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A NEW CHARACTERIZATION OF TAME 2-SPHERES IN E^3

BY

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ABSTRACT. It is shown in Theorem 1 that a 2-sphere S in E^3 is tame from $A = \text{Int } S$ if and only if for each compact set $F \subset A$ there exists a 2-sphere S' with complementary domains $A' = \text{Int } S'$, $B' = \text{Ext } S'$, such that $F \subset A' \subset \overline{A'} \subset A$ and for each $x \in S'$ there exists a path in $\overline{B'}$ of diameter less than $\rho(F, S)$ which runs from x to a point $y \in S$. Furthermore, the theorem holds when A is replaced by B, A' by B', B' by A', and Int by Ext. Two applications of this characterization are given. Theorem 2 states that a 2-sphere is tame from the complementary domain C if for arbitrarily small $\epsilon > 0$, S has a metric ϵ-envelope in C which is a 2-sphere. Theorem 3 answers affirmatively the following question: Is a 2-sphere $S \subset E^3$ tame in E^3 if there exists an $\epsilon > 0$ such that if $a, b \in S$ satisfy $\rho(a, b) < \epsilon$, then there exists a path in S of spherical diameter $\rho(a, b)$ which connects a and b?

1. **Introduction.** Bing's original characterization of a tame 2-sphere S in E^3 states that tameness from the complementary domain C is equivalent to the existence of an arbitrarily small ϵ-homeomorphism from S into C. Since any 2-sphere in E^3 can be homeomorphically approximated by a polyhedral 2-sphere, it may be assumed that the ϵ-homeomorphism of Bing's theorem carries S onto a polyhedral (hence tame) 2-sphere $S' \subset C$.

Theorem 1 was developed while the author was attempting to remove the restriction that S' be "tied" so strongly to S by the ϵ-homeomorphism. The end result is Theorem 1, which modifies Bing's theorem in the following way: S' is no longer required to be an ϵ-homeomorphic image of S, but instead must have the property that each point of S be a distance less than ϵ from S', and furthermore, from each point x of S' there must exist a path of diameter less than ϵ which leads to a point $y \in S$ and which lies in the closure of the component of $C - S'$ whose boundary is $S \cup S'$. (This path need not depend continuously on x.)

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(1) This paper contains part of the author's Ph. D. dissertation, which has the same title.

Although the statement of Theorem 1 is somewhat more complicated than that of Bing's characterization, it is sometimes easier to construct the 2-sphere S' described in Theorem 1 than it is to apply Bing's or other related characterizations. Theorems 2 and 3 are applications of Theorem 1 which illustrate this fact.

Most of the terms used in this paper are defined in the excellent survey article by Burgess and Cannon [3]. The exceptions are terms coined by the author, which are defined at the point of their first appearance. We use ρ throughout for the usual metric on E^3 and $N(X, \epsilon)$ for the ϵ-neighborhood of a set X in E^3.

2. **The principal result.**

Theorem 1. *Let S be a 2-sphere embedded in E^3. Let A and B denote the interior and exterior of S respectively, and let ρ be the usual metric on E^3. Then the following statements are equivalent*:

(1) *S is tame from A.*

(2) *For each compact set $F \subset A$, there exists a 2-sphere S', embedded in E^3 and having complementary domains $A' = \text{Int } S'$, $B' = \text{Ext } S'$, such that* (i) $F \subset A' \subset \overline{A'} \subset A$, *and* (ii) *for each $x \in S'$ there exists a path $\overline{xy}$ contained in $\overline{B'}$ having initial point $x \in S'$, terminal point $y \in S$, and whose diameter is less than $\rho(F, S)$.*

Statements (1) and (2) are also equivalent if we replace A by B, A' by B', B' by A', and Int by Ext in these statements.

The original statement and proof of Theorem 1 appearing in (7) requires that the 2-sphere S' be tamely embedded. The referee has suggested the following simpler proof which uses the 0-ulc property to remove this restriction:

Proof. The proof will first deal with tameness from A.

