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PRIMITIVE SATISFACTION AND EQUATIONAL PROBLEMS FOR LATTICES AND OTHER ALGEBRAS

BY

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ABSTRACT. This paper presents a general method of solving equational problems in all equational classes of algebras whose congruence lattices are distributive, such as those consisting of lattices, relation algebras, cylindric algebras, orthomodular lattices, lattice-ordered rings, lattice-ordered groups, Heyting algebras, other lattice-ordered algebras, implication algebras, arithmetic rings, and arithmetical algebras.

Introduction. Most algebraic studies concern the members of an equational class specified in advance, such as the class of all groups, of all rings, or of all lattices. Here by an equational class (variety, primitive class) is meant a class of algebras determined by a list of algebraic identities (laws, identical relations, universally quantified polynomial equations).

Many algebraic problems, however, have this converse form: For a given class $\mathbf{K}$ of algebras with the "same" operations, characterize the equational closure $\mathbf{K}^e$, the smallest equational class containing $\mathbf{K}$. For example, if $\mathbf{K}$ consists merely of the group of integers, then $\mathbf{K}^e$ can be described as consisting of all abelian groups.

It is natural to attempt to describe $\mathbf{K}^e$ by giving an explicit list of defining identities, or equivalently, a list of identities which are valid in all members of $\mathbf{K}$ and which imply all other such identities. Such a list is called a set of "equational axioms" for $\mathbf{K}$, or an "equational basis" for $\mathbf{K}$. The problem of finding such a list will be called the "equational axiom problem" for $\mathbf{K}$, to be abbreviated $\mathrm{AP}(\mathbf{K})$. For example, in [2], $\mathrm{AP}(\mathbf{K})$ was solved for many classes $\mathbf{K}$ of lattices, such as the classes of all lattices of at most a given width, breadth, or length. In [26], $\mathrm{AP}(\{L\})$ was solved for each finite lattice L. In [8] and [9], such problems were studied for orthomodular lattices, and in [4], for Heyting algebras. A

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given problem AP(K) may or may not have a finite solution. Indeed, there may exist no finite solution even for certain classes of lattices ([1], [26]), or even for certain classes K consisting of a single finite algebra ([23], [33]).

In a series of papers ([45], [44], [47]), R. Wille has developed a strikingly different description of the members of K^e for any class K of lattices defined by noncontainment of a copy of a fixed, finite partly ordered set. (Such classes include the width, breadth, and length cases mentioned above.) Wille's description, phrased in terms of "primitive subsets," is an almost pictorial condition involving weak projectivities and is especially suitable for computation.

In the present paper, Wille's description of K^e by primitive subsets [47] and the author's description of K^e by equational axioms [2] will be simultaneously generalized from the case of lattices to a setting which will be seen to be the most natural: the case where K lies inside any equational class E of algebras whose lattices of congruence relations are distributive. Such "congruence-distributive" equational classes E were characterized in a fundamental paper of Jónsson [20]. Examples include not only the class of lattices but also any equational class of lattice-ordered algebras, such as the Heyting algebras of intuitionistic logic [38], orthomodular lattices [18], lattice-ordered groups [11], and lattice-ordered rings ([11], [19]). Further examples include equational classes consisting of arithmetic rings [43], arithmetical algebras [36], in fact any equational class with a median polynomial (see §3), and the class of implication algebras [28]. As in [2], the classes K to be treated most directly are those definable by first-order sentences which are universally quantified disjunctions of polynomial equations (UDE's). For lattice-ordered algebras, such sentences can express a bound on width, length, breadth, or other parameter; for arbitrary algebras, such sentences can express cardinality restrictions, disjunctions of identities, and so on. Moreover, if K is *any* subclass of a congruence-distributive equational class E and if K is determined by some set of first-order sentences, then the class K_0 of homomorphic images of subalgebras of algebras in K satisfies $K_0^e = K^e$ and *is* definable by UDE's, which in practice can often be found explicitly. (See the proof of Corollary 1.9 below.)

Except in the cases of lattices, Heyting algebras, arithmetic rings, and orthomodular lattices, only a few isolated axiom problems AP(K) for congruence-distributive equational classes seem to have been previously studied or solved.

