrm{Cl}(x)$ if there can be no confusion. If A is a subset of H, $[A : 1]$ denotes the number of elements in A and if A and B are two subsets, $[A: B] = [A: 1]/[B: 1]$. A p element is an element whose order is a power of p and a p group is a group in which every element is a p element. If n is any integer and q is any prime n_q denotes the q part (or q factor) of n. All unexplained terminology and notation is as in Huppert [5].

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1. F. R. DeMeyer and G. J. Janusz [3] defined a finite group G with center Z to be of central type if there is an irreducible character χ on G so that $\chi(1)^2 = [G: Z]$. They proved the following: If G is a group of central type then there is a 2-cocycle α on $\overline{G} = G/Z$ so that $K(\overline{G})_\alpha$ has center K, where K denotes the set of complex numbers. Herbert Pahling [6] showed that if G is a group of central type with center Z, then for every $x \in G$, $x \notin Z$, there is an element $g \in G$ so that $1 \neq x^{-1}g^{-1}xg \in Z$; and conversely, if $[G,G] \cap Z$ is cyclic and for every $x \in G$, $x \notin Z$, there is an element $g \in G$ so that $1 \neq x^{-1}g^{-1}xg \in Z$, then G is a group of central type. In this section, results will be proved which connect the above results.

Let G be a group with center Z and let $\mathrm{Cl}(x)$ be the conjugacy class in G containing x. The condition in Pahling's results suggests the study of the elements $x \in G$, for which $x^{-1}\,\mathrm{Cl}(x) \cap Z = \{1\}$. In order to make the results of this section as general as later applications require, we study a larger set.

Definition. Let A be a subgroup of the center Z of a group G and let λ be a linear character on A. Define

$$T(G,\lambda) = \{x \in G: x^{-1}\,\mathrm{Cl}(x) \cap A \subseteq \mathrm{kernel}(\lambda)\}.$$

If $x \in T(G,\lambda)$, then $\mathrm{Cl}(x) \subseteq T(G,\lambda)$ and $x \cdot a \in T(G,\lambda)$ for all $a \in A$. The results of this section give relationships between the irreducible constituents of λ^G and the elements of the set $T(G,\lambda)$.

Lemma 1.1. *Let G be a group with center Z and let A be a subgroup of Z. If λ is a linear character on A, then there is a 2-cocycle α on G/Z so that the center of the projective group algebra $K(G/A)_\alpha$ has dimension t over K where t is the number of conjugacy classes of G/A contained in $T(G)/A$. Moreover, a basis of the center of $K(G/A)_\alpha$ consists of elements of the form $\sum_{x\in C} U_x$, where C is the natural image in G/A of a conjugacy class of G contained in $T(G,\lambda)$.*

We prove Lemma 1.1 first in the case that λ restricted to $[G,G] \cap A$ is faithful. We carry out the proof of Lemma 1.1 by a sequence of assertions.

(1.2) If $\gamma \in \overline{G} = G/A$, let γ^* be an element of G, chosen to represent the coset γ. It is possible to choose the coset representatives in such a way that if $\beta, \gamma \in \overline{G}$ with β the natural image of an element of $T(G,\lambda)$, then $(\gamma^{-1}\beta\gamma)^* = (\gamma^*)^{-1}\beta^*\gamma^*$.

Proof. Let $C_1, \dots, C_t$ be distinct conjugacy classes in $\overline{G}$, contained in $T(G,\lambda)/A$, where $C_1 = \{\overline{1}\}$. Choose $(\overline{1})^* = 1$. For each $2 \le i \le t$, fix an element $\beta \in C_i$ and choose β^*. For every $\gamma \in \overline{G}$, define $(\gamma^{-1}\beta\gamma)^* = (g)^{-1}\beta^*g$, where $\bar{g} = \gamma$. We must show that $(\gamma^{-1}\beta\gamma)^*$ is well defined.

First of all it is clear that the definition of $(\gamma^{-1}\beta\gamma)^*$ is independent of the choice of g, since $A \subseteq Z$. Suppose $\delta^{-1}\beta\delta = \gamma^{-1}\beta\gamma$ for some $\delta \in \overline{G}$. Let $d \in G$, so that $\bar{d} = \delta$. Then $d^{-1}\beta^*d = g^{-1}\beta^*g \cdot a$ for some $a \in A$, and

$$a^{-1} = (\beta^*)^{-1}(gd^{-1})^{-1}\beta^*gd^{-1} \in A \cap [G,G].$$

Since $\beta \in C_i$, $\beta^* \in T(G,\lambda)$ and $(\beta^*)^{-1}(gh^{-1})^{-1}\beta^*gh^{-1} = a \in \mathrm{kernel}(\lambda)$. Since λ restricted to $[G,G] \cap A$ is assumed to be faithful, $a = 1$, and $d^{-1}\beta^*d = g^{-1}\beta^*g$. Hence $(\gamma^{-1}\beta\gamma)^*$ is well defined. Choose the representatives of other elements of $\overline{G}$ arbitrarily. This completes the proof of (1.2).

Choose coset representatives of the elements of $\overline{G}$ as described in (1.2). We define a 2-cocycle α on $\overline{G}$ by

$$\alpha(\delta,\gamma) = \lambda(((\delta\gamma)^*)^{-1}\delta^*\gamma^*).$$

We isolate the following computation.

(1.3) If β is in the image of an element in $T(G,\lambda)$, then for every $\gamma \in \overline{G}$, $\alpha(\gamma^{-1},\beta)\alpha(\gamma^{-1}\beta,\gamma) = \lambda((\gamma^{-1})^*\gamma^*)$.

Proof. By (1.2), $(\gamma^{-1}\beta\gamma)^* = (\gamma^*)^{-1}\beta^*\gamma^*$. Then

$$\begin{aligned}\alpha(\gamma^{-1},\beta)\alpha(\gamma^{-1}\beta,\gamma) &= \lambda(((\gamma^{-1}\beta)^*)^{-1}(\gamma^{-1})^*\beta^*) \cdot \lambda(((\gamma^{-1}\beta\gamma)^*)^{-1}(\gamma^{-1}\beta)^*\gamma^*)\\ &= \lambda(((\gamma^{-1}\beta)^*)^{-1}(\gamma^{-1})^*\beta^*) \cdot \lambda((\gamma^*)^{-1}(\beta^*)^{-1}\gamma^*(\gamma^{-1}\beta)^*\gamma^*)\\ &= \lambda(((\gamma^{-1}\beta)^*)^{-1}(\gamma^{-1})^*\beta^*(\beta^*)^{-1}\gamma^*(\gamma^{-1}\beta)^*)\\ &= \lambda((\gamma^{-1})^*\gamma^*).\end{aligned}$$

(1.4) Suppose that $\sum_{\gamma\in\overline{G}} c_\gamma U_\gamma$ is in the center of $K(\overline{G})_\alpha$ where $c_\gamma \in K$. If γ is not an element of the image of $T(G,\lambda)$ in $\overline{G}$, then $c_\gamma = 0$. If γ is in the image of $T(G,\lambda)$, then $c_{\delta^{-1}\gamma\delta} = c_\gamma$ for every $\delta \in \overline{G}$.

Proof. $\sum_{\gamma\in\Gamma} c_\gamma U_\gamma$ is in the center of $K(\overline{G})_\alpha$ if and only if for every $\delta \in \overline{G}$,

$$U_{\delta^{-1}}\Big(\sum_{\gamma\in\overline{G}} c_\gamma U_\gamma\Big)U_\delta = U_{\delta^{-1}} U_\delta\Big(\sum_{\gamma\in\overline{G}} c_\gamma U_\gamma\Big)$$
$$= \lambda(\delta^*(\delta^{-1})^*)\Big(\sum_{\gamma\in\overline{G}} c_\gamma U_\gamma\Big).$$

Computing, we have

$$U_{\delta^{-1}}\Big(\sum_{\gamma\in\overline{G}} c_\gamma U_\gamma\Big)U_\delta = \sum_{\gamma\in\overline{G}} c_\gamma \alpha(\delta^{-1},\gamma)\alpha(\delta^{-1}\gamma,\delta)U_{\delta^{-1}\gamma\delta}$$
$$= \sum_{\gamma\in\overline{G}} (c_{\delta\gamma\delta^{-1}})\alpha(\delta^{-1},\delta\gamma\delta^{-1})\alpha(\gamma\delta^{-1},\delta)U_\gamma.$$

Hence, for every δ and γ in $\overline{G}$,

$$(c_{\delta\gamma\delta^{-1}})\alpha(\delta^{-1},\delta\gamma\delta^{-1})\alpha(\gamma\delta^{-1},\delta) = \lambda(\delta^*(\delta^{-1})^*)c_\gamma. \tag{1.5}$$

If γ is not in the image of $T(G,\lambda)$, there is an element $g \in G$, so that $g\gamma^*g^{-1} = \gamma^*a$ for some $a \neq 1$ in A. Let $\delta = \bar{g}$. Then

$$\alpha(\delta^{-1},\delta\gamma\delta^{-1})\alpha(\gamma\delta^{-1},\delta) = \lambda(((\gamma\delta^{-1})^*)^{-1}(\delta^{-1})^*(\delta\gamma\delta^{-1})^*(\gamma^*)^{-1}(\gamma\delta^{-1})^*\delta^*)$$
$$= \lambda(((\gamma\delta^{-1})^*)^{-1}(\delta^{-1})^*\gamma^*(\gamma^*)^{-1}(\gamma\delta^{-1})^*\delta^*)$$
$$= \lambda((\delta^{-1})^*\delta^*\cdot a)$$
$$\neq \lambda((\delta^{-1})^*\delta^*).$$

Since equation (1.5) must hold, $c_\gamma = 0$ for every γ not in the image of $T(G,\lambda)$.

If γ is in the image of $T(G,\lambda)$, then $\delta\gamma\delta^{-1}$ is also and by (1.3), for every $\delta \in \overline{G}$,

$$\alpha(\delta^{-1},\delta\gamma\delta^{-1})\alpha(\gamma\delta^{-1},\delta) = \lambda((\delta^{-1})^*\delta^*).$$

Thus $c_{\delta\gamma\delta^{-1}} = c_\gamma$ for every $\delta \in \overline{G}$.

If $C_1,\ldots,C_t$ are distinct conjugacy classes in $\overline{G}$ contained in $T(G,\lambda)/A$, then the elements $\sum_{\gamma\in C} U_\gamma$ for $C = C_i$ for $1 \le i \le t$ form a linearly independent set of elements in the center of $K(\overline{G})_\alpha$. By (1.4) these elements form a basis of the center of $K(\overline{G})_\alpha$. This completes the proof of Lemma 1.1 when λ restricted to $[G,G] \cap A$ is faithful.

If λ restricted to $[G,G] \cap A$ is not faithful, let $N = [G,G] \cap \text{kernel}(\lambda)$. Let $G' = G/N$, $A' = A/N$ and let λ' be a linear character on G' defined by $\lambda'(aN) = \lambda(a)$ for any $a \in aN$. Then $T(G',\lambda')$ in G' is the natural image of $T(G,\lambda)$ and the number of conjugacy classes of G'/A' contained in $T(G',\lambda')/A'$ is the same as the number of classes of G/A in $T(G,\lambda)/A$. Since G/A is isomorphic to G'/A', if α' is a 2-cocycle on G'/A' as defined in the previous case, then there is a corresponding 2-cocycle α on G/A, so that $K(G/A)_\alpha$ and $K(G'/A')_{\alpha'}$ are isomorphic K-algebras. This completes the proof of Lemma 1.1.

Lemma 1.1 allows us to count the number of inequivalent irreducible constituents of λ^G, where λ is a linear character on a subgroup A of the center of G.

(1.6) The number of inequivalent irreducible constituents of λ^G is t, the number of conjugacy classes of G/A contained in $T(G,\lambda)/A$.

Proof. Let α be a 2-cocycle on G/A as defined in the proof of Lemma 1.1. By [4, pp. 163–179], $K(G/A)_\alpha$ is isomorphic to $\sum_{i=1}^t \mathrm{Hom}_K(M_i, M_i)$ where M_i is an irreducible left $K(G/A)_\alpha$ module.

For each i, let T_i^* be a projective representation of G/A corresponding to M_i. If $g \in G$ and $\bar{g}$ is its image in G/A, then $g = (\bar{g})^* a(g)$ for some element $a(g) \in A$. Define $T_i(g) = \lambda(a(g))T_i^*(\bar{g})$. Let g and d be elements of G. Then $g = (\bar{g})^*a(g)$, $d = (\bar{d})^*a(d)$, $gd = (\bar{g}\bar{d})^*a(gd)$ and

$$\begin{aligned} T_i(g)T_i(d) &= \lambda(a(g))T_i^*(\bar{g})\lambda(a(d))T_i^*(\bar{d}) \\ &= \lambda(a(g)a(d))\alpha(\bar{g},\bar{d})T_i^*(\bar{g}\bar{d}) \\ &= \lambda(a(g)a(d))\lambda(((\bar{g}\bar{d})^*)^{-1}(\bar{g})^*(\bar{d})^*)T_i^*(\bar{g}\bar{d}) \\ &= \lambda(a(gd))T_i^*(\bar{g}\bar{d}) = T_i(gd). \end{aligned}$$

Hence T_i is an ordinary representation of G. If $\phi_i(g) = \mathrm{trace}(T_i(g))$ for $g \in G$, then ϕ_i is an irreducible character on G for $1 \le i \le t$, and $\phi_i|_A = \phi_i(1)\lambda$.