(1) $\Longrightarrow$ (2): The proof is obvious.

(2) $\Longrightarrow$ (1): Suppose that statement (2) holds. The proof that S is tame from A depends on two lemmas.

Lemma 1. *The 2-sphere S' of statement* (2) *may be chosen to be polyhedral.*

Proof. We first use the validity of statement (2) to choose a 2-sphere S'', embedded in E^3 and having complementary domains $A'' = \text{Int } S''$, $B'' = \text{Ext } S''$, such that (i) $F \subset A'' \subset \overline{A''} \subset A$, and (ii) for each $x \in S''$ there exists a path $\overline{xy}$ contained in $\overline{B''}$ having initial point $x \in S''$, terminal point $y \in S$, and whose diameter is less than $\epsilon = (1/2)\rho(F, S)$. We next choose disks $D_1, \cdots, D_n$ of diameter less than $\epsilon/3$ whose union is S''. Since the sets A'' and B'' are 0-ulc ([8, p. 66]; cf. also [3, Theorems 4.1.2 and 4.1.3]), there are arcs $\alpha_1 = \overline{p_1q_1}, \cdots, \alpha_n = \overline{p_nq_n}$ in E^3 of diameter less than ϵ such that, for each i, $p_i \in A''$, $q_i \in S$, and $\alpha_i \cap S''$ is a single point of Int D_i. Let $\delta > 0$ be smaller than any of the numbers $\rho(S'', S)$, $\epsilon/3$, $\rho(S'', F)$, $\rho(p_i, S'')$, and $\rho(S'' - D_i, \alpha_i)$ $(i = 1, \cdots, n)$. Let $h: S'' \longrightarrow S'$ be a homeomorphism from S'' onto a polyhedral 2-sphere S' in E^3 which moves no point of S'' as far as δ [2]. We claim that S' satisfies the requirements of statement (2) with respect to S and F; this we see as follows. Since $\delta < \rho(S'', S)$ and $\delta < \rho(S'', F)$, it follows that $F \subset A' \subset \overline{A'} \subset A$. Let $x \in b(D_i) \subset S'$ be an arbitrary point of S'. Since $\delta < \rho(p_i, S'')$, $p_i \in A'$. Thus $\alpha_i \cap S' \neq \emptyset$. Since $\delta < \rho(S'' - D_i, \alpha_i)$, $\alpha_i \cap S' \subset b(D_i)$. Thus there is a subarc α of α_i which lies in B' and connects $b(D_i)$ with S. There is an arc β in $b(D_i)$ joining x to α. Then $\alpha \cup \beta$ contains an arc from x to S which lies in B' and has diameter less than $2\epsilon = \rho(F, S)$. This completes the proof of Lemma 1.

Lemma 2. *Let S' be a polyhedral 2-sphere embedded in E^3 with complementary domains $A' = \operatorname{Int} S'$ and $B' = \operatorname{Ext} S'$. Let D be a polyhedral disk embedded in E^3 with $\operatorname{Bd} D \subset A'$ such that D intersects S' transversely. Let C be a component of $\overline{B'} - D$ such that there is an arc $\overline{ab}$ from $\operatorname{Bd} D$ to C which except for its endpoints $a \in \operatorname{Bd} D$ and $b \in C$, misses $D \cup \overline{B'}$. Then given $\epsilon > 0$, there exists a nonsingular polyhedral disk D' such that*

(a) $\operatorname{Bd} D' = \operatorname{Bd} D$ *and*

(b) $D' \subset A' \cap N[D \cup (S' - C), \epsilon]$.