The first section will be devoted to a generalization of Wille's theory; it will be shown that if a subclass K of a congruence-distributive equational class E is defined by UDE's, then K^e consists of all algebras in E which satisfy these sentences "primitively." As corollaries, Wille's theorem on primitive sets and Jónsson's theorem of [20] on subdirectly irreducible algebras are obtained. Jónsson's theorem, in fact, can also be regarded as a kind of characterization of K^e, in terms of residual satisfaction (Corollary 1.10).

The second section will show how the notion of primitive satisfaction leads directly to solutions of axiom problems for congruence-distributive equational classes $\mathbf{E}$ in which the intersection of two principal congruence relations is always principal. Typical examples of such classes include those consisting of Heyting algebras, relation algebras, arithmetic rings, vector groups, F-rings, and cylindric algebras of finite dimension, in addition to the more elementary cases of distributive lattices, Boolean algebras, and n-valued Post algebras. In addition to producing easy explicit solutions of axiom problems, the methods of this section shed light on the property of being finitely subdirectly irreducible and on the existence of finite solutions to axiom problems. In particular, for such equational classes, an easy proof is given of the fact that finite algebras have finite equational bases.

§3 shows, for congruence-distributive equational classes, how to express a finite meet of principal congruence relations as a join of principal congruence relations.

In §4, the results of §1 and §3 are used to solve the axiom problems $\mathrm{AP}(\mathbf{K})$ explicitly for all $\mathbf{K}$ studied in §1. Finally, in §5 a method is given for reducing finitely many identities to a single identity, relative to any congruence-distributive equational class.

The notion of primitive satisfaction can be further developed in terms of congruence-valued logic [3].

For the necessary concepts of lattice theory and universal algebra, see Birkhoff [7], Grätzer [12], [13], and Jónsson [21]. Tarski [41] presents a useful and comprehensive survey of equational logic, the broader context of this paper.

For algebras, let us use the notation $\mathfrak{A} = \langle A; o_\gamma, \gamma < \nu\rangle$. (This formality will be relaxed in the case of lattices L.) All algebras in a given discussion will be assumed to be of a fixed type and all polynomial symbols (formal polynomials, terms) $\mathbf{p}$ will be assumed to be formed from corresponding operation symbols, with variables as indicated. The lattice of congruence relations of an algebra $\mathfrak{A}$ will be denoted by $\Theta(\mathfrak{A})$. The smallest member of $\Theta(\mathfrak{A})$ which identifies a and $b \in A$ will be written $\theta(a, b)$; congruence relations of this form are said to be *principal*. $\mathfrak{A}$ is subdirectly irreducible (SI) when $\Theta(\mathfrak{A})$ has a least nonzero element and is finitely subdirectly irreducible (FSI) when 0 in $\Theta(\mathfrak{A})$ is meet-irreducible.

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1. **Primitive satisfaction.** In this section, a generalization of Wille's theory of primitive sets [47] will be presented.

By way of background, recall these facts about congruence relations ψ in a lattice L:

(1) $a \equiv b \pmod{\psi}$ if and only if $a \vee b \equiv a \wedge b$. For this reason, it is customary to restrict attention to pairs $\langle a, b\rangle$ with $a \geq b$. Such a pair is often called a (formal) quotient and is rewritten as a/b.

(2) For quotients a/b, c/d, $a \equiv b \pmod{\psi}$ implies $c \equiv d \pmod{\psi}$ if either (i) c/d is a subquotient of a/b, i.e., $a \geq c \geq d \geq b$, or (ii) c/d is a transpose of a/b, i.e., $a \wedge d = b$ and $a \vee d = c$, or $b \wedge c = d$ and $b \vee c = a$.

(3) If c/d can be reached from a/b by forming a succession of subintervals and transposes, then c/d is said to be *weakly projective* into a/b; an equivalent condition is the existence of $r_1, \cdots, r_k \in L$ such that if $\phi\colon L \to L$ is defined by $\phi(x) = \cdots (((x \vee r_1) \wedge r_2) \vee r_3) \cdots r_k$, then $c/d = \phi(a)/\phi(b)$. For such a/b, c/d, $a \equiv b \pmod{\psi}$ implies $c \equiv d \pmod{\psi}$, by (2).