Let ζ be an irreducible constituent of λ^G and let M be a corresponding KG module. Since $\zeta|_A = \zeta(1)\lambda$, M is an irreducible left $K(G/A)_\alpha$ module. By [4, Theorem 25.10, p. 166], M is isomorphic to a minimal left ideal of $K(G/A)_\alpha$ and M is isomorphic to M_i for some $1 \le i \le t$. Thus if ζ is an irreducible constituent of λ^G, then $\zeta = \phi_i$ for some i.

Let $\phi_1, \ldots, \phi_u$ be a maximum number of inequivalent characters from the set $\{\phi_i \mid 1 \le i \le t\}$. Since $(\phi_i|_A, \lambda) = (\lambda^G, \phi_i) = \phi_i(1)$, $[G\colon A] = \lambda^G(1) = \sum_{i=1}^u \phi_i(1)^2 = \sum_{i=1}^u d_i^2$. However $[G\colon A] = \sum_{i=1}^t d_i^2$ and hence $u = t$ and $\phi_1, \ldots, \phi_t$ are inequivalent irreducible constituents of λ^G.

Let G be a group with center Z. If χ is an irreducible character on G such that $\chi(1)^2 = [G\colon Z]$, then $\chi|_Z = \chi(1)\lambda$ and $\lambda^G = \chi(1)\chi$ for some linear character λ on Z. Conversely, if λ is a linear character on Z such that $\lambda^G = \chi(1)\chi$ for some irreducible character χ on G, then $\chi(1)^2 = [G\colon Z]$. Hence G is a group of central type if and only if there is a linear character λ on Z such that $t = 1$ where t is the number in (1.6). Since t is the number of conjugacy classes of G/Z contained in $T(G,\lambda)$, then $t = 1$ if and only if $T(G,\lambda) = Z$. These remarks verify the results in §1 of [6]. Since for every linear character on Z one can define a 2-cocycle on G/Z as in the proof of Lemma 1.1, Theorem 1 of [3] follows from Lemma 1.1 and the above remarks.

There is another relationship which exists between the elements of the set $T(G,\lambda)$ and the irreducible constituents of λ^G which will be useful later.

(1.7) If $x \notin T(G,\lambda)$ and ϕ is any irreducible constituent of λ^G, then $\phi(x) = 0$. If $x \in T(G,\lambda)$ then there is an irreducible constituent of λ^G for which $\phi(x) \neq 0$.

Proof. Suppose $x \notin T(G,\lambda)$. Then there is an element $g \in G$, such that $g^{-1}xg = xa$, $a \in A$, $\lambda(a) \neq 1$. Then $\phi(x) = \phi(g^{-1}xg) = \phi(xa) = \lambda(a)\phi(x)$. Since $\lambda(a) \neq 1$, $\phi(x) = 0$.

Let $\lambda^G = \sum_{i=1}^{t} \phi_i(1)\phi_i$ and let T_i, $1 \leq i \leq t$, be inequivalent irreducible representations of G, T_i corresponding to ϕ_i. For each i and each conjugacy class C of G, $\sum_{x\in C} T_i(x)$ is a scalar matrix by Schur's lemma [4, 27.3, p. 181]. Let $\sum_{x\in C} T_i(x) = k \cdot T_i(1)$, $k \in K$. The trace of $k \cdot T_i(1)$ is $k \cdot \phi_i(1)$, and

$$k \cdot \phi_i(1) = \sum_{x\in C} \operatorname{trace}(T_i(x)) = \sum_{x\in C} \phi_i(x) = n \cdot \phi_i(x_0),$$

where n is the number of elements in C and x_0 is any element of C. Thus if $\phi_i(x_0) = 0$ for any $x_0 \in C$, then $\sum_{x\in C} T_i(x)$ is the zero matrix.

Let T be a representation of A corresponding to λ. Then for every $g \in G$, $T^G(g)$ is similar to $\bigoplus \sum_{i=1}^{t} \phi_i(1)T_i(g)$. Let $x \in T(G,\lambda)$ and suppose that $\phi_i(x) = 0$ for all $1 \leq i \leq t$. If C is the conjugacy class of G containing x, then $\sum_{y\in C} T_i(y)$ is the zero matrix for every i, and hence $\sum_{y\in C} T^G(y)$ is the zero matrix. For all g, $h \in G$, there is i,j so that $(\sum_{y\in C} T^G(y))_{ij} = \sum_{y\in C} \dot\lambda(g^{-1}yh)$, and thus $\sum_{y\in C} \dot\lambda(g^{-1}yh) = 0$ for all g, h in G. In particular $\sum_{y\in C} \dot\lambda(x^{-1}y) = 0$. If $x^{-1}y \in A$ for any $y \in C$, then $x^{-1}y \in x^{-1}\,\mathrm{Cl}(x) \cap A$. Since $x \in T(G,\lambda)$, $x^{-1}\,\mathrm{Cl}(x) \cap A \subseteq \operatorname{kernel}(\lambda)$. If n is the number of y's in C for which $x^{-1}y \in A$, then $\sum_{y\in C} \dot\lambda(x^{-1}y) = n \cdot 1 = 0$. Since $x \in C$, $n \geq 1$, contradicting the statement that $n = 0$. Hence for some i, $\phi_i(x) \neq 0$.

We can summarize the results of this section in the following theorem.

Theorem 1.8. *Let λ be a linear character defined on a subgroup A of the center of a finite group G. Let*

$$T(G,\lambda) = \{x \in G\colon x^{-1}g^{-1}xg \in \mathit{kernel}(\lambda) \text{ if } x^{-1}g^{-1}xg \in [G,G] \cap A\}$$

and let t be the number of conjugacy classes of G/A contained in $T(G,\lambda)/A$. Then λ^G has t inequivalent irreducible constituents. If ϕ is an irreducible constituent of λ^G and $x \notin T(G,\lambda)$, then $\phi(x) = 0$. If $x \in T(G,\lambda)$, then there is an irreducible constituent ϕ of λ^G for which $\phi(x) \neq 0$.

2. In this section we study groups which are not of central type but share properties with normal subgroups of index p of groups of central type.

Definition. Let H be a group with center Z. We call H p-special if there is a linear character λ on Z, such that

(a) λ^H has p inequivalent irreducible constituents $\phi_1, \ldots, \phi_p$ all of the same degree, and

(b) if $\sum_{i=1}^{p} m_i\phi_i(x) = 0$ for nonnegative integers m_i and some $x \in H$, then either $\phi_i(x) = 0$ for all i or $m_i = m_j$ for all i, j.

We will also say H is p-special with respect to λ.

Note. Let H be p-special with respect to λ and $\lambda^H = e(\phi_1 + \cdots + \phi_p)$. If $x \in H$, such that $\phi_i(x) = 0$ for some i, then by condition (b), $\phi_j(x) = 0$ for all j. If $T(H,\lambda) = \{x \in H: x^{-1}\,\mathrm{Cl}_H(x) \cap Z \subseteq \mathrm{kernel}(\lambda)\}$ then $\phi_i(x) \neq 0$, $1 \leq i \leq p$, if and only if $x \in T(H,\lambda)$ by Theorem 1.8.

Theorem 2.1. *If G is a group of central type with center $Z(G)$ and H is a normal subgroup of G of index p, then H is of central type if $Z(H) \neq Z(G)$ and H is p-special if $Z(H) = Z(G)$.*

Proof. Suppose χ is an irreducible character on G, so that $\chi(1)^2 = [G\colon Z(G)]$. If $Z(G) \not\subseteq H$, then $\chi|_H$ is irreducible since elements of $Z(G)$ are represented by scalar matrices by Schur's lemma. Hence $(\chi|_H(1))^2 = \chi(1)^2 = [G\colon Z(G)] \leq [H\colon Z(H)]$. Therefore, $[Z(G)\colon Z(H)] = p$ and H is of central type. If $Z(G) \subseteq H$, then $Z(G) \subseteq Z(H)$ and $[H\colon Z(H)] < [G\colon Z(G)]$. Hence $\chi|_H$ cannot be irreducible. Since H is a normal subgroup of G of index p, by Clifford's theorem [4, Theorem 49.2, p. 343] either $\chi|_H = p\phi$ where ϕ is irreducible on H or $\chi|_H = \phi_1 + \cdots + \phi_p$ where $\phi_1, \ldots, \phi_p$ are conjugate irreducible characters on H. If $\chi|_H = p\phi$, then, by Frobenius reciprocity [4, Theorem 38.8, p. 271], $\phi^G = p\chi + \cdots$ and $\phi^G(1) = p\phi(1) \geq p\chi(1)$ which is impossible. If $\chi|_H = \phi_1 + \cdots + \phi_p$, then H has an irreducible character of degree $([G\colon Z(G)]/p^2)^{1/2}$ and

$$[H\colon Z(H)] \geq [G\colon Z(G)]/p^2 = [H\colon Z(H)] \cdot [Z(H)\colon Z(G)]/p.$$

If $[Z(H)\colon Z(G)] = p$, then H has an irreducible character of degree $[H\colon Z(H)]$ and H is of central type.

If $Z(H) = Z(G)$, H is not of central type. Let $Z = Z(G) = Z(H)$, and $\chi|_Z = \chi(1)\lambda$ where λ is a linear character on Z. Then $\lambda^H = \phi_1(1)\phi_1 + \cdots + \phi_p(1)\phi_p$ and since the $\phi_1, \ldots, \phi_p$ are conjugate characters, they all have the same degree. Suppose $\sum_{i=1}^p m_i\phi_i(x) = 0$ for nonnegative integers m_i and some $x \in H$. If $x \notin T(H,\lambda)$ then, by Theorem 1.8, $\phi_i(x) = 0$ for every i. If $x \in Z$ then $\phi_i(x) = \phi_i(1)\lambda(x)$ and $0 = \sum_{i=1}^p m_i\phi_i(x) = \sum_{i=1}^p m_i\phi_1(1)\lambda(x)$. Hence $\sum_{i=1}^p m_i = 0$ or $m_i = 0$ for each i. Now suppose $x \in T(H,\lambda)$, $x \notin Z$. Since G is of central type, $T(G,\lambda) = Z$ and since $x \notin Z$, there is an element $g \in G$ such that $x^{-1}g^{-1}xg = z$ and $\lambda(z) \neq 1$. Since $x \in T(H,\lambda)$, $g \notin H$ and by relabeling if necessary, we can assume that $\phi_i = \phi_p^{g^i}$ and $\phi_p(x) \neq 0$. Since $g^p \in H$, and $x^{-1}g^{-p}xg^p = z^p$, $\lambda(z^p) = 1$, and $\lambda(z^i)$ is a primitive pth root of 1 if $i \not\equiv 0$ (modulo p). Then

$$\begin{aligned}
0 &= \sum_{i=1}^p m_i\phi_i(x) \\
&= \sum_{i=1}^p m_i\phi_p(g^{-i}xg^i) \\
&= \sum_{i=1}^p m_i\lambda(x^{-1}g^{-i}xg^i)\phi_p(x) \\
&= \left(\sum_{i=1}^p m_i\lambda(z^i)\right)\phi_p(x).
\end{aligned}$$

Since $\phi_p(x) \neq 0$, $\sum_{i=1}^{p} m_i \lambda(z^i) = 0$ and since $\lambda(z^i)$, $0 \leq i \leq p-1$, are p distinct pth roots of 1, $m_i = m_j$ for all i and j. This completes the proof of Theorem 2.1.

Note. If H is a p-special group and H is a normal subgroup of index p in a group of central type and $x \in T(H,\lambda)$ then there is an element $g \in G$, so that $x^{-1}g^{-1}xg = z \in Z$ and $\lambda(z) \neq 1$. If we define an automorphism σ on H by $\sigma(h) = g^{-1}hg$, then

(a) σ^p is an inner automorphism of H;

(b) $\sigma(z) = z$ for all $z \in Z$;

(c) $\sigma(x) = x \cdot z$ where $z \in Z$, $\lambda(z) \neq 1$ for some $x \in T(H,\lambda)$.

Thus if a p-special group H is a normal subgroup of index p of a group of central type, then there must be an automorphism of H satisfying conditions (a), (b), and (c). We next show these conditions are sufficient.

Theorem 2.2. *Let H be a finite group with center Z and let λ be a linear character on Z such that H is p-special with respect to λ. Suppose there is an automorphism σ of H, so that*

(a) *σ^p is an inner automorphism of H;*

(b) *$\sigma(z) = z$ for all $z \in Z$;*

(c) *$\sigma(x) = x \cdot z$ where $z \in Z$, $\lambda(z) \neq 1$ for some $x \in T(H,\lambda)$ where $T(H,\lambda) = \{x \in H: x^{-1}h^{-1}xh \in Z$ iff $x^{-1}h^{-1}xh \in \mathrm{kernel}(\lambda)\}$.*

Then H is a normal subgroup of index p of a group of central type.

Proof. Let G be the group generated by elements $h \in H$ and an element g where $hg^n = g^n\sigma^n(h)$ for any integer n. Then $Z(G) = Z$ and since σ^p is an inner automorphism of H, H is a normal subgroup of index p of G.