Proof. Since D intersects S' transversely, $D \cap S'$ is the union of finitely many disjoint simple closed curves. The proof is by induction on the number n of those curves. For $n = 0$, we may set $D' = D$. Suppose inductively that the lemma is true for fewer than n curves of intersection and that $n > 0$. Since $n > 0$, $D \cap S' \neq \emptyset$; and we may choose a component J of $D \cap S'$ such that the interior of the subdisk D_J of D bounded by J misses S'. We consider two cases.

Case 1. $D_J \subset \overline{B'}$. In this case there is a disk S_J in S' which is bounded by J and misses C; i.e., $S_J \subset (\overline{B'} - C)$. By standard cut and paste techniques it is possible to cut off D near S_J in A'. (Consider the disks in D bounded by curves in $S_J \cap D$. Those which are not contained in the interior of any other such disk are replaced by disks in A' parallel to subdisks of S_J.) Call the new disk thus obtained D_1 and note that D_1 meets S' transversely and in fewer components than did D. Let C_1 denote the component of $\overline{B'} - D_1$ which contains C. Note that if $D_1 - D$ is chosen sufficiently close to S_J, then the arc $\overline{ab}$, except for its endpoints, misses $D_1 \cup \overline{B'}$ and for some $\epsilon_1 > 0$, $A' \cap N[D_1 \cup (S' - C_1), \epsilon_1] \subset A' \cap N[D \cup (S' - C), \epsilon]$. By inductive hypothesis, there is a nonsingular polyhedral disk D' such that

(a) $\operatorname{Bd} D' = \operatorname{Bd} D_1 = \operatorname{Bd} D$ and

(b) $D' \subset A' \cap N[D_1 \cup (S' - C_1), \epsilon_1] \subset A' \cap N[D \cup (S' - C), \epsilon]$, as desired.

Case 2. $D_J \subset \overline{A'}$. In this case let S_J be the disk in S' bounded by J whose interior is separated in A' from the arc $\overline{ab}$ by the disk D_J. Let S_J' be a polyhedral disk in S' which is very close to S_J homeomorphically and which contains S_J in its interior. Let S''_J be a polyhedral disk in $\overline{A'} - D$ which is very close to D_J homeomorphically, lies, except for its boundary, in A', and has the same boundary as S'_J. Let S_1 be the polyhedral 2-sphere $(S' - S'_J) \cup S''_J$. Let $A_1 = \text{Int } S_1$, $B_1 = \text{Ext } S_1$, and C_1 = component of $\overline{B_1} - D$ which contains C. It is easy to check that $\text{Bd } D \subset A_1 \subset A'$ (since $\text{Int } \overline{ab} \subset A_1$); that D intersects S_1 transversely and in fewer components than it intersected S; that $\overline{ab}$ is an arc from Bd D to C_1 which, except for its endpoints, misses $D \cup \overline{B_1}$; and finally, that if S''_J is close enough to D_J, then $A_1 \cap N[D \cup (S_1 - C_1), \epsilon_1] \subset A' \cap N[D \cup (S' - C), \epsilon]$ for some $\epsilon_1 > 0$. Again the inductive hypothesis applies and supplies a polyhedral disk D' such that

(a) $\text{Bd } D' = \text{Bd } D$ and

(b) $D' \subset A_1 \cap N[D \cup (S_1 - C_1), \epsilon_1] \subset A' \cap N[D \cup (S' - C), \epsilon]$, as desired.

Cases 1 and 2 complete the inductive proof of Lemma 1.

We now return to the proof that (2) $\Rightarrow$ (1).

Bing has proved [1] that a 2-sphere S in E^3 is tame from a complementary domain A if A is 1-ULC. His proof is easily seen to be valid if the following condition replaces the 1-ULC condition.

Condition *. Suppose E is a disk in S and D is a polyhedral disk in E^3 such that $\text{Bd } D \subset A$ and $D \cap S \subset \text{Int } E$. Suppose further that Bd D can be joined to $S - E$ by an arc $\alpha = \overline{uv}$ which lies, except for its endpoints $u \in \text{Bd } D$ and $v \in S - E$, in $E^3 - (S \cup D)$. Then, given $\epsilon > 0$, Bd D bounds a disk D' in $A \cap N[D \cup E, \epsilon]$.