For convenience, let us say that quotients $a_1/b_1, \cdots, a_N/b_N$ *clash* in L if there exists a quotient c/d which is weakly projective into a_i/b_i for each i and which is nontrivial (i.e., $c \neq d$).

Wille [47] defines a finite subset $Q = \{q_1, \cdots, q_n\}$ of a lattice L to be *primitive* if (in our terminology) all nontrivial quotients of the form $q_i \vee q_j/q_j$ $(i, j = 1, \cdots, n)$ clash (taken together). Wille's main theorem can be phrased as follows.

1.1. Theorem (Wille [47], rephrased). *Let P be a finite partly ordered set, and let $\mathbf{K}$ be the class of all lattices which contain no subset order-isomorphic to P. Then $\mathbf{K}^e$ consists of all lattices which contain no primitive subset order-isomorphic to P.*

In [2], the author showed how to describe such a class $\mathbf{K}$ using a sentence which is a universally quantified disjunction of equations (UDE), i.e., a sentence of the form

$$(\forall x_1) \cdots (\forall x_m) \mathrm{f}_1(\mathrm{x}) = \mathrm{g}_1(\mathrm{x}) \bigvee \cdots \bigvee \mathrm{f}_N(\mathrm{x}) = \mathrm{g}_N(\mathrm{x});$$

here $\bigvee$ denotes disjunction (OR), the f_i and g_i are polynomial symbols in variables $x_1, \cdots, x_m$, and x abbreviates $x_1, \cdots, x_m$. Moreover, for lattices such a sentence can be reduced (if necessary) to an equivalent, new one for which $(\forall \mathrm{x}) f_i(\mathrm{x}) \leq g_i(\mathrm{x})$ holds in all lattices. This analysis suggests that Wille's result can be generalized as follows, a fact observed independently by C. Herrmann [17] and the author.

1.2. Theorem. *Let S be a lattice UDE, say $S = (\forall x_1) \cdots (\forall x_m) f_1(\mathrm{x}) = \mathrm{g}_1(\mathrm{x}) \bigvee \cdots \bigvee \mathrm{f}_N(\mathrm{x}) = \mathrm{g}_N(\mathrm{x})$, where $(\forall \mathrm{x}) \mathrm{f}_i(\mathrm{x}) \leq \mathrm{g}_i(\mathrm{x})$ holds in all lattices. Let $\mathbf{K}$ be the class of lattices satisfying S. Then $\mathbf{K}^e$ consists of all lattices L which* *do not contain elements* $c_1, \cdots, c_m$ *such that the quotients* $g_1(c)/f_1(c), \cdots, g_N(c)/f_N(c)$ *clash.*

The proof will be given below as 1.7.

The following fact seems to be well known. (See for example [39, 13.3], [47], [2, proof of Lemma 3.2].)

1.3. **Proposition.** *In a lattice* L, *quotients* $a_1/b_1, \cdots, a_N/b_N$ *clash if and only if* $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N) > 0$ *in* $\Theta(L)$.

This fact suggests the following generalization of Wille's theory, a generalization on which the remainder of this paper is based.

1.4. **Definition.** *An algebra* $\mathfrak{A}$ *satisfies the UDE*

$$S = (\forall x_1) \cdots (\forall x_m) f_1(x) = g_1(x) \;\mathbb{V}\; \cdots \;\mathbb{V}\; f_N(x) = g_N(x)$$

primitively (in symbols, $\mathfrak{A} \models^{P} S$*) if* $\mathfrak{A}$ *obeys the congruence condition*

$$\text{(1.4*)} \qquad \begin{array}{l} \text{for any } a_1, \cdots, a_m \in A, \\ \theta(f_1(a), g_1(a)) \cap \cdots \cap \theta(f_N(a), g_N(a)) = 0 \text{ in } \Theta(\mathfrak{A}). \end{array}$$

If Σ *is a set of UDE's,* $\mathfrak{A}$ *satisfies* Σ *primitively* ($\mathfrak{A} \models^{P} \Sigma$) *when* $\mathfrak{A} \models^{P} S$ *for all* $S \in \Sigma$.

1.5. **Theorem (Primitive Satisfaction Theorem).** *Let* $\mathbf{E}$ *be a congruence-distributive equational class, and let* Σ *be a set of UDE's. If* $\mathbf{K}$ *consists of all members of* $\mathbf{E}$ *which satisfy* Σ, *then* $\mathbf{K}^e$ *consists precisely of those algebras in* $\mathbf{E}$ *which satisfy* Σ *primitively.*

Before proving this theorem (see 1.12 below) let us pause to examine several basic facts about primitive satisfaction and to draw several corollaries.