Since H is p-special with respect to λ, $\lambda^H = e(\phi_1 + \cdots + \phi_p)$ where the ϕ_i's are inequivalent irreducible characters on H and $\phi_i(1) = e$ for all i. Let χ be an irreducible constituent of λ^G. By Theorem 1.8, $\chi(y) \neq 0$ only if $y \in T(G,\lambda)$. If x is the element given in part (c) of Theorem 2.2, $x \notin T(G,\lambda)$ and hence $\chi(x) = 0$. Since χ is a constituent of λ^G, $\chi|_H = \sum_{i=1}^{p} m_i\phi_i$ where the m_i's are nonnegative integers. Then $\chi(x) = 0 = \sum_{i=1}^{p} m_i\phi_i(x)$. Since H is p-special and $x \in T(H,\lambda)$, $m_i = m_p$ for all i. Hence $\chi|_H = m_p \sum_{i=1}^{p} \phi_i$ and $\chi(1) = m_p \cdot p \cdot e$ or

$$\chi(1)^2 = m_p^2 \cdot p^2 \cdot e^2 = m_p^2 \cdot p^2 \cdot [H: Z]/p$$
$$= m_p^2 \cdot p \cdot [H: Z] = m_p^2 \cdot [G: Z].$$

Since $\chi(1)^2 \leq [G: Z]$, $m_p = 1$, $\chi(1)^2 = [G: Z]$ and G is a group of central type.

Example 2.3. Let $H = \langle x,y \mid x^8 = y^2 = 1, y^{-1}xy = x^{-1}\rangle$; H is the dihedral group of order 16 and H is 2-special. $Z(H) = \{1, x^4\}$. Let λ be defined on $Z(H)$ by $\lambda(x^4) = -1$. If ω is a primitive 8th root of 1, define $\sigma_i(x) = \omega^i$ for $0 \leq i \leq 7$. Let $X = \langle x \rangle$. Let $\phi_1|_X = \sigma_1 + \sigma_7$ with $\phi_1(h) = 0$ for all $h \notin X$ and let $\phi_2|_X = \sigma_3 + \sigma_5$ with $\phi_2(h) = 0$ for all $h \notin X$. Then ϕ_1 and ϕ_2 are inequivalent irreducible characters on H and are constituents of λ^H. $T(H,\lambda) = \{1,x,x^3,x^4,x^5, x^7\}$. If we define σ by $\sigma(x) = x^5$ and $\sigma(y) = y$, then σ satisfies the hypothesis of Theorem 2.2.

Example 2.4. Let p be any prime, and let $H = \langle a,b,c,d,e \mid a^p = b^p = c^p = d^p = e^p = 1,\ b^{-1}ab = ad,\ c^{-1}ac = a,\ c^{-1}bc = be,\ d \in Z(H), e \in Z(H)\rangle$. Then $Z(H) = \langle d,e\rangle$. Let λ be a linear character on $Z(H)$, defined by $\lambda(d) = \omega$, $\lambda(e) = 1$, where ω is a primitive pth root of 1. Let $C = \langle c,d,e\rangle$. Let $\sigma_i(c^s \cdot z) = \omega^{is}\lambda(z)$. Then $\lambda^C = \sum_{i=1}^{p} \sigma_i$. Define ϕ_i on H by $\phi_i\,|_C = p\sigma_i$ and $\phi_i(h) = 0$ for all $h \notin C$, $1 \le i \le p$. Then $\phi_1, \ldots, \phi_p$ are inequivalent irreducible constituents of λ^H and H is p-special with respect to λ. $T(H,\lambda) = \langle c,d,e\rangle$. If σ is defined by $\sigma(a) = a$, $\sigma(b) = b$, $\sigma(c) = cd$, $\sigma(d) = d$, $\sigma(e) = e$, then σ satisfies the hypothesis of Theorem 2.2.

Example 2.5. Let p be any prime, $p \neq 2$ and let $H = \langle x,y,u,v,z \mid x^{p^2} = z^{p^2} = y^p = u^p = v^p = 1,\ z \in Z(H),\ y^{-1}xy = x^{p+1},\ u^{-1}xu = xyz,\ u^{-1}yu = yz^p,\ v^{-1}xv = x,\ v^{-1}yv = y,\ v^{-1}uv = uz^p\rangle$. The center of H is $\langle z\rangle$. Let λ be a faithful linear character on $Z(H)$. Let $X = \langle x\rangle \cdot \langle z\rangle$ and let ω be a primitive p^2 root of 1. If $\sigma_i(x^s \cdot z) = \omega^{si}\lambda(z)$, then $\lambda^X = \sum_{i=1}^{p^2} \sigma_i$. Define ϕ_i on H by

$$\phi_i\,|_X = p\sigma_{pi} + \sum_{u=1}^{p}\sum_{v=1}^{p-1} \sigma_{pu+v}$$

and $\phi_i(h) = 0$ for all $h \notin X$, $1 \le i \le p$. Then $\phi_0, \ldots, \phi_{p-1}$ are inequivalent irreducible constituents of λ^H, and H is p-special with respect to λ. $T(H,\lambda) = \{x^s z^i \colon 0 \le i \le p^2 - 1, s \text{ relatively prime to } p\}$. If σ is defined by $\sigma(x) = x \cdot z^p$, $\sigma(y) = y$, $\sigma(u) = u$, $\sigma(v) = v$, $\sigma(z) = z$, then σ satisfies the hypothesis of Theorem 2.2.

We will study p-special groups by studying the set $T(H,\lambda)$.

Lemma 2.6. *Let H be a p-special group with respect to λ on the center Z, and suppose $[H,H] \cap \text{kernel}(\lambda) = \{1\}$. Let $x \in T(H,\lambda)$, $x \notin Z$ and n be the minimum number so that $x^n \in Z$. Then $y \in T(H,\lambda)$ if and only if either $y \in Z$ or y is conjugate to $x^s \cdot z$ for some s relatively prime to n and some $z \in Z$.*

Proof. If $x \in H$, let $\langle x\rangle$ denote the subgroup of H generated by x. Let $x \in T(H,\lambda)$, $x \notin Z$, $X = \langle x\rangle \cdot Z$ and $n = [X\colon Z]$. Let $\lambda^H = e(\phi_1 + \cdots + \phi_p)$, $\lambda^X = \sigma_1 + \cdots + \sigma_n$ and $\sigma_i^H = \sum_{j=1}^{p} k_{ji}\phi_j$.

Suppose $y \in T(H,\lambda)$, $y \notin Z$, and y is not conjugate to any element of X. Then

$$\sigma_i^H(y) = 0 = \sum_{j=1}^{p} k_{ji}\phi_j(y).$$

Since $y \in T(H,\lambda)$ and H is p-special, $k_{ji} = k_{pi}$ for all j. Hence $\sigma_i^H = k_{pi}\sum_{j=1}^{p} \phi_j$ and $\sigma_i^H(1) = [H\colon X] = k_{pi} \cdot p \cdot e$. Hence $k_{pi} = k_{pp}$ for all i and $\phi_j\,|_X = \sum_{i=1}^{n} k_{ji}\sigma_i = k_{pp}\sum_{i=1}^{n} \sigma_i = k_{pp}\lambda^X$. Hence $\phi_j(x) = 0$ for all $1 \le j \le p$ and $x \notin T(H,\lambda)$ by Theorem 1.8, which contradicts our choice of x. Thus if $y \in T(H,\lambda)$, $y \notin Z$, then y is conjugate to some element of X.

Suppose $x^s \in T(H,\lambda)$ and s and n are not relatively prime. By the same argument as in the preceding paragraph, x is conjugage to some element of $\langle x^s \rangle \cdot Z$. However, this is impossible since $[\langle x^s\rangle \cdot Z\colon Z] < [\langle x \rangle \cdot Z\colon Z]$. Therefore $x^s \in T(H,\lambda)$ only if s and n are relatively prime.

Suppose s and n are relatively prime and $x^s \notin T(H,\lambda)$. Then there is an element $h \in H$ such that $x^{-s}h^{-1}x^s h = z \in Z$ and $z \neq 1$. Since s and n are relatively prime, there is an integer t so that $st \equiv 1\,(\text{modulo } n)$ and $x^{st} = x \cdot z_1$ for some $z_1 \in Z$. Since $x^{-s}h^{-1}x^s h = z$, $z^t = (x^{-s}h^{-1}x^s h)^t = x^{-st}h^{-1}x^{st}h = x^{-1}h^{-1}xh$. If $z^t = 1$, then $z = x^{-s}h^{-1}x^s h = (x^{-1}h^{-1}xh)^s = 1$, which contradicts the choice of h. Hence $x \notin T(H,\lambda)$, which contradicts our choice of x. Therefore if s and n are relatively prime, then $x^s \in T(H,\lambda)$. If $x^s \in T(H,\lambda)$ then $x^s \cdot z \in T(H,\lambda)$ for all $z \in Z$, and any conjugate of $x^s \cdot z$ is an element of $T(H,\lambda)$.

Lemma 2.7. *Let H be a p-special group with respect to λ on the center Z and assume $[H,H] \cap \mathit{kernel}(\lambda) = \{1\}$. Then every element of $T(H,\lambda)/Z$ has order a power of p in H/Z.*

Proof. Let $\lambda^H = e(\phi_1 + \cdots + \phi_p)$. Let S_p be a p Sylow subgroup of H, $R = S_p \cdot Z$ and let γ be an irreducible constituent of λ^R. By Schur's lemma, elements of Z are represented by scalar matrices, and hence γ restricted to S_p is irreducible and $\gamma(1)$ is a power of p. Since γ^H is a constituent of λ^H, $\gamma^H = \sum_{i=1}^{p} m_i\phi_i$ for some nonnegative integers m_i. Now

$$\gamma^H(1) = [H\colon R]\gamma(1) = \sum_{i=1}^{p} m_i e$$

and

$$[H\colon R]^2\gamma(1)^2 = \left(\sum_{i=1}^{p} m_i\right)^2 e^2 = \left(\sum_{i=1}^{p} m_i\right)^2 [H\colon Z]/p.$$

By taking p-parts, we get the equation

$$\gamma(1)^2 = \left(\sum_{i=1}^{p} m_i\right)_p^2 [R\colon Z]/p.$$

Since $\gamma(1)^2 \leq [R\colon Z]$, $\gamma(1)^2 = [R\colon Z]/p$. Thus $\lambda^R = e_p(\gamma_1 + \cdots + \gamma_p)$ where $\gamma_1, \ldots, \gamma_p$ are inequivalent irreducible characters on R and $\gamma_i(1)^2 = e_p^2 = [R\colon Z]/p$ for all i.

Suppose $x \in T(H,\lambda)$ and x is not conjugate to any element of R. Then $\gamma_i^H(x) = 0$ for all i. Let $\gamma_i^H = \sum_{j=1}^{p} k_{ij}\phi_j$. Then

$$\gamma_i^H(x) = 0 = \sum_{j=1}^{p} k_{ij}\phi_j(x).$$

Since $x \in T(H,\lambda)$, $\phi_j(x) \neq 0$ for some j and since H is p-special, $k_{ij} = k_{ip}$ for all j. Hence

$$\gamma_i^H = k_{ip}(\phi_1 + \cdots + \phi_p) \quad \text{and} \quad \gamma_i^H(1) = [H: R]\gamma_i(1) = k_{ip} \cdot p \cdot e.$$

By taking p-parts, we have $e_p = \gamma_i(1) = (k_{ip})_p \cdot p \cdot e_p$ which is clearly impossible. Thus if $x \in T(H,\lambda)$, x is conjugate to an element of R. Since $R = S_p \cdot Z$, the order of xZ in H/Z is a power of p.

Theorem 2.8. *Let H be a group with center Z, and let λ be a linear character on Z. Let*

$$T(H,\lambda) = \{x \in H\colon x^{-1}\,\mathrm{Cl}_H(x) \cap Z \subseteq \mathit{kernel}(\lambda)\}.$$

If H is p-special with respect to λ then for any p Sylow subgroup S of H, there is an $x \in S$ such that

(a) $T(R,\lambda)\ \bigcup_{i=0}^{p-1} \mathrm{Cl}_R\,(x^i) \cdot Z$ *where* $R = S \cdot Z$;

(b) $T(H,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_H\,(x^i) \cdot Z$;

(c) *for $i \not\equiv 0$ (modulo p), $\mathrm{Cl}_H(x^i) \cdot Z = \mathrm{Cl}_H\,(x^j) \cdot Z$ if and only if $i \equiv j$ (modulo p).*

Proof. We prove the theorem first in the case that $[H,H] \cap \mathrm{kernel}(\lambda) = \{1\}$. Let S be a p Sylow subgroup and let $R = S \cdot Z$. As in the proof of Lemma 2.7, λ^R has p inequivalent irreducible constituents. By Theorem 1.8, $T(R,\lambda)/Z$ consists of p distinct conjugacy classes of R/Z. Let $x \in T(R,\lambda)$, $x \notin Z$. Since $R = S \cdot Z$, we can choose $x \in S \cap T(R,\lambda)$. As in the proof of Lemma 2.6, $\mathrm{Cl}_R\,(x^i) \cdot Z \subseteq T(R,\lambda)$ for all i relatively prime to p. Since R/Z is a p group, $x^r \cdot Z$ and $x^s \cdot Z$ are conjugate in R/Z only if $r \equiv s$ (modulo p). Since $T(R,\lambda)/Z$ consists of exactly p distinct conjugacy classes of H/Z,

$$T(R,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_r(x^i) \cdot Z.$$

Moreover, for $i \not\equiv 0$ (modulo p), $\mathrm{Cl}_R(x^i) \cdot Z = \mathrm{Cl}_R(x^j) \cdot Z$ if and only if $i \equiv j$ (modulo p).