Theorem 1 will be established once we prove that statement (2) of Theorem 1 implies that Condition * is satisfied. Suppose therefore that E, D, α, and ϵ are given as in Condition *.

We first wish to apply statement (2). To this end, choose a point $w \in \text{Int } \alpha$, and let $\alpha_1 = \overline{uw}$ and $\alpha_2 = \overline{wv}$ denote the two arcs into which w divides $\alpha = \overline{uv}$. Choose a positive number δ such that

$$\delta < \min\{\epsilon/2, \rho(\alpha_2, D \cup E), \rho(S, \text{Bd } D \cup \alpha_1)\}.$$

Let $F = A - N(S, \delta)$. Statement (2) of Theorem 1 implies that there is a 2-sphere S' in E^3 having complementary domains $A' = \text{Int } S'$, $B' = \text{Ext } S'$, such that (i) $F \subset A' \subset \overline{A'} \subset A$, and (ii) for each $x \in S'$ there exists a path $\overline{xy}$ contained in $\overline{B'}$ having initial point $x \in S'$, terminal point $y \in S$, and whose diameter is less than $\rho(F, S)$.

We now wish to apply Lemma 2. Lemma 1 shows that we may choose S' to be polyhedral and to meet D transversely. Note that $\text{Bd } D \subset F \subset A'$. Let C be the component of $\overline{B'} - D$ which contains $S - E$. Let x be the first point of $\alpha = \overline{uv}$ which lies in S'. We must necessarily have $x \in \text{Int } \alpha_2$ since $\delta < \rho(S, \alpha_1)$. We claim that $x \in C$. Indeed, let $\overline{xy}$ be a path contained in $\overline{B'}$ which connects

x to S and has diameter less than δ. Since $x \in \alpha_2$ and $\delta < \rho(\alpha_2, D \cup E)$, $y \in S - E \subset C$. Thus $x \in \overline{xy} \subset C$, as claimed. We have established therefore that there is an arc $\overline{ux}$ from Bd D to C which, except for its endpoints, misses $D \cup \overline{B'}$. Thus Lemma 2 applies and yields a polyhedral disk D' such that

(a) Bd D' = Bd D and

(b) $D' \subset A' \cap N[D \cup (S' - C), \delta]$.

Since $A' \subset A$, we will be done once we have shown that $N[D \cup (S' - C), \delta] \subset N[D \cup E, \epsilon]$, or, since $\delta < \epsilon/2$, that $(S' - C) \subset N[D \cup E, \delta]$. Let $z \in S' - C$ and let β be an arc in $\overline{B'}$ of diameter less than δ which joins z to S. If β misses both D and E, then z is in the same component of $\overline{B'} - D$ as is $S - E$; i.e., $z \in C$, a contradiction. Hence $\beta \cap (D \cup E) \neq \emptyset$, and $z \in N[D \cup E, \delta]$. This completes the proof that statement (2) of Theorem 1 implies Condition *. As noted earlier, this also completes the proof that S is tame from A.

We can use the foregoing proof to deal with tameness from B by simply forming the one-point compactification S^3 of E^3 and then removing a point from A to form E^3 again. □

3. Applications of Theorem 1. The following theorem is an almost immediate result of Theorem 1. If X, Y are subsets of E^3, define the *metric ϵ-envelope* of Y in X to be the set $\{x \in X \mid \rho(x, Y) = \epsilon\}$.

Theorem 2. *Let S be a 2-sphere embedded in E^3 with A = Int S, B = Ext S. Let ρ be the usual E^3 metric. Suppose that for each $\alpha > 0$ there exists a real ϵ with $0 < \epsilon < \alpha$ such that the metric ϵ-envelope of S in A is a 2-sphere embedded in E^3. Then S is tame from A. The implication also holds if A is replaced by B in the last two statements.*

Proof. S will be proven tame from A by showing that it satisfies condition (2) of Theorem 1. The proof for tameness from B is similar.