1.6. **Observations.** (1) *If* S *is an identity (i.e.,* $N = 1$*), then primitive satisfaction for* S *reduces to ordinary satisfaction for* S.

(2) *If* $\mathfrak{A}$ *is SI, then primitive satisfaction for* $\mathfrak{A}$ *agrees with ordinary satisfaction, since* 0 *in* $\Theta(\mathfrak{A})$ *is meet-irreducible.*

(3) *In general, ordinary satisfaction implies primitive satisfaction, i.e.,* $\mathfrak{A} \models S$ *entails* $\mathfrak{A} \models^{P} S$.

Two theorems stated above as known facts can be drawn as corollaries of the Primitive Satisfaction Theorem 1.5:

1.7. **Proof of Theorem 1.2.** Combine Proposition 1.3 with Theorem 1.5.

1.8. **Proof of Theorem 1.1 (Wille's Theorem).** As in Lemma 2.2 of [2], let $P = \{p(1), \cdots, p(n)\}$; for $i = 1, \cdots, n$, let $g_i(x)$ denote the lattice polynomial $g_i(x_1, \cdots, x_n) = \bigvee_j x_j$ with j ranging over $\{j\colon p(j) \le p(i)\}$, and let S be the sentence $(\forall x_1)\cdots(\forall x_n)\ \bigvee_{i,k}\ [g_i(\mathrm{x}) = g_i(\mathrm{x}) \vee g_k(\mathrm{x})]$ with (i, k) ranging over $\{(i, k)\colon p(k) \not\leq p(i)\}$. Then according to that lemma, S describes $\mathbf{K}$, relative to the class of lattices; indeed, S fails for an evaluation of x_i at c_i $(1 \leq i \leq n)$ in a lattice L precisely when the elements $g_i(\mathrm{c})$ form a copy of P in L. Thus the condition prohibited in Theorem 1.2 occurs in L precisely when L contains a *primitive* subset order-isomorphic to P.

As a further corollary, an alternate proof of Jónsson's key result of [20] can be derived. While less direct than the original proof, it does provide an entirely different path to that result.

1.9. Corollary (Jónsson [20, Corollary 3.2]). *If* $\mathbf{E}$ *is a congruence-distributive equational class and* $\mathbf{K}$ *is a subclass of* $\mathbf{E}$, *then every SI member* $\mathfrak{A}$ *of* $\mathbf{K}^e$ *is a homomorphic image of a subalgebra of an ultraproduct of algebras from* $\mathbf{K}$; *in symbols,* $\mathfrak{A} \in \mathbf{HSU}(\mathbf{K})$. *Thus* $\mathbf{K}^e$ *consists of (isomorphic copies of) subdirect products of algebras in* $\mathbf{HSU}(\mathbf{K})$.

Proof. Let $\mathbf{K}_0 = \mathbf{HSU}(\mathbf{K})$. It can be checked that $\mathbf{K}_0$ is closed under $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{U}$; $\mathbf{K}_0$ is therefore definable by set of positive universal sentences [12, Corollary 1, p. 252, and Corollary 2, p. 275] or equivalently, by a set Σ of UDE's. Moreover, $\mathbf{K}_0^e = \mathbf{K}^e$. If $\mathfrak{A} \in \mathbf{K}_0^e$ is SI, then $\mathfrak{A}$ satisfies Σ primitively by the Primitive Satisfaction Theorem 1.5, so $\mathfrak{A}$ satisfies Σ in the ordinary sense by Observation 1.6(2). Thus $\mathfrak{A} \in \mathbf{K}_0$, as asserted.

B. Jónsson has also pointed out that his theorem just quoted can, in the present context, be phrased in terms of residual satisfaction:

1.10. Corollary (Jónsson [20, Theorem 3.3], rephrased). *Let* $\mathbf{E}$ *be a congruence-distributive equational class, and let* $\mathbf{K}$ *be a subclass of* $\mathbf{E}$ *determined by some set* Σ *of positive universal sentences. Then* $\mathbf{K}^e$ *consists of those members of* $\mathbf{E}$ *which satisfy* Σ *residually, i.e., which are subdirect products of algebras which satisfy* Σ.