Let $y \in T(H,\lambda)$, $y \notin Z$. By Lemma 2.7, yZ is a p element in H/Z, and since all p Sylow subgroups of H are conjugates, $\mathrm{Cl}_H(y) \cap R \neq \varnothing$. Let $y' \in \mathrm{Cl}_H(y) \cap R$. Then $y' \in T(H,\lambda) \cap R$ and $y' \in T(R,\lambda)$. Then $y' \in \mathrm{Cl}_R(x^i) \cdot Z$ for some $1 \leq i \leq p-1$. Hence $y' = r^{-1}x^i r \cdot z$ for some $r \in R$, $z \in Z$, and $\mathrm{Cl}_H(y') \cdot Z = \mathrm{Cl}_H(x^i) \cdot Z$. But $y \in \mathrm{Cl}_H(y')$ and hence $y \in \mathrm{Cl}_H(x^i) \cdot Z$. Thus for every $y \in T(h,\lambda)$, $y \notin Z$, $y \in \mathrm{Cl}_H(x^i) \cdot Z$ for some i. Therefore

$$T(H,\lambda) \subseteq \bigcup_{i=0}^{p-1} \mathrm{Cl}_H(x^i) \cdot Z.$$

Since λ^H has p inequivalent irreducible constituents, by Theorem 1.8, $T(H,\lambda)/Z$ consists of exactly p distinct conjugacy classes of H/Z. Therefore

$$T(H,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_H(x^i) \cdot Z$$

and

$$\mathrm{Cl}_H(x^i) \cdot Z \neq \mathrm{Cl}_H(x^j) \cdot Z$$

for $i \neq j$, $0 \leq i, j \leq p-1$. Since for $i \not\equiv 0$ (modulo p), $\mathrm{Cl}_R(x^i) \cdot Z = \mathrm{Cl}_R(x^j) \cdot Z$ if $i \equiv j$ (modulo p), then $\mathrm{Cl}_H(x^i) \cdot Z = \mathrm{Cl}_H(x^j) \cdot Z$ for $i \equiv j$ (modulo p), $i \not\equiv 0$ (modulo p). Hence, for $i \not\equiv 0$ (modulo p), $\mathrm{Cl}_H(x^i) \cdot Z = \mathrm{Cl}_H(x^j) \cdot Z$ if and only if $i \equiv j$ (modulo p). This completes the proof of Theorem 2.8 in the case that λ is faithful on $[H,H] \cap Z$.

If λ is not faithful on $[H,H] \cap Z$, let $N = [H,H] \cap \mathrm{kernel}(\lambda)$. Let $\bar{H} = H/N$, $\bar{Z} = Z/N$, and $\bar{\lambda}$ be a linear character on $\bar{Z}$ defined by $\bar{\lambda}(zN) = \lambda(z)$ for any $z \in zN$. If $\lambda^H = e(\phi_1 + \cdots + \phi_p)$, define $\bar{\phi}_i(hN) = \phi_i(h)$ for any $h \in hN$. Then $\bar{\phi}_1, \ldots, \bar{\phi}_p$ are inequivalent irreducible constituents of $\bar{\lambda}^{\bar{H}}$, each of degree e. Let $\bar{S}$ be any p Sylow subgroup of $\bar{H}$, and let $\bar{R} = \bar{S} \cdot \bar{Z}$.

If $Z(\bar{H}) \neq \bar{Z}$, let

$$\bar{\phi}_i \,|_{Z(\bar{H})} = \bar{\phi}_i(1)\sigma_i = e\sigma_i$$

where σ_i is a linear character on $Z(\bar{H})$. Then

$$\begin{aligned} \bar{\lambda}^{\bar{H}} \,|_{Z(\bar{H})} &= [\bar{H}: Z(\bar{H})]\bar{\lambda}^{Z(\bar{H})} \\ &= e(\bar{\phi}_1 \,|_{Z(\bar{H})} + \cdots + \bar{\phi}_p \,|_{Z(\bar{H})}) \\ &= e^2(\sigma_1 + \cdots + \sigma_p) \end{aligned}$$

Hence

$$\bar{\lambda}^{Z(\bar{H})} = \sigma_1 + \cdots + \sigma_p, \quad [Z(\bar{H}): \bar{Z}] = p, \quad \text{and} \quad e^2 = [\bar{H}: Z(\bar{H})].$$

Hence $\bar{H}$ is of central type. By Theorem 2 of [3], S is of central type and $Z(\bar{S}) = Z(\bar{H}) \cap \bar{S}$. Since $[Z(\bar{H}): \bar{Z}] = p$,

$$[Z(\bar{S}): \bar{Z} \cap \bar{S}] = [Z(\bar{H}) \cap \bar{S}: \bar{Z} \cap \bar{S}] = p.$$

Let $\bar{x} \in Z(\bar{S})$, $\bar{x} \notin \bar{Z}$. Since $\bar{S}$ is of central type, $\bar{R} = \bar{S} \cdot \bar{Z}$ is of central type and $Z(\bar{R}) = Z(\bar{H})$. Now $Z(\bar{R}) \subseteq T(\bar{R},\bar{\lambda})$ and since $T(\bar{R},\bar{\lambda})/\bar{Z}$ contains p conjugacy classes of $\bar{R}/\bar{Z}$ by Theorem 1.8 and $[Z(\bar{R}): \bar{Z}] = p$, $Z(\bar{R}) = T(\bar{R},\bar{\lambda})$ and $T(\bar{R},\bar{\lambda}) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_{\bar{R}}(\bar{x}^i) \cdot \bar{Z}$. Also $T(\bar{H},\bar{\lambda}) = Z(\bar{H})$ and $T(\bar{H},\bar{\lambda}) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_{\bar{H}}(\bar{x}^i) \cdot \bar{Z}$. Moreover $\mathrm{Cl}_{\bar{H}}(\bar{x}^i) \cdot \bar{Z} = \mathrm{Cl}_{\bar{H}}(\bar{x}^j) \cdot \bar{Z}$ if and only if $i \equiv j$ (modulo p).

If $Z(\overline{H}) = \overline{Z}$ and $\sum_{i=1}^{p} m_i\overline{\phi}_i(xN) = 0$ for some $xN \in \overline{H}$ and some nonnegative integers m_i, then $\sum_{i=1}^{p} m_i\phi_i(x) = 0$ for some $x \in H$. Since H is p-special, either $\phi_i(x) = 0$ for all i, or $m_i = m_j$ for all i, j. Hence $\overline{H}$ is p-special. Since $\overline{\lambda}$ is faithful on $[\overline{H},\overline{H}] \cap \overline{Z}$, for any p Sylow subgroup $\overline{S}$ of $\overline{H}$, there is an $\overline{x} \in \overline{S}$ so that

(a) $T(\overline{R},\overline{\lambda}) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_{\overline{R}}(\overline{x}^i) \cdot \overline{Z}$ where $\overline{R} = \overline{S} \cdot \overline{Z}$.

(b) $T(\overline{H},\overline{\lambda}) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_{\overline{H}}(\overline{x}^i) \cdot \overline{Z}$.

(c) For $i \not\equiv 0$ (modulo p), $\mathrm{Cl}_{\overline{H}}(\overline{x}^{-i}) \cdot \overline{Z} = \mathrm{Cl}_{\overline{H}}(\overline{x}^j) \cdot \overline{Z}$ if and only if $i \equiv j$ (modulo p). Hence, regardless of whether $Z(\overline{H}) = \overline{Z}$ or not, for any p Sylow subgroup $\overline{S}$ of $\overline{H}$, there is an $\overline{x} \in \overline{S}$, so that conditions (a), (b), and (c) are satisfied.

Let S be a p Sylow subgroup of H. If $\overline{S}$ is the natural image of S in $\overline{H}$, then $\overline{S}$ is a p Sylow subgroup of $\overline{H}$. Let $R = S \cdot Z$ and $\overline{R} = \overline{S} \cdot \overline{Z}$. Since $N \subseteq \mathrm{kernel}(\lambda)$, it can be easily verified that

$$\overline{T(H,\lambda)} = T(\overline{H},\overline{\lambda}) \quad \text{and} \quad \overline{T(R,\lambda)} = T(\overline{R},\overline{\lambda}).$$

Let $x \in S$, such that $xN = \overline{x}$ and $\overline{X} \in \overline{S}$, satisfying conditions (a), (b), and (c). Then

(a) $T(R,\lambda) = \bigcup_{i=1}^{p-1} \mathrm{Cl}_R(x^i) \cdot Z$.

(b) $T(H,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_H(x^i) \cdot Z$.

(c) For $i \not\equiv 0$ (modulo p), $\mathrm{Cl}_H(x^i) \cdot Z = \mathrm{Cl}_H(x^j) \cdot Z$. if and only if $i \equiv j$ (modulo p).

This completes the proof of Theorem 2.8.

Let H be p-special with respect to λ and let $\lambda^H = e(\phi_1 + \cdots + \phi_p)$. In the previous proofs, the condition

(b) if $\sum_{i=1}^{p} m_i\phi_i(x) = 0$ for some nonnegative integers m_i and some $x \in H$, then either $\phi_i(x) = 0$ for all i, or $m_i = m_j$ for all i, j is used often. The following example shows that this condition is necessary in Theorem 2.5.

Example 2.9. Let $H = \langle x,y,w \mid x^5 = y^4 = w^2 = 1, w^{-1}yw = y^3, y^{-1}xy = x, w^{-1}xw = x^4\rangle$. Then $Z(H) = \langle y^2\rangle$ and $[H: Z] = 20$. Let $Y = \langle xy\rangle$, let ω_0 be a 10th root of -1, and let ω be a primitive 10th root of 1. Let $\sigma_i((xy)^s) = \omega_0^s\omega^{si}$. If λ is defined on $Z(H)$ by $\lambda(y^2) = -1$, then $\lambda^Y = \sum_{i=0}^{9} \sigma_i$. Let $\phi_1, \ldots, \phi_5$ be defined on H by the following: $\phi_1|_Y = \sigma_1 + \sigma_4$, $\phi_2|_Y = \sigma_2 + \sigma_3$, $\phi_3|_Y = \sigma_7 + \sigma_8$, $\phi_4|_Y = \sigma_6 + \sigma_9$, $\phi_5|_Y = \sigma_0 + \sigma_5$, and $\phi_i(h) = 0$ for all $h \notin Y$ and $1 \le i \le 5$. Then $\phi_1, \ldots, \phi_5$ are inequivalent irreducible characters on H, each of degree 2, and $\lambda^H = 2(\phi_1 + \cdots + \phi_5)$. Notice that $\phi_5(x^sy) = 0$ while $\phi_i(x^sy) \neq 0$ for $i \neq 5$. Hence condition (b) is not satisfied.

Throughout the remainder of this section we will be working toward a converse of Theorem 2.8. We will need the following algebraic facts [4, Example 1, p. 13]:

(2.10) If p is a prime, $p \neq 2$, and α is a positive integer, then

(a) $(p + 1)^{p^{\alpha-1}} = ap^\alpha + 1$ where $a \equiv 1$ (modulo p).

(b) for every $0 \le a \le p^{\alpha-1} - 1$, there is a unique $0 \le t \le p^{\alpha-1} - 1$ so that $(p + 1)^t \equiv ap + 1$ (modulo p^α).

Lemma 2.11. *If H is a group with center Z and*

$$T(H) = \{x \in H\colon x^{-1}\,\mathrm{Cl}(x) \cap Z = \{1\}\} = \bigcup_{i=0}^{p-1} \mathrm{Cl}\,(x^i)\cdot Z$$

for some x which has order a power of p, then for all $1 \le i \le p-1$ and all positive integers a

$$\mathrm{Cl}(x^{(ap+1)i})\cdot Z = \mathrm{Cl}(x^i)\cdot Z.$$

Proof. Let s be any number relatively prime to p. If $x^s \notin T(H)$ then there is an element $h \in H$, so that $h^{-1}x^s h = x^s\cdot z$ where $z \neq 1$. Since s is relatively prime to p, there is an integer a so that $x^{as} = x$, and

$$h^{-1}x^{as}h = h^{-1}xh = x\cdot z^a.$$

Since $z \neq 1$, $z^a \neq 1$. But this implies that $x \notin T(H)$, which is a contradiction. Therefore $x^s \in T(H)$ for all s relatively prime to p.

Assume $p \neq 2$. Let α be the minimum number so that $x^{p^\alpha} \in Z$. Let A be the multiplicative group of integers, modulo p^α. Let $A_1 = \{a \in A\colon x \in \mathrm{Cl}(x^a)\cdot Z\}$. Then

(2.12) A_1 is a subgroup of A and $[A\colon A_1]$ divides $p-1$.

Proof. Suppose a, $b \in A_1$. Then there are h_1, $h_2 \in H$, z_1, $z_2 \in Z$ that $x = h_1^{-1}x^a h_1 z_1$ and $x = h_2^{-1}x^b h_2 z_2$. Then

$$\begin{aligned} x^{a^{-1}b} &= (h_1^{-1}xh_1z_1^{a^{-1}})^b = h_1^{-1}x^bh_1z_1^{a^{-1}b} \\ &= h_1^{-1}(h_2xh_2^{-1}z_2^{-1})h_1z_1^{a^{-1}b} \\ &= (h_2^{-1}h_1)^{-1}xh_2^{-1}h_1z_2^{-1}z_1^{a^{-1}b} \end{aligned}$$

or

$$x = h_2^{-1}h_1x^{a^{-1}b}(h_2^{-1}h_1)^{-1}z_2z_1^{-a^{-1}b}.$$

Thus $a^{-1}b \in A_1$ and A_1 is a subgroup of A.