Let $F \subset A$ be a compact set. By hypothesis there exists a real number ϵ such that $0 < \epsilon < \rho(F, S)$ and the set $S' = \{x \in A \mid \rho(x, S) = \epsilon\}$ is a 2-sphere in A. Let A' = Int S', B' = Ext S'. We first show that $F \subset A' \subset \overline{A'} \subset A$. Since $S' \subset A$, $\overline{B}$ is contained in one of the complementary domains of S'. Since B is unbounded, we must have $B \subset \overline{B} \subset B'$. It follows immediately that $A' \subset \overline{A'} \subset A$. To show that $F \subset A'$, let x be a point of F. Since $\rho(x, S) > \epsilon$, $x \notin S'$. Let β be any path in E^3 which starts at x and ends at some point $z \in S \subset \overline{B} \subset B'$. Since $\rho(x, S) > \epsilon$ and $\rho(z, S) = 0$, there is some point y on β such that $\rho(y, S) = \epsilon$ by the intermediate value property. Then $y \in S'$, which implies that every path from x to $z \in B'$ must intersect S'. Since B' is path-connected and $x \notin S'$, we conclude that $x \in A'$. Therefore $F \subset A' \subset \overline{A}' \subset A$.

It remains to show that for each $x \in S'$ there exists a path $\overline{xy}$ contained in $\overline{B'}$ having initial point $x \in S'$, terminal point $y \in S$, and whose diameter is less than $\rho(F, S)$. Let $x \in S'$. Since S is compact and $\rho(x, S) = \epsilon$, there is a point $y \in S$ such that $\rho(x, y) = \epsilon$. Consider the path $\overline{xy}$ formed by the straight line segment running from x to y. The diameter of this path is $\epsilon < \rho(F, S)$, and all we need to show is that it lies in $\overline{B'}$. Suppose not. Then some point w strictly between x and y on $\overline{xy}$ must lie in A'. Since $y \in S \subset B'$, there is a point z strictly between w and y on $\overline{xy}$ which lies on S'. But this results in a contradiction because $z \in S' \Longrightarrow \rho(z, S) = \epsilon$ but z is strictly between x and y on $\overline{xy}$ which implies that $\rho(z, y) < \epsilon \Longrightarrow \rho(z, S) < \epsilon$. □

The converse of Theorem 2 is clearly not true. Theorem 2 gives rise to the following question: If the metric ϵ-envelope of a set X in E^3 is a 2-sphere S, is S tame? Partial answers can be obtained if X lies in one of the complementary domains C. In this case it is clear that for each point $x \in S$, there exists a round tangent ball in $S \cup C$ which touches S only at x. Loveland [6, p. 396] has asked if this makes S tame, and Cannon [4, pp. 444–445] proved that S is tame from $E^3 - C$ under this condition. If $X \subset \operatorname{Int} S$ and $\epsilon > \operatorname{diam} X$ then each point of S is visible from a point $x \in X$. Cobb [5] shows that S is then tame in E^3. His proof appears in [3, pp. 326–327].

Define the *spherical diameter* of a set $X \subset E^3$ to be the diameter of the smallest closed round ball containing X. (For a given set, the ratio r of the spherical diameter to the usual diameter satisfies $1 \leq r \leq \sqrt{3/2}$.)

All of the commonly known wild spheres appear to have the property that one can find two points x, y on the sphere which are arbitrarily close together such that any arc on the sphere having x and y as endpoints must have a spherical diameter greater than $\rho(x, y)$. Must every wild sphere have this property? It seems reasonable that such a property might result from the rather severe entanglement in E^3 which is characteristic of wild spheres. The following lemmas are used in Theorem 3 which answers the question affirmatively. The proof of Lemma 3 is a simple geometrical argument and is therefore omitted.