Thus, in the context of Theorem 1.5, primitive satisfaction coincides with residual satisfaction. The proof is simply that $\mathbf{HSU}(\mathbf{K}) = \mathbf{K}$ if $\mathbf{K}$ is defined by positive universal sentences. The virtue of the separately named concept of primitive satisfaction is that it is well adapted to computation and production of identities, as subsequent sections will show. Primitive satisfaction and residual satisfaction do not coincide if the hypothesis of congruence-distributivity is omitted.

Let us now turn to a proof of the Primitive Satisfaction Theorem 1.5.

One approach, as suggested by B. Jónsson, is to combine the theorem quoted as Corollary 1.10 with Observation 1.6(3) and a proof that primitive satisfaction is a residual property, i.e., is preserved under subdirect products.

The following proof, which depends on an analysis of the behavior of congruence relations under surjections of algebras, is still brief and permits the derivation of the statements 1.9, 1.10 as legitimate corollaries.

If $\phi: \mathfrak{A} \to \mathfrak{B}$ is a homomorphism of algebras and $\theta \in \Theta(\mathfrak{A})$, let $\phi^*(\theta)$ denote the smallest congruence relation ψ on $\mathfrak{B}$ such that $a_1 \equiv a_2$ (θ) implies $\phi(a_1) \equiv \phi(a_2)$ (ψ).

1.11. Lemma. *Let* $\phi: \mathfrak{A} \to \mathfrak{B}$ *be a surjection. Then* $\phi^*: \Theta(\mathfrak{A}) \to \Theta(\mathfrak{B})$:

(a) *preserves principal congruence relations, i.e.,* $\phi^*(\theta(r, s)) = \theta(\phi(r), \phi(s))$,

(b) *preserves arbitrary joins,*

(c) *for arbitrary meets, obeys* $\phi^*(\bigcap_\alpha \psi_\alpha) \le \bigcap_\alpha \phi^*(\psi_\alpha)$.

If, in addition, $\Theta(\mathfrak{A})$ *is distributive, then*

(d) ϕ^* *is a lattice homomorphism.*

Proof. First let us observe that according to a version of the Correspondence Theorem (cf. [12, Theorem 1, p. 57, and Theorem 3, p. 61]), ϕ^* is the composition $\iota\rho$ of the retraction $\rho: \Theta(\mathfrak{A}) \to \Theta(\mathfrak{A})$ given by $\rho(\theta) = \theta \vee \ker \phi$ [congruence kernel] and the isomorphism ι of the interval $[\ker \phi, 1]$ in $\Theta(\mathfrak{A})$ onto $\Theta(\mathfrak{B})$, as given by $\iota(\psi) = (\phi \times \phi)(\psi)$. For (a): A congruence relation ψ in $[\ker \phi, 1]$ identifies r and s in $\mathfrak{A}$ iff $\iota\psi$ identifies $\phi(r)$ and $\phi(s)$ in $\mathfrak{B}$. The smallest such ψ is $\ker \phi \vee \theta(r, s)$, and the smallest such $\iota\psi$ is $\theta(\phi(r), \phi(s))$, so these correspond under ι. Thus $\phi^*\theta(r, s) = \theta(\phi(r), \phi(s))$. For (b): In the factorization $\phi^* = \iota\rho$, ρ is a complete join homomorphism and ι is an isomorphism. For (c): Any isotone function between lattices obeys such a rule. For (d): If c is a fixed element of a distributive lattice, the retraction $x \mapsto x \vee c$ is a lattice endomorphism. Thus in the factorization $\phi^* = \iota\rho$, ρ is a lattice homomorphism and ι is an isomorphism.