If $A_i = \{a \in A\colon x^i \in \mathrm{Cl}(x^a)\cdot Z\}$, then the mapping $a \to ai$ is a one-to-one mapping of A_1 onto A_i, for $1 \le i \le p-1$. For every $a \in A$, $x^a \in T(H)$ and $x^a \notin Z$. Therefore

$$x^a \in \bigcup_{i=1}^{p-1} \mathrm{Cl}(x^i)\cdot Z,\quad A = \bigcup_{i=1}^{p-1} A_i,\quad \text{and}\quad [A\colon 1] \le (p-1)[A_1\colon 1].$$

Therefore $[A\colon A_1] \le p-1$. Since $[A\colon 1] = (p-1)p^{\alpha-1}$, $[A\colon A_1]$ divides $p-1$. This completes the proof of (2.12).

Since $[A\colon A_1]$ divides $p-1$, for every $a \in A$, $a^{p-1} \in A_1$. Since

$$\{(p+1)^{t(p-1)}\colon 0 \le t \le p^{\alpha-1}-1\} = \{(p+1)^t\colon 0 \le t \le p^{\alpha-1}-1\},$$

for every t, there is t_1 so that $(p+1)^t = (p+1)^{t_1(p-1)}$. Therefore $(p+1)^t \in A_1$ for $0 \le t \le p^{\alpha-1}-1$. By (2.10)(b) for every $0 \le a \le p^{\alpha-1}-1$, $ap+1 \in A_1$. Hence $x \in \mathrm{Cl}(x^{ap+1}) \cdot Z$ and $\mathrm{Cl}(x) \cdot Z = \mathrm{Cl}(x^{ap+1}) \cdot Z$. Since $(ap+1) \in A_1$, $(ap+1)i \in A_i$. Hence $x^i \in \mathrm{Cl}(x^{(ap+1)i}) \cdot Z$ or $\mathrm{Cl}(x^i) \cdot Z = \mathrm{Cl}(x^{(ap+1)i}) \cdot Z$.

If $p = 2$, then $\mathrm{Cl}(x^0) \cdot Z \cap \mathrm{Cl}(x^1) \cdot Z = \varnothing$. For any a, since $x^{2a+1} \in T(H)$, $x^{2a+1} \in \mathrm{Cl}(x) \cdot Z$. Therefore

$$\mathrm{Cl}(x) \cdot Z = \mathrm{Cl}(x^{2a+1}) \cdot Z.$$

We now prove the following crucial lemma:

Lemma 2.13. *Let H be a group with center Z and assume $[H,H] \cap Z$ is cyclic. Let λ be a linear character on Z, with λ faithful on $[H,H] \cap Z$. Assume $\lambda^H = e(\phi_1 + \cdots + \phi_p)$ where $\phi_1, \ldots, \phi_p$ are inequivalent irreducible characters on H and $T(H,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}(x^i) \cdot Z$ for some $x \in H$, where xZ has order a power of p. Then H is p-special.*

Proof. By Schur's lemma, $\phi_i|_Z = \phi_i(1)\lambda$ and since $(\phi_i, \lambda^H) = e$, $\phi_i(1) = e$, $1 \le i \le p$. If $p = 2$, assume $m_1\phi_1(y) + m_2\phi_P(y) = 0$ for some $y \in H$ and nonnegative integers m_1 and m_2. If $y \notin T(H,\lambda)$, then $\phi_i(y) = 0$, $i = 1, 2$, by Theorem 1.8. If $y \in Z$, then $\phi_i(y) = e\lambda(y)$ and $0 = m_1\phi_1(y) + m_2\phi_2(y) = e(m_1+m_2)\lambda(y)$. Thus $m_1 = m_2 = 0$. If $y \in T(H,\lambda)$, $y \notin Z$, then $\phi_i(y) \ne 0$ for $i = 1$ or $i = 2$. Since

$$0 = \lambda^H(y) = e(\phi_1(y) + \phi_2(y)),$$

$\phi_2(y) = -\phi_1(y)$. Then

$$0 = m_1\phi_1(y) + m_2\phi_2(y) = (m_1 - m_2)\phi_1(y).$$

Since $\phi_1(y) \ne 0$, $m_1 = m_2$. Thus H is p-special if $p = 2$.

Assume $p \ne 2$. Let α be the minimum number so that $x^{p^\alpha} \in Z$. Let ω_0 be any p^αth root of $\lambda(x^{p^\alpha})$ and let ω be a primitive p^αth root of 1. Define $\sigma_i(x^s \cdot z) = \omega_0^s\lambda(z) \cdot \omega^{is}$. If $X = \langle x \rangle \cdot Z$, then σ_i is independent of the way elements of X are represented, and σ_i, $0 \le i \le p^\alpha - 1$, is a linear character on X. Since $\sigma_i(z) = \lambda(z)$ for all $z \in Z$, $(\sigma_i, \lambda^X) = 1$. Hence

$$\lambda^X = \sum_{v=0}^{p^\alpha-1} \sigma_v.$$

We show the following:

(2.14) For a suitable ω_0, there are integers K and $k_{u,j}$, $0 \le u \le p-1$, $1 \le j \le p$, such that

$$\phi_j|_X = \sum_{u=0}^{p-1} k_{uj}\sigma_{p^{\alpha-1}u} + K\sum_{v=0}^{p^\alpha-1} \sigma_v.$$

Moreover, $K = e/p^{\alpha}$ if $\alpha \neq 1$.

Proof. If $\alpha = 1$, then $\lambda^X = \sigma_0 + \cdots + \sigma_{p-1}$. Since

$$\lambda^H|_X = [H: X]\lambda^X = e\left(\sum_{j=1}^{p} \phi_j|_X\right) = [H: X]\left(\sum_{i=0}^{p-1} \sigma_i\right),$$

then $\phi_j|_X = \sum_{u=0}^{p-1} k_{uj}\sigma_u$ for integers k_{uj}. If $K = 0$, then (2.14) follows if $\alpha = 1$.

If $\alpha \geq 2$, by Lemma 2.11, $x^{p+1} \in \text{Cl}(x) \cdot Z$. Hence there is $h \in H$, $z_0 \in Z$, so that

$$h^{-1}xh = x^{p+1}z_0. \tag{2.15}$$

We wish to compute $h^{-t}xh^t$.

(2.16) If $h^{-1}xh = x^{\beta} \cdot z$ for any integer β and $z \in Z$, then

$$h^{-t}xh^t = x^{\beta^t}z^{e(t)}$$

where $e(t) = (\beta^t - 1)/(\beta - 1)$.

Proof. If $t = 1$, the assertion follows by hypothesis. Suppose the assertion is true for $t = k$. Then

$$\begin{aligned} h^{-(k+1)}xh^{(k+1)} &= h^{-1}h^{-k}xh^kh \\ &= h^{-1}x^{\beta^k}hz^{e(k)} = (h^{-1}xh)^{\beta^k}z^{e(k)} \\ &= (x^{\beta}z)^{\beta^k}z^{e(k)} = x^{\beta^{k+1}}z^{\beta^k+e(k)} \end{aligned}$$

and $\beta^k + e(k) = \beta^k + (\beta^k - 1)/(\beta - 1) = (\beta^{k+1} - 1)/(\beta - 1) = e(k + 1)$. By induction the assertion holds for all positive integers t.

By (2.10), $(p + 1)^{p^{\alpha-1}} = ap^{\alpha} + 1$ where $a \equiv 1$ (modulo p). From equation (2.15) we get

$$\begin{aligned} h^{-p^{\alpha-1}}xh^{p^{\alpha-1}} &= x^{(p+1)^{p^{\alpha-1}}}z_0^{e(p^{\alpha-1})} \\ &= x^{ap^{\alpha}+1} \cdot z_0^{ap^{\alpha-1}} \\ &= x(x^{p^{\alpha}}z_0^{p^{\alpha-1}})^a. \end{aligned}$$

Since $x \in T(H,\lambda)$, $(x^{p^{\alpha}}z_0^{p^{\alpha-1}})^a = 1$. Since $a \equiv 1$ (modulo p), $x^{p^{\alpha}}z_0^{p^{\alpha-1}} = 1$. Choose ω_0 such that $\lambda(z_0) = \omega_0^{-p}$.

If $s \not\equiv 0$ (modulo p) and $g^{-1}x^sg \in X$, then $g^{-1}xg \in X$ and $g^{-1}xg = x^iz$ for some $z \in Z$. Since λ^H has p inequivalent irreducible constituents, by Theorem 1.8, $T(H,\lambda)/Z$ consists of p distinct conjugacy classes of H/Z. Since

$$T(H,\lambda) = \bigcup_{i=0}^{p-1} \text{Cl}(x^i) \cdot Z,$$

$\mathrm{Cl}(x^i)\cdot Z \neq \mathrm{Cl}(x^j)\cdot Z$ if $i \neq j$, $0 \leq i,j \leq p-1$. Thus $\mathrm{Cl}(x^i)\cdot Z \cap \mathrm{Cl}(x^j)\cdot Z = \varnothing$ if $i \neq j$, $0 \leq i,j \leq p-1$. Since $g^{-1}xg = x^i z$, $i \equiv 1$ (modulo p) by Lemma 2.11. Then $i = ap+1$ for some $0 \leq a \leq p^{\alpha-1}-1$ and by (2.10) there is an integer t such that

$$i \equiv (p+1)^t \qquad (\text{modulo } p^\alpha).$$

By (2.15) and (2.16), $h^{-t}xh^t = x^{(p+1)^t}\cdot z_0^{e(t)}$ and $g^{-1}xg = x^i z = x^{(p+1)^t}\cdot z' = h^{-t}xh^t z_0^{-e(t)}\cdot z'$, for some $z' \in Z$. Hence

$$h^t g^{-1} x g h^{-t} = x z_0^{-e(t)}\cdot z'.$$

Since $x \in T(H,\lambda)$, $z_0^{-e(t)}z' = 1$ and

$$gh^{-t} \in C = \{h \in H\colon hx = xh\}.$$

Thus if $g^{-1}x^s g \in X$ for any $s \not\equiv 0$ (modulo p) then $g = ch^t$ for some $c \in C$ and $0 \leq t \leq p^{\alpha-1}-1$. Then for $s \not\equiv 0$ (modulo p)

$$\begin{aligned}
\sigma_i^H(x^s) &= (1/[X\colon 1]) \sum_{g\in H} \dot\sigma_i(g^{-1}x^s g)\\
&= (1/[X\colon 1]) \sum_{c\in C}\sum_{t=0}^{p^{\alpha-1}-1} \sigma_i(h^{-t}c^{-1}x^s c h^t)\\
&= [C\colon X]\sum_{t=0}^{p^{\alpha-1}-1} \sigma_i(x^{(p+1)^t s}\cdot z_0^{e(t)\cdot s})\\
&= [C\colon X]\sum_{t=0}^{p^{\alpha-1}-1} \omega_0^{(p+1)^t\cdot s}\,\omega_0^{-pe(t)\cdot s}\,\omega^{i(p+1)^t s}\\
&= [C\colon X]\sum_{t=0}^{p^{\alpha-1}-1} \omega_0^s\,\omega^{i(p+1)^t s}.
\end{aligned}$$

By (2.10) as t goes from 0 to $p^{\alpha-1}-1$, if $(p+1)^t = ap+1$, a varies from 0 to $p^{\alpha-1}-1$ (modulo $p^{\alpha-1}$). Thus

$$\begin{aligned}
\sigma_i^H(x^s) &= [C\colon X]\sum_{t=0}^{p^{\alpha-1}-1} \omega_0^s\,\omega^{i(p+1)^t s}\\
&= [C\colon X]\omega_0^s \sum_{a=0}^{p^{\alpha-1}-1} \omega^{i(ap+1)s}\\
&= [C\colon X]\omega_0^s\,\omega^{is} \sum_{a=0}^{p^{\alpha-1}-1} (\omega^{ips})^a.
\end{aligned}$$

If $i \not\equiv 0$ (modulo $p^{\alpha-1}$), since $s \not\equiv 0$ (modulo p) and $\alpha \geq 2$, $\omega^{isp} \neq 1$. Then

$$\sum_{a=0}^{p^{\alpha-1}-1} (\omega^{isp})^a = (1-\omega^{isp^\alpha})/(1-\omega^{isp}) = 0$$

since ω is a p^αth root of 1.