Lemma 3. *Let P be a solid, closed, rectangular parallelepiped. Let R be the union of all closed, round balls having a diameter which is an edge, a face diagonal, or a principal diagonal of P. Let a, b be distinct points in P. Then any path α from a to b which has spherical diameter equal to $\rho(a, b)$ must lie in R.*

Lemma 4. *Suppose M is a Euclidean polyhedron in E^3 which is connected but not simply connected. Let β be a positive real number. Then there exist two simple closed curves $K \subset M$, $H \subset E^3 - M$ such that neither is null-homotopic in the complement of the other. Furthermore H may be chosen to lie in the β-neighborhood of M.*

Proof. Let U_β denote the β-neighborhood of M. Let $N \subset U_\beta$ be a regular neighborhood of M. Each component of Bd N is a p.l. 2-manifold without boundary. Since M is not simply connected, neither is N. Therefore, some component C of Bd N is not simply connected. The fundamental theorem of compact surfaces states that C is either a 2-sphere or the connected sum of a finite number of tori. The first possibility is ruled out, so there is a subset $\hat{C}$ of C which is a torus T minus the interior of a disk $D \subset T$. Select two polygonal simple closed curves H and K on $\hat{C}$ which intersect each other transversely and at a single point on $\hat{C}$. A simple linking argument shows that either H pushed slightly into $U_\beta - N$ links K homologically in E^3 or K pushed slightly into $U_\beta - N$ links H homologically in E^3. Interchanging the names of H and K if necessary, we may assume the former. K can be homotopically pushed into M via a collapsing of N into M. At this point neither of K, H is null-homotopic in the complement of the other, although K may not simple. However, some subset of K is a simple closed curve satisfying the non null-homotopic condition. Taking K to be this curve completes the proof. □

Theorem 3. *Let $S \subset E^3$ be a 2-sphere with ρ the usual E^3 metric. Suppose there exists an $\epsilon > 0$ such that any two points, a, $b \in S$ satisfying $\rho(a, b) < \epsilon$ can be joined by a path in S of spherical diameter $= \rho(a, b)$. Then S is tame in E^3.*

Proof. The proof deals with tameness from $A = \text{Int } S$. That S is tame from $B = \text{Ext } S$ can be proved similarly. Let $F \subset A$ be compact with $\eta = \rho(F, S)$. Consider the solid, closed cubes in E^3 of edge length $e < \min[\eta/4, \epsilon/\sqrt{6}]$ whose vertices have coordinates of the form (me, ne, pe) where m, n, p are integers. The word "cube" will refer to one of these cubes unless otherwise stated. The nonempty union T of all cubes lying entirely in A contains F, and S is accessible from each point of Bd T via a path in $\overline{A - T}$ of diameter $< \eta$. The object is to change T into a polyhedral 3-cell B which retains these two properties. Then Bd B will be a 2-sphere S' satisfying statement (2) of Theorem 1, proving that S is tame from A. T will first be modified into T'' so that each component of T'' becomes a polyhedral 3-cell. B is then easily constructed by connecting the components with slightly thickened polygonal arcs in $A - \text{Int } T''$. Each component T_i of T has a connected complement and therefore can fail to be a 3-cell by either not being simply connected or by not being a 3-manifold-with-boundary. The first of these difficulties is corrected by removing from each T_i neighborhoods of *constriction points* of T_i as shown in Figure 1. Such a point x is a vertex of exactly two cubes M, N lying in T_i with $M \cap N = \{x\}$. The resulting modified version T' of T retains the two properties of T mentioned previously. For purposes of continuity, the proof that each component T'_j of T' is simply connected will be deferred until later. Figure 2 shows how each T'_j is then made into a polyhedral 3-manifold with boundary, hence a polyhedral 3-cell B_j, by attaching to the concave "troughs" of Bd T'_j small cubes of edge length e/m

where $m > 2$ satisfies $e/m < \rho(T, S)$. The B_j are disjoint and their union T'' retains the two properties of T mentioned previously. Connecting the B_j with fattened polygonal arcs in $A - \text{Int } T''$ provides the desired 3-cell B, and the proof is complete except for the argument below.