Since $\lambda^H = \sum_{i=0}^{p^\alpha-1} \sigma_i^H = e(\phi_1 + \cdots + \phi_p)$,

$$\sigma_i^H = \sum_{j=1}^{p} n_{ij}\phi_j$$

for some nonnegative integers n_{ij}. Consider σ_i^H, for $i \not\equiv 0$ (modulo $p^{\alpha-1}$). If $y \in H$, $\sigma_i^H(y) = 0$ for all $y \notin T(H,\lambda)$ since each $\phi_j(y) = 0$ by Theorem 1.8. If $y \in T(H,\lambda)$, $y \notin Z$, then y is conjugate to $x^s \cdot z$ for some $1 \leq s \leq p-1$ and some $z \in Z$. However

$$\sigma_i^H(x^s \cdot z) = \sigma_i^H(x^s) \cdot \lambda(z) = 0.$$

Since σ_i^H is a class function, $\sigma_i^H(y) = 0$ for all $y \in H$, $y \notin Z$. If $y \in Z$, $\sigma_i^H(y) = [H\colon X]\lambda(y)$. Hence σ_i^H is a multiple of λ^H or

$$\sigma_i^H = (e/p^\alpha)(\phi_1 + \cdots + \phi_p).$$

Thus $n_{ij} = e/p^\alpha$ for all $1 \leq j \leq p$ and all $i \not\equiv 0$(modulo $p^{\alpha-1}$). Let $K = e/p^\alpha$. Since $\lambda^H|_X = e(\phi_1|_X + \cdots + \phi_p|_X) = [H\colon X]\lambda^X = [H\colon X]\sum_{i=0}^{p^\alpha-1}\sigma_i$, we have

$$\begin{aligned}
\phi_j|_X &= \sum_{i=0}^{p^\alpha-1} n_{ij}\sigma_i \\
&= \sum_{u=0}^{p-1}\sum_{v=0}^{p^{\alpha-1}-1} (n_{up^{\alpha-1}+v,j})\sigma_{up^{\alpha-1}+v} \\
&= \sum_{u=0}^{p-1} (n_{up^{\alpha-1},j})\sigma_{up^{\alpha-1}} + \sum_{u=0}^{p-1}\sum_{v=1}^{p^{\alpha-1}-1} (n_{up^{\alpha-1}+v,j})\sigma_{up^{\alpha-1}+v} \\
&= \sum_{u=0}^{p-1} (n_{up^{\alpha-1},j})\sigma_{up^{\alpha-1}} + K\sum_{u=0}^{p-1}\sum_{v=1}^{p^{\alpha-1}-1} \sigma_{up^{\alpha-1}+v} \\
&= \sum_{u=0}^{p-1} (n_{up^{\alpha-1},j} - K)\sigma_{up^{\alpha-1}} + K\sum_{v=0}^{p^\alpha-1} \sigma_v.
\end{aligned}$$

Let $k_{uj} = (n_{up^{\alpha-1},j}) - K$. Then (2.14) follows.

To show H is p-special, suppose $\sum_{j=1}^{p} m_j\phi_j(y) = 0$ for some $y \in H$ and nonnegative integers m_j. If $y \notin T(H,\lambda)$, $\phi_j(y) = 0$ for all j by Theorem 1.8. If $y \in Z$, then

$$0 = \sum_{j=1}^{p} m_j\phi_j(y) = \sum_{j=1}^{p} m_j \cdot e\lambda(y).$$

Thus $m_j = 0$ for all $1 \leq j \leq p$. If $y \in T(H)$, $y \notin Z$, then y is conjugate to $x^s \cdot z$ for some $1 \leq s \leq p-1$, and $z \in Z$. Then

$$\sum_{j=1}^{p} m_j\phi_j(x^s \cdot z) = 0.$$

Using (2.14), we find

$$\begin{aligned}0 &= \sum_{j=1}^{p} m_j \phi_j(x^s \cdot z)\\ &= \sum_{j=1}^{p} m_j\left(\sum_{u=0}^{p-1} k_{uj}\sigma_{p^{a-1}u}(x^s \cdot z)\right) + K\left(\sum_{j=1}^{p} m_j\right)\sum_{v=0}^{p^a-1}\sigma_v(x^s \cdot z).\end{aligned}$$

Since $1 \leq s \leq p-1$,

$$\sum_{v=0}^{p^a-1}\sigma_v(x^s \cdot z) = \lambda^X(x^s \cdot z) = 0.$$

Hence

$$\begin{aligned}0 &= \sum_{u=0}^{p-1}\left(\sum_{j=1}^{p} m_j k_{uj}\right)\sigma_{p^{a-1}u}(x^s \cdot z)\\ &= \sum_{u=0}^{p-1}\left(\sum_{j=1}^{p} m_j k_{uj}\right)\omega_0^s\lambda(z)\cdot\omega^{p^{a-1}us}\end{aligned}$$

and

$$0 = \sum_{u=0}^{p-1}\left(\sum_{j=1}^{p} m_j k_{uj}\right)\omega^{p^{a-1}us}.$$

Since each $\omega^{p^{a-1}us}$ is a pth root of 1, the above equation implies that $\sum_{j=1}^{p} m_j k_{uj} = \sum_{j=1}^{p} m_j k_{0j}$ for all $0 \leq u \leq p-1$. Then

$$\begin{aligned}\sum_{j=1}^{p} m_j\phi_j\,|_X &= \sum_{u=0}^{p-1}\left(\sum_{j=1}^{p} m_j k_{uj}\right)\sigma_{p^{a-1}u} + K\left(\sum_{j=1}^{p} m_j\right)\left(\sum_{v=0}^{p^a-1}\sigma_v\right)\\ &= \left(\sum_{j=1}^{p} m_j k_{0j}\right)\left(\sum_{u=0}^{p-1}\sigma_{p^{a-1}u}\right) + K\left(\sum_{j=1}^{p} m_j\right)\left(\sum_{v=0}^{p^a-1}\sigma_v\right).\end{aligned}$$

For all $1 \leq s \leq p-1$ and $z \in Z$, $\sum_{j=1}^{p} m_j\phi_j(x^s \cdot z) = 0$. Since $T(H,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}(x^i)\cdot Z$, $\sum_{j=1}^{p} m_j\phi_j(y) = 0$ for all $y \notin Z$. Thus $\sum_{j=1}^{p} m_j\phi_j$ is a multiple of λ^H or

$$\sum_{j=1}^{p} m_j\phi_j = \left(\sum_{j=1}^{p} m_j/p\right)(\phi_1 + \cdots + \phi_p).$$

Thus $m_j = (\sum_{j=1}^{p} m_j)/p$ or $m_j = m_i$ for all i, j. Hence H is p-special.

We can describe the Sylow subgroups of p-special groups.

Theorem 2.17. *Let H be a p-special group. Then*

(a) *for any prime $q \neq p$, and any q Sylow subgroup S_q of H, S_q is of central type with $Z(S_q) = Z \cap S_q$.*

(b) *If S_p is any p Sylow subgroup of H, then either S_p is of central type with $[Z(S_p)\colon Z \cap S_p] = p$ or S_p is p-special with $Z(S_p) = Z \cap S_p$.*

Proof. Let λ be a linear character on Z, such that H is p-special with respect to λ. Let $\lambda^H = e(\phi_1 + \cdots + \phi_p)$ where $\phi_1, \ldots, \phi_p$ are inequivalent irreducible characters on H. Let q be any prime, let S_q be a q Sylow subgroup of H, and let $R_q = S_q \cdot Z$. Let

$$\lambda^{R_q} = \gamma_1(1)\gamma_1 + \cdots + \gamma_s(1)\gamma_s$$

where $\gamma_1, \ldots, \gamma_s$ are inequivalent irreducible characters on R_q. Since $\gamma_i|_{S_q}$ is irreducible, $\gamma_i(1)$ is a power of q. Since γ_i is a constituent of λ^{R_q},

$$\gamma_i^H = \sum_{j=1}^{p} k_{ij}\phi_j$$

for some integers k_{ij}. Then

$$\gamma_i^H(1) = [H\colon R]\gamma_i(1) = \sum_{j=1}^{p} k_{ij}\phi_j(1) = e\left(\sum_{j=1}^{p} k_{ij}\right).$$

By taking q parts we get

$$\gamma_i(1) = e_q\left(\sum_{j=1}^{p} k_{ij}\right)_q.$$

If $q \neq p$, since $e^2 = [H\colon Z]/p$, we get

$$e_q^2 = ([H\colon Z]/p)_q = [H\colon Z]_q = [R_q\colon Z].$$

Hence $\gamma_i(1)^2 \geq [R_q\colon Z]$. However

$$\gamma_i(1)^2 \leq [R_q\colon Z(R_q)] \leq [R_q\colon Z].$$

Hence $\gamma_i(1)^2 = [R_q\colon Z(R_q)]$, $Z(R_q) = Z$, and R_q is of central type. Thus S_q is of central type and $Z(S_q) = Z \cap S_q$.

If $q = p$, then

$$e_p^2 = ([H\colon Z]/p)_p = ([H\colon Z]_p)/p = [R_p\colon Z]/p.$$

Since $\gamma_i(1)^2 \leq [R_p\colon Z(R_p)] \leq [R_p\colon Z]$ and

$$\gamma_i(1)^2 = e_p^2\left(\sum_{j=1}^{p} k_{ij}\right)_p^2 = ([R_p\colon Z]/p)\left(\sum_{j=1}^{p} k_{ij}\right)_p^2,$$

we have that $(\sum_{j=1}^{p} k_{ij})_p^2 = 1$. Hence

$$\gamma_i(1)^2 = [R_p\colon Z]/p \quad \text{for all } i.$$

Let $e_p^2 = [R_p: Z]/p$. Then $\lambda^{R_p} = e_p(\gamma_1 + \cdots + \gamma_p)$. If $Z(R_p) \neq Z$, $[Z(R_p): Z]$ must be a power of p. Since $\gamma_i(1)^2 = [R_p: Z]/p$, $[Z(R_p): Z] = p$. Then $\gamma_i(1)^2 = [R_p: Z(R_p)]$ for each i and R_p is of central type. Hence in the case that $Z(R_p) \neq Z$, S_p is of central type with $[Z(S_p): Z \cap S_p] = p$.

Assume $Z(R_p) = Z$. As in the proof of Theorem 2.8, there is an $x \in S_p$, such that $T(R_p,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_{R_p}(x^i) \cdot Z$. Since $\lambda^{R_p} = e_p(\gamma_1 + \cdots + \gamma_p)$, if λ is faithful on $[R_p, R_p] \cap Z$, by Lemma 2.13, R_p is p-special. If λ is not faithful on $[R_p, R_p] \cap Z$, let $N = [R_p, R_p] \cap \mathrm{kernel}(\lambda)$. Let $\overline{R}_p = R_p/N$, $\overline{Z} = Z/N$, $\overline{\lambda}(zN) = \lambda(z)$ for any $z \in zN$, and $\overline{\gamma}_i(rN) = \gamma_i(r)$ for any $r \in rN$. Then

$$\overline{\lambda}^{\overline{R}_p} = e_p(\overline{\gamma}_1 + \cdots + \overline{\gamma}_p) \quad \text{and} \quad T(\overline{R}_p,\overline{\lambda}) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_{\overline{R}_p}(\overline{x}^i) \cdot \overline{Z}.$$

By Lemma 2.13, $\overline{R}_p$ is p-special. Suppose $\sum_{j=1}^{p} m_j \gamma_j(y) = 0$ for some $y \in R_p$ and nonnegative integers m_j. Then

$$0 = \sum_{j=1}^{p} m_j \overline{\gamma}_j(\overline{y})$$

and since $\overline{R}_p$ is p-special, either $\overline{\gamma}_j(\overline{y}) = 0$ for all j, or $m_i = m_j$ for all i, j. Hence, either $\gamma_j(y) = 0$ for all j, or $m_i = m_j$ for all i, j and R_p is p-special. Thus, if $Z(R_p) = Z$, then R_p is p-special. Hence, S_p is p-special with $Z(S_p) = Z \cap S_p$.

We can describe simply which of the two possibilities in (b) occurs in the case that $p \neq 2$. Example 2.3 shows that this characterization does not hold when $p = 2$.

Corollary 2.18. *Let H be a group with center Z. Assume $[H,H] \cap Z$ is cyclic and λ is a linear character on Z, faithful on $[H,H] \cap Z$. Let $T(H) = T(H,\lambda) = \{h \in H: h^{-1}\,\mathrm{Cl}_H(h) \cap Z = \{1\}\}$. Assume H is p-special with respect to λ for some prime $p \neq 2$. If $x \in T(H)$, $x \notin Z$, and $X = \langle x \rangle \cdot Z$, then X is not normal in H. If S is a p Sylow subgroup of H, $R = S \cdot Z$, Let*

$$T(R) = T(R,\lambda) = \{r \in R: r^{-1}\,\mathrm{Cl}_R(r) \cap Z = \{1\}\}.$$

Let $x \in T(R)$, $x \notin Z$. Then R is of central type if and only if $X = \langle x \rangle \cdot Z$ is normal in R.

Proof. Since H is p-special with respect to λ, by Theorem 2.8 there is $x_0 \in S$ so that

(a) $T(R) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_R(x_0^i) \cdot Z$;

(b) $T(H) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_H(x_0^i) \cdot Z$.

By Lemma 2.10, for all integers a and all $1 \leq i \leq p-1$,

$$\mathrm{Cl}_R(x_0^{(ap+1)i}) \cdot Z = \mathrm{Cl}_R(x_0^i) \cdot Z$$

and

$$\mathrm{Cl}_H(x_0^{(ap+1)i}) \cdot Z = \mathrm{Cl}_H(x_0^i) \cdot Z.$$

Also $\lambda^H = e(\phi_1 + \cdots + \phi_p)$ where $\phi_1, \ldots, \phi_p$ are inequivalent irreducible characters on H and $\lambda^R = e_p(\gamma_1 + \cdots + \gamma_p)$ where $\gamma_1, \ldots, \gamma_p$ are inequivalent irreducible characters on R.

Let α be the minimum number such that $x_0^{p^\alpha} \in Z$. If $x \in T(R)$, $x \notin Z$, then $x = r^{-1}x_0^i rz$ for some $r \in R$, $z \in Z$, $1 \leq i \leq p^\alpha - 1$, with i relatively prime to p. Let $X_0 = \langle x_0 \rangle \cdot Z$. If X is normal in R, then $X \doteq X_0$, and for each i, $\mathrm{Cl}_R(x_0^i) \subseteq X_0$. Hence

$$T(R) \subseteq X_0 = X.$$

Similarly, if there is an $x \in T(H)$, $x \notin Z$, such that $X = \langle x \rangle \cdot Z$ is normal in H, then

$$T(H) \subseteq X.$$

Since λ^R has p inequivalent irreducible constituents, $T(R)/Z$ contains p distinct conjugacy classes. Hence $\mathrm{Cl}_R(x) \cdot Z = \mathrm{Cl}_R(x^i) \cdot Z$ only if $i \equiv 1\,(\text{modulo}\, p)$. Similarly $\mathrm{Cl}_H(x) \cdot Z = \mathrm{Cl}_H(x^i) \cdot Z$ only if $i \equiv 1\,(\text{modulo}\, p)$. To avoid doing the same argument twice, we prove the following:

(2.19) Let G be a group with center Z. Assume $[G,G] \cap Z$ is cyclic and λ is a linear character on Z, faithful on $[G,G] \cap Z$. Suppose $\lambda^G = e(\zeta_1 + \cdots + \zeta_p)$ where $\zeta_1, \ldots, \zeta_p$ are inequivalent irreducible characters on G. Let

$$T(G) = \{g \in G\colon g^{-1}\,\mathrm{Cl}_G(g) \cap Z = \{1\}\}$$

and assume $T(G) \neq Z$. If $x \in T(G)$, such that $\mathrm{Cl}_G(x) \cdot Z = \mathrm{Cl}_G(x^j) \cdot Z$ if and only if $j \equiv 1\,(\text{modulo}\, p)$, then $T(G) \nsubseteq \langle x \rangle \cdot Z$.

Proof. Suppose there is an $x \in T(G)$ so that $T(G) \subseteq \langle x \rangle \cdot Z = X$, and $\mathrm{Cl}_G(x) \cdot Z = \mathrm{Cl}_G(x^j) \cdot Z$ if and only if $j \equiv 1\,(\text{modulo}\, p)$.

By Theorem 1.8, $T(G)/Z$ contains p conjugacy classes and thus

$$T(G) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_G(x^i) \cdot Z.$$

Let α be the minimum number so that $x^{p^\alpha} \in Z$. Let ω_0 be a p^αth root of $\lambda(x^{p^\alpha})$ and let ω be a primitive p^αth root of 1. Define $\sigma_i(x^s \cdot z) = \omega_0^s \omega^{si} \lambda(z)$. As in the proof of Lemma 2.12, $\lambda^X = \sum_{i=0}^{p^\alpha-1} \sigma_i$. Since $p \neq 2$, by (2.14) there are integers K and k_{uj}, $0 \leq u \leq p-1$, $1 \leq j \leq p$, such that

$$\zeta_j\,|_X = \sum_{u=0}^{p-1} k_{uj}\sigma_{p^{\alpha-1}u} + K \sum_{v=0}^{p^\alpha-1} \sigma_v.$$

If $\alpha \neq 1$, $K = e/p^\alpha$ and

$$\zeta_j(1) = e = \left(\sum_{u=0}^{p-1} k_{uj}\right) + Kp^\alpha = \left(\sum_{u=0}^{p-1} k_{uj}\right) + e.$$

Hence $\sum_{u=0}^{p-1} k_{uj} = 0$. Since $T(G) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_G(x^i) \cdot Z \subseteq X$, $g^{-1}x^ig \in X$ for every $g \in G$ and every i. Therefore X is normal in G. By Clifford's theorem [4, Theorem 49.2, p. 343] since

$$\zeta_j\,|_X = \sum_{u=0}^{p-1} k_{uj}\sigma_{p^{\alpha-1}u} + K\sum_{v=0}^{p^\alpha-1} \sigma_v$$

for every $0 \leq u \leq p-1$, either $k_{uj} + K = 0$ or $k_{uj} = 0$. Since $\sum_{u=0}^{p-1} k_{uj} = 0$, $k_{uj} = 0$ for every u, and every j. Then

$$\zeta_j\,|_X = K\sum_{v=0}^{p^\alpha-1} \sigma_v$$

and $\zeta_j(x) = 0$ for every j. By Theorem 1.8, $x \notin T(G)$ which is a contradiction. Therefore $\alpha = 1$. If $g \in G$, $g^{-1}xg \in X$ and hence $g^{-1}xg = x^i \cdot z$. Since $\mathrm{Cl}_G(x) \cdot Z = \mathrm{Cl}_G(x^j) \cdot Z$ only if $j \equiv 1$ (modulo p), $i \equiv 1$ (modulo p). Since $\alpha = 1$, we can assume $i = 1$. Since $x \in T(G)$, and $g^{-1}xg = x \cdot z$, $z = 1$. Therefore for all $g \in G$, $g^{-1}xg = x$ and $x \in Z(G)$. Since $Z \subseteq T(G) \subseteq X \subseteq Z(G) = Z$, $T(G) = Z$, which contradicts the hypothesis. This completes the proof of (2.19).

Returning to the proof of Corollary 2.18, we have that if there is an $x \in T(H)$, $x \notin Z$ such that $\langle x\rangle \cdot Z$ is normal in H, then $T(H) \subseteq \langle x\rangle \cdot Z$ and by (2.19), $T(H) = Z(H)$ which is impossible, since $x \notin Z = Z(H)$. If there is an $x \in T(R)$, $x \notin Z$, such that $\langle x\rangle \cdot Z$ is normal in H, then $T(R) \subseteq \langle x\rangle \cdot Z$ and by (2.19), $T(R) = Z(R)$. If $T(R) = Z(R)$, then $Z(R) \neq Z$ and R is of central type with $[Z(R){:}\ Z] = p$.

If R is of central type, then $[Z(R){:}\ Z] = p$. Since λ^R has p inequivalent irreducible constituents, by Theorem 1.8, $T(R)/Z$ contains p conjugacy classes of R/Z. Clearly, $Z(R) \subseteq T(R)$. Thus $Z(R) = T(R)$. If $x \in T(R)$, $x \notin Z$, then $X = \langle x\rangle \cdot Z = Z(R)$ and X is normal in R.

There is a close relationship between the structure of $T(H,\lambda)$ and the structure of the Sylow subgroups of H.

Theorem 2.20. *Let H be a group with center Z and assume that $[H,H] \cap Z$ is cyclic. Let*

$$T(H) = \{x \in H{:}\ x^{-1}\,\mathrm{Cl}_H(x) \cap Z = \{1\}\}.$$

Let q be any prime and let S be any q Sylow subgroup of H. Then S is of central type with $Z(S) = S \cap Z$ if and only if $T(H)/Z$ contains no element of order a power of q.

Proof. Let S be a q Sylow subgroup of H, and suppose S is of central type with $Z(S) = S \cap Z$. If $R = S \cdot Z$, then R is of central type and $Z(R) = Z$. Let x be any element of H, $x \notin Z$ so that xZ has order a power of q. Since all q Sylow subgroups of H are conjugates, there is a conjugate R' of R so that $x \in R'$. Since $[H,H] \cap Z$ is cyclic, $[R',R'] \cap Z$ is cyclic. Since R' is isomorphic to R, R' is of central type. By Theorem 4 of [6], $T(R') = Z(R')$. Since $Z(R') = Z$ and $x \notin Z$, there is an $r \in R'$ so that $r^{-1}xr = x \cdot z$, $z \neq 1$, $z \in Z$. Thus $x \notin T(H)$.

Conversely, suppose $T(H)/Z$ contains no element of order a power of q. Let λ be a linear character on Z, such that λ is faithful on $[H,H] \cap Z$. Let $\lambda^H = \phi_1(1)\phi_1 + \cdots + \phi_t(1)\phi_t$ where $\phi_1, \ldots, \phi_t$ are inequivalent irreducible characters on H. Let S be any q Sylow subgroup of H, $R = S \cdot Z$ and let $\lambda^R = \gamma_1(1)\gamma_1 + \cdots + \gamma_s(1)\gamma_s$ where $\gamma_1, \ldots, \gamma_s$ are inequivalent irreducible characters on R. Since γ_j is a constituent of λ^R and $\lambda^H |_R = [H\colon R]\lambda^R$,

$$\gamma_j^H = \sum_{i=1}^{t} k_{ij}\phi_i, \qquad 1 \leq j \leq s, \tag{2.21}$$

and

$$\phi_i |_R = \sum_{j=1}^{s} k_{ij}\gamma_j, \qquad 1 \leq i \leq t, \tag{2.22}$$

for some nonnegative integers k_{ij}. Let K_i be the greatest common divisor of k_{ij}, $1 \leq j \leq s$, and let $k_{ij} = K_i k'_{ij}$. Let M be the least common multiple of $\phi_i(1)$, $1 \leq i \leq t$.

Since $T(H)/Z$ does not contain a q element, if $r \in R$, $r \notin Z$, then $r \notin T(H)$ and hence, by Theorem 1.8, $\phi_i(r) = 0$ for all i. Thus for all $r \in R$, and all i, u

$$(M/\phi_i(1))\phi_i(r) = (M/\phi_u(1))\phi_u(r)$$

or

$$(M/\phi_i(1))\phi_i |_R = (M/\phi_u(1))\phi_u |_R .$$

Thus, by equation (2.21), $(M/\phi_i(1))k_{ij} = (M/\phi_u(1))k_{uj}$ for all i, j, u, or $(M/\phi_i(1))K_i k'_{ij} = (M/\phi_u(1))K_u k'_{uj}$. Thus

$$k'_{ij} = (\phi_i(1)K_u/\phi_u(1)K_i)k'_{uj}.$$

Since k'_{ij}, $1 \leq j \leq s$, have no common divisors, and k'_{uj}, $1 \leq j \leq s$, have no common divisors

$$\phi_i(1)K_u = \phi_u(1)K_i \quad \text{or} \quad \phi_u(1)/K_u = \phi_i(1)/K_i \quad \text{for all } i, u.$$

From equation (2.22), we have

$$\phi_i |_R = K_i \sum_{u=1}^{s} k'_{ij}\gamma_j$$

and hence K_i divides $\phi_i(1)$. Thus

$$\phi_i(1)/K_i = L \tag{2.23}$$

for all i where L is an integer independent of i. For every i,

$$\sum_{j=1}^{s} k'_{ij}\gamma_j(r) = 0 \quad \text{if } r \notin Z$$

and

$$\sum_{j=1}^{s} k'_{ij}\gamma_j(r) = L\lambda(r) \quad \text{if } r \in Z.$$

Hence $\sum_{j=1}^{s} k'_{ij}\gamma_j$ is a multiple of λ^R or

$$\sum_{j=1}^{s} k'_{ij}\gamma_j = (L/[R\colon Z])(\gamma_1 = (1)\gamma_1 + \cdots + \gamma_s(1)\gamma_s).$$

Thus $k'_{ij} = (L/[R\colon Z])\gamma_j(1)$ for all i, j. Let q^α be the minimum value of $\gamma_j(1)$, $1 \leq j \leq s$. Then $q^\alpha L/[R\colon Z]$ is an integer. If $[R\colon Z] = q^\beta$, then L is divisible by $q^{\beta-\alpha}$. By equation (2.23), $\phi_i(1)$ is divisible by $q^{\beta-\alpha}$ for all i. By equation (2.21),

$$\gamma_j^H(1) = [H\colon R]\gamma_j(1) = \sum_{i=1}^{t} k_{ij}\phi_i(1), \qquad 1 \leq j \leq s.$$

Since $[H\colon R]$ is relatively prime to q, and each $\phi_i(1)$ is divisible by $q^{\beta-\alpha}$, each $\gamma_j(1)$ is divisible by $q^{\beta-\alpha}$. For some j, $q^\alpha = \gamma_j(1)$, and hence $\alpha \geq \beta - \alpha$ or $2\alpha \geq \beta$. However

$$\gamma_j(1)^2 \leq [R\colon Z(R)] \leq [R\colon Z] = q^\beta.$$

Therefore $2\alpha \leq \beta$, $\beta = 2\alpha$,

$$\gamma_j(1)^2 = [R\colon Z(R)] \quad \text{and} \quad Z(R) = Z.$$

Hence R is of central type and $Z(R) = Z$. Thus S is also of central type with $Z(S) = S \cap Z$.

We can now characterize p-special groups in terms of the structure of the group. Notice that this theorem is the converse of Theorem 2.8, in the case that λ is faithful on $[H, H] \cap Z$.

Theorem 2.24. *Let H be a group with center Z and assume that $[H, H] \cap Z$ is cyclic. Let*

$$T(H) = \{x \in H\colon x^{-1}\,\mathrm{Cl}_H(x) \cap Z = \{1\}\}.$$

Let S be any p Sylow subgroup of H, $R = S \cdot Z$, and let

$$T(R) = \{x \in R\colon x^{-1}\,\mathrm{Cl}_R(x) \cap Z = \{1\}\}.$$

Assume there is an $x \in S$ *such that*

(a) $T(R) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_R(x^i) \cdot Z$.

(b) $T(H) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_H(x^i) \cdot Z$.