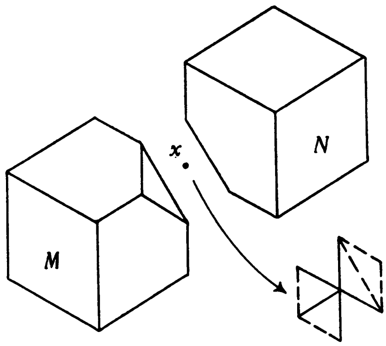

Figure 1

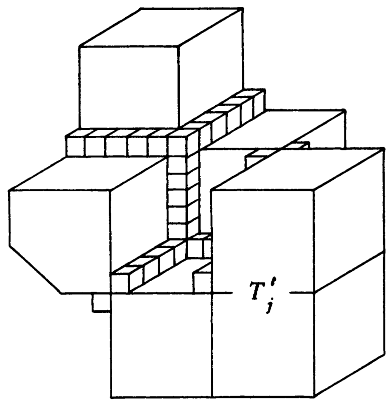

Figure 2

Proof that each T_j' is simply connected. Assume not. Choose $\beta < \rho(T, S)$. By Lemma 4, there exist polygonal simple closed curves $H_1 \subset E^3 - T'$ and $K_1 \subset T_j'$ such that H_1 lies in the β-neighborhood N_j of T_j' and neither curve is null-homotopic in the complement of the other. H_1 can be chosen to miss T by moving it, if necessary, in $N_j - T_j'$ so that it fails to intersect any of the neighborhoods which earlier were removed from each T_i. β has been chosen small enough to assure that each cube in the union V of all cubes intersecting H_1 will contain points of S, and only the boundary of each cube will intersect T. Because $H_1 \subset \text{Int } V$, H_1 can be moved slightly in $V - T$ so that it intersects no edge of any cube. Thus H_1 is now the union of m polygonal arcs laid end-to-end with each arc γ_i satisfying the following: $\gamma_i \subset Q_i \cup Q_{i+1}$, where Q_i, Q_{i+1} are cubes from V which intersect in a common face, and the endpoints of γ_i lie in Q_i, Q_{i+1} respectively. For each i, Q_i and Q_{i+1} each contain points of S, so H_1 can be further moved in $V - T$ so that the arcs γ_i change into straight line segments α_i satisfying $\alpha_i \subset Q_i \cup Q_{i+1}$ with the endpoints x_i, x_{i+1} of α_i being points of S lying in Q_i, Q_{i+1} respectively. Since $\rho(x_i, x_{i+1}) \leq e\sqrt{6} < \epsilon$, the hypothesis of the theorem implies that there is a path $\Gamma_i \subset S$ of spherical diameter $\rho(x_i, x_{i+1})$ joining x_i and x_{i+1}. Because the closed path $\Gamma = (\bigcup_{i=1}^m \Gamma_i) \subset S$ is null-homotopic in S, hence in $E^3 - K_1$, there is some k such that the path $G_k = \alpha_k \cup \Gamma_k$ is not null-homotopic in $E^3 - K_1$. The spherical diameter of G_k is $\rho(x_k, x_{k+1})$. By Lemma 3, G_k lies in the 3-cell R formed by the union of all closed round balls whose diamter is an edge, a face diagonal, or a principal diagonal of $Q_k \cup Q_{k+1}$. Figure 3 indicates that all cubes except Q_k and Q_{k+1} (hence all cubes in T) fall into 6 classes depending upon the nature of their intersection with R. The darkened edges of each representative cube L are those which miss Int R, and E_L denotes the union of these edges. If two such cubes L, M meet in a common edge or face, then $E_L \cup E_M$ is arc-connected.