(c) *For* $i \not\equiv 0$ *(modulo* p*),* $\mathrm{Cl}_H(x^i) \cdot Z = \mathrm{Cl}_H(x^j) \cdot Z$ *if and only if* $i \equiv j$ *(modulo* p*).*

Then H *is* p*-special.*

Proof. Let λ be a linear character on Z, with λ faithful on $[H,H] \cap Z$, Let $\lambda^H = \phi_1(1)\phi_1 + \cdots + \phi_t(1)\phi_t$ where $\phi_1, \ldots, \phi_t$ are inequivalent irreducible characters on H.

Let q be a prime, $q \neq p$, let S_q be a q Sylow subgroup of H, and let $R_q = S_q \cdot Z$. By Theorem 2.20, S_q and R_q are of central type since (b) implies that $T(H)/Z$ contains no q elements. Also $T(R_q) = Z$. Since $T(R_q)/Z$ contains only one conjugacy class, by Theorem 1.8, λ^{R_q} has only one irreducible constituent. Let $\lambda^{R_q} = \zeta_q(1)\zeta_q$. Then

$$\lambda^H = \zeta_q(1)\zeta_q^H = \phi_1(1)\phi_1 + \cdots + \phi_t(1)\phi_t.$$

Thus $\zeta_q(1)$ divides $\phi_i(1)$ for each i. Since $\zeta_q(1)^2 = [R_q\colon Z]$, each $\phi_i(1)^2$ is divisible by $[R_q\colon Z] = [H\colon Z]_q$, where $[H\colon Z]_q$ denotes the q factor of $[H\colon Z]$.

Let S be any p Sylow subgroup of H and let $R = S \cdot Z$. Since $T(R)/Z$ contains p conjugacy classes, λ^R has p inequivalent irreducible constituents by Theorem 1.8. Since each irreducible constituent of λ^R has degree a power of p, and their squares add up to $[R\colon Z]$ which is also a power of p, all irreducible constituents of λ^R must have the same degree. Let e_p be this common degree. Then $\lambda^R = e_p(\gamma_1 + \cdots + \gamma_p)$ where $\gamma_1, \ldots, \gamma_p$ are inequivalent irreducible characters on R. Then

$$\lambda^H = e_p(\gamma_1^H + \cdots + \gamma_p^H) = \phi_1(1)\phi_1 + \cdots + \phi_t(1)\phi_t.$$

Thus e_p divides each $\phi_i(1)$. Since $e_p^2 = [R\colon Z]/p$, each $\phi_i^2(1)$ is divisible by $[R\colon Z]/p = [H\colon Z]_p/p$, where $[H\colon Z]_p$ denotes the p part of $[H\colon Z]$. Then each $\phi_i(1)^2$ is divisible by $[H\colon Z]_q$ for all primes $q \neq p$ and by $[H\colon Z]_p/p$ or $\phi_i(1)^2$ is divisible by $[H\colon Z]/p$. Since

$$[H\colon Z] = \sum_{i=1}^{t} \phi_i(1)^2,$$

$t \leq p$. Since $t(H) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_H(x^i) \cdot Z$ and $\mathrm{Cl}_H(x^i) \cdot Z \neq \mathrm{Cl}_H(x^j) \cdot Z$ for $1 \leq i$, $j \leq p - 1$ by (c), $T(H)/Z$ contains p conjugacy classes. Hence, by Theorem 1.8, λ^H has p inequivalent irreducible constituents. Hence $t = p$, and if $e^2 = [H\colon Z]/p$, then $\phi_i(1) = e$ for all i and

$$\lambda^H = e(\phi_1 + \cdots + \phi_p).$$

By Lemma 2.13 H is p-special.

The condition on $[H,H] \cap Z$ can be dropped and we have the following theorem, which is the converse of Theorem 2.8 in all cases.

Theorem 2.25. *Let H be a group with center Z. Let λ be a linear character on Z and let*

$$T(H,\lambda) = \{x \in H: x^{-1}\,\mathrm{Cl}_H(x) \cap Z \subseteq \mathit{kernel}(\lambda)\}.$$

Let S be any p Sylow subgroup of H and let $R = S \cdot Z$. Let

$$T(R,\lambda) = \{x \in R: x^{-1}\,\mathrm{Cl}_R(x) \cap Z \subseteq \mathit{kernel}(\lambda)\}.$$

Assume there is $x \in S$ such that

(a) $T(R,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_R(x^i) \cdot Z$.

(b) $T(H,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_H(x^i) \cdot Z$.

(c) *For $i \not\equiv 0$ (modulo p), $\mathrm{Cl}_H(x^i) \cdot Z = \mathrm{Cl}_H(x^j) \cdot Z$ if and only if $i \equiv j$ (modulo p).*

Then H is p-special with respect to λ.

Proof. Let $N = [H,H] \cap \mathrm{kernel}(\lambda)$. Let $\overline{H} = H/N$, $\overline{R} = R/N$, $\overline{x} = xN$, and $\overline{\lambda}(zN) = \lambda(z)$ for any $z \in zN$. Then

(a′) $\overline{T(R,\lambda)} = T(\overline{R},\overline{\lambda}) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_{\overline{R}}(\overline{x}^i) \cdot \overline{Z}$.

(b′) $\overline{T(H,\lambda)} = T(\overline{H},\overline{\lambda}) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_{\overline{H}}(\overline{x}^i) \cdot \overline{Z}$.

(c′) For $i \not\equiv 0$ (modulo p), $\mathrm{Cl}_{\overline{H}}(\overline{x}^i) \cdot \overline{Z} = \mathrm{Cl}_{\overline{H}}(\overline{x}^j) \cdot \overline{Z}$ if and only if $i \equiv j$ (modulo p).

By Theorem 2.24, $\overline{H}$ is p-special and

$$\overline{\lambda}^{\overline{H}} = e(\zeta_1 + \cdots + \zeta_p)$$

where $\zeta_1, \ldots, \zeta_p$ are inequivalent irreducible characters on $\overline{H}$. Let $\lambda^H = \phi_1(1)\phi_1 + \cdots + \phi_t(1)\phi_t$ where $\phi_1, \ldots, \phi_t$ are inequivalent irreducible characters on H. If $x \in N$, then $\phi_i(x) = \phi_i(1)\lambda(x) = \phi_i(1)$. Define $\overline{\phi}_i$ by $\overline{\phi}_i(xN) = \phi_i(x)$ for any $x \in xN$. Then

$$\overline{\lambda}^{\overline{H}} = \overline{\phi}_1(1)\overline{\phi}_1 + \cdots + \overline{\phi}_t(1)\overline{\phi}_t = e(\zeta_1 + \cdots + \zeta_p).$$

Hence $\overline{\phi}_i(1) = \phi_i(1) = e$ for every i, $t = p$, and by relabeling if necessary $\overline{\phi}_i = \zeta_i$, $1 \le i \le t$. Suppose $\sum_{i=1}^{p} m_i\phi_i(y) = 0$ for some $y \in H$ and nonnegative integers m_i. If $\overline{y} = yN$, then $\sum_{i=1}^{p} m_i\overline{\phi}_i(\overline{y}) = 0$. Since $\overline{H}$ is p-special, either $\overline{\phi}_i(\overline{y}) = 0$ for all i, or $m_i = m_j$ for all i, j. Hence either $\phi_i(y) = 0$ for all i or $m_i = m_j$ for all i, j. Hence H is p-special with respect to λ.

We can rewrite Theorems 2.8 and 2.25 in a slightly different form.

Corollary 2.26. *Let H be a group with center Z. Let λ be a linear character on Z. Let*

$$T(H,\lambda) = \{x \in H: x^{-1}\,\mathrm{Cl}_H(x) \cap Z \subseteq \mathit{kernel}(\lambda)\}.$$

Let S be any p Sylow subgroup of H, let $R = S \cdot Z$ and let

$$T(R,\lambda) = \{x \in R\colon x^{-1}\,\mathrm{Cl}_R(x) \cap Z \subseteq kernel(\lambda)\}.$$

Then H is p-special if and only if

(a) *every element of $T(H,\lambda)/Z$ has order a power of p and $T(H,\lambda)/Z$ consists of p conjugacy classes of H/Z, and*

(b) *$T(R,\lambda)/Z$ consists of p conjugacy classes of R/Z.*

Proof. If H is p-special, then conditions (a) and (b) follow at once from Theorem 2.8.

Suppose conditions (a) and (b) hold. Let $x \in T(R,\lambda)$, $x \notin Z$. Then as in the proof of Lemma 2.11, $x^i \in T(R,\lambda)$ for all $1 \le i \le p-1$. Since R/Z is a p group, $\mathrm{Cl}_R(x^i) \cdot Z \neq \mathrm{Cl}_R(x^j) \cdot Z$ for $i \neq j$, $1 \le i,j \le p-1$. Since $\bigcup_{i=0}^{p-1} \mathrm{Cl}_R(x^i) \cdot Z \subseteq T(R,\lambda)$ and $T(R,\lambda)/Z$ contains only p conjugacy classes of R/Z, we have

$$T(R,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_R\,(x^i) \cdot Z.$$

Let $y \in T(H,\lambda)$, $y \notin Z$. Since yZ has order a power of p and all p Sylow subgroups of H are conjugate, $\mathrm{Cl}_H(y) \cap R \neq \varnothing$. Let $y' \in \mathrm{Cl}_H(y) \cap R$. Since $y' \in T(H,\lambda) \cap R$, $y' \in T(R,\lambda)$. Since $y' \notin Z$, $y' \in \mathrm{Cl}_R(x^i) \cdot Z$ for some i. Then

$$\mathrm{Cl}_R(y') \cdot Z = \mathrm{Cl}_R(x^i) \cdot Z \quad \text{and} \quad \mathrm{Cl}_H(y') \cdot Z = \mathrm{Cl}_H(x^i) \cdot Z.$$

Since $y \in \mathrm{Cl}_H(y') \cdot Z$, $y \in \mathrm{Cl}_H(x^i) \cdot Z$. Thus for every $y \in T(H,\lambda)$, $y \notin Z$,

$$y \in \bigcup_{i=0}^{p-1} \mathrm{Cl}_H(x^i) \cdot Z.$$

Hence $T(H,\lambda) \subseteq \bigcup_{i=0}^{p-1} \mathrm{Cl}_H(x^i) \cdot Z$. Since $T(H,\lambda)/Z$ consists of p conjugacy classes of H/Z,

$$T(H,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_H(x^i) \cdot Z$$

and $\mathrm{Cl}_H(x^i) \cdot Z \neq \mathrm{Cl}_H(x^j) \cdot Z$, $i \neq j$, $1 \le i,j \le p-1$. By Lemma 2.11 for all integers a and $i \not\equiv 0\,(\text{modulo } p)$,

$$\mathrm{Cl}_H\,(x^{(ap+1)i}) \cdot Z = \mathrm{Cl}_H(x^i) \cdot Z.$$

Hence, for $i \not\equiv 0$ (modulo p),

$$\mathrm{Cl}_H\,(x^i) \cdot Z = \mathrm{Cl}_H(x^j) \cdot Z$$

if and only if $i \equiv$ (modulo p). Thus H is p-special by Theorem 2.25.

A word of caution is in order here. One might be tempted to replace (a) of Theorem 2.25 by the statement that either R is of central type with $[Z(R): Z] = p$ or R is p-special with $Z = Z(R)$. However, these statements are not equivalent. If R is p-special with $Z = Z(R)$, then by Theorem 2.8, $T(R,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_R(x^i) \cdot Z$ for some $x \in S$, the p Sylow subgroup of R. However, if R is of central type with $[Z(R): Z] = p$, it does not follow that $T(R,\lambda) = \bigcup_{i=0}^{p-1} \mathrm{Cl}_R(x^i) \cdot Z$ for some x, or even that $T(R,\lambda)/Z$ consists of p conjugacy classes of R/Z.

Example 2.27. Let $S = \langle x,y,z_0 \mid x^3 = y^3 = z_0^3 = 1, y^{-1}xy = xz_0, y^{-1}z_0y = z_0, x^{-1}z_0x = z_0\rangle$ and assume S is the p Sylow subgroup of a group H with center Z. Let $R = S \cdot Z$, and let λ be a linear character on Z. Let ω be a primitive cube root of 1, and define $\sigma_i(z_0^s \cdot z) = \lambda(z)\omega^{si}$, where $z \in Z$. It can be shown that σ_i is independent of the way elements of $Z(R) = \langle z_0\rangle \cdot Z$ are represented and

$$\lambda^{Z(R)} = \sigma_0 + \sigma_1 + \sigma_2.$$

Then

$$T(R,\lambda) = \{x \in R: x^{-1}\,\mathrm{Cl}_R(x) \cap Z \subseteq \mathrm{kernel}(\lambda)\} = R.$$

However, for $i \neq 0$

$$T(R,\sigma_i) = \{x \in R: x^{-1}\,\mathrm{Cl}_R(x) \cap Z(R) \subseteq \mathrm{kernel}(\sigma_i)\} = Z(R).$$

Hence, for $i = 1$ or $i = 2$, σ_i^R has only one irreducible constituent by Theorem 1.8. If $\sigma_i^R = \zeta_i(1)\zeta_i$, then $\zeta_i(1)^2 = [R: Z(R)]$ and R is of central type. However, by Theorem 1.8, σ_0^R has 9 inequivalent irreducible constituents. Therefore λ^R has a total of 11 inequivalent irreducible constituents and $T(R,\lambda)/Z$ consists of 11 conjugacy classes of R/Z.

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