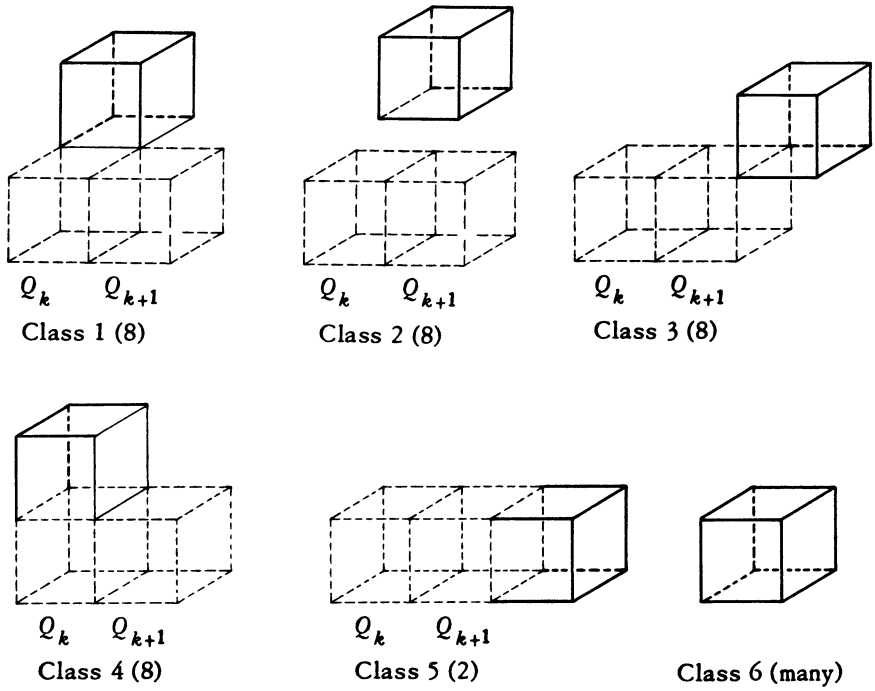

Figure 3

The previous removal of neighborhoods in T of constriction points assures that the closed curve $K_1 \subset T_j' \subset T$ mentioned previously can be chosen to intersect no vertex of any cube, and therefore can be traversed by passing through a sequence of cubes in T, each cube N_i intersecting its predecessor in a common face or edge. Since $E_{N_i} \cup E_{N_{i-1}}$ is arc-connected, a further adjustment of K_1 in T makes $K_1 \cap N_i \subset E_{N_i}$ for each i. K_1 now misses Int R. Now $G_k \subset R$, G_k misses K_1, and is not null-homotopic in $E^3 - K_1$. But R is a 3-cell, so G_k can be contracted in R radially inward to a point without hitting K_1. This contradiction completes the argument that T_j' is simply connected. □

The converse of this theorem is clearly false. It might be possible to strengthen the theorem by showing that there is some constant $K > 1$ such that tameness is implied if the path from x to y mentioned in the hypothesis has a spherical diameter equal to $K\rho(x, y)$. This leads to the problem of finding the least upper bound for such a constant. The proof of Theorem 3 breaks down if $K > 1$ because then it can no longer be guaranteed that $E_M \cap E_N$ is connected when the cubes M, N

intersect in a common edge or face. The theorem might be true if "diameter" replaces "spherical diameter", but neither a proof nor a counterexample has been found.

Another question related to Theorem 3 is the following: If S is a 2-sphere in E^3 and C is one of its complementary domains, define a *chord of* C to be a straight line segment lying in C having its endpoints in S. Is S tame from C if there exists an $\epsilon > 0$ such that for each chord of C of length $l < \epsilon$ there is an arc in S of spherical diameter $= l$ which connects the endpoints of the chord?

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