1.12. **Proof of Theorem 1.5.** Let $\mathbf{K}^{\mathrm{P}}$ be the class of all algebras in $\mathbf{E}$ which satisfy Σ primitively. We must show that $\mathbf{K}^{\mathrm{P}} = \mathbf{K}^{\mathrm{e}}$. This equality will follow immediately if it can be shown that $\mathbf{K}^{\mathrm{P}}$ is an equational class. Indeed, by Observation 1.6(3), $\mathbf{K} \subseteq \mathbf{K}^{\mathrm{P}}$, so $\mathbf{K}^{\mathrm{e}} \subseteq \mathbf{K}^{\mathrm{P}}$; on the other hand, the inclusion $\mathbf{K}^{\mathrm{P}} \subseteq \mathbf{K}^{\mathrm{e}}$ follows from the fact that $\mathbf{K}^{\mathrm{P}}$ is generated by its SI members [6], which by Observation 1.6(2) are in $\mathbf{K}$, hence are in $\mathbf{K}^{\mathrm{e}}$.

To show $\mathbf{K}^{\mathrm{P}}$ is equational, let us verify that $\mathbf{K}^{\mathrm{P}}$ is closed under formation of subalgebras, direct products, and homomorphic images [5]. It suffices to check that primitive satisfaction of each individual UDE $S \in \Sigma$ is preserved under these three constructions. Write

$$S = (\forall x_1) \cdots (\forall x_m) \mathrm{f}_1(\mathrm{x}) = \mathrm{g}_1(\mathrm{x}) \mathbin{\text{W}} \cdots \mathbin{\text{W}} \mathrm{f}_N(\mathrm{x}) = \mathrm{g}_N(\mathrm{x}).$$

Subalgebras. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$, where $\mathfrak{A} \models^{P} S$. For $b_1, b_2 \in B$, observe that principal congruence relations obey $\theta_{\mathfrak{B}}(b_1, b_2) \subseteq \theta_{\mathfrak{A}}(b_1, b_2) \cap B^2 \subseteq \theta_{\mathfrak{A}}(b_1, b_2)$. Then for any $c_1, \cdots, c_m \in B$, $\theta_{\mathfrak{B}}(f_1(c), g_1(c)) \cap \cdots \cap \theta(f_N(c), g_N(c)) \subseteq \theta_{\mathfrak{A}}(f_1(c), g_1(c)) \cap \cdots \cap \theta_{\mathfrak{A}}(f_N(c), g_N(c)) = 0$, so that $\mathfrak{B} \models^{P} S$.

Direct products. Suppose $\mathfrak{B} = \Pi_{i \in I} \mathfrak{A}_i$, where $\mathfrak{A}_i \models^{P} S$. For given $c_1, \cdots, c_m \in B$, let $\psi = \bigcap_{j=1}^{N} \theta(f_j(c), g_j(c))$. If $\pi_i: \mathfrak{B} \to \mathfrak{A}_i$ is any coordinate projection, (c) and (a) of Lemma 1.11, the fact that π_i is a homomorphism, and the definition of $\mathfrak{A}_i \models^{P} S$ yield in turn

$$\pi_i^*(\psi) \leq \bigcap_{j=1}^{N} \pi_i^*\theta(f_j(c), g_j(c)) = \bigcap_{j=1}^{N} \theta(\pi_i f_j(c), \pi_i g_j(c)) = \bigcap_{j=1}^{N} \theta(f_j(d), g_j(d)) = 0,$$

where $d_1 = \pi_i(c_1), \cdots, d_m = \pi_i(c_m)$. Thus $\psi \leq \ker \pi_i$ for each i. Since $\bigcap_{i \in I} \ker \pi_i = 0$ in $\Theta(\mathfrak{B})$, $\psi = 0$. Since c was arbitrary, $\mathfrak{B} \models^{P} S$.

Homomorphisms. Let $\phi: \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, where $\mathfrak{A} \models^{P} S$. For any given $c_1, \cdots, c_m \in B$, choose $a_1, \cdots, a_m \in A$ with $\phi(a_i) = c_i$. By Lemma 1.11(a) and (d) [here the distributivity of $\Theta(\mathfrak{A})$ is used], we obtain $\bigcap_{j=1}^{N} \theta(f_j(c), g_j(c)) = \bigcap_{j=1}^{N} \phi^*\theta(f_j(a), g_j(a)) = \phi^*[\bigcap_{j=1}^{N} \theta(f_j(a), g_j(a))] = \phi^*(0) = 0$. Therefore $\mathfrak{B} \models^{P} S$. The proof is thus complete.

2. Classes with the principal intersection property. How can the notion of primitive satisfaction, which involves congruence relations, be used to find identities which solve axiom problems AP(K)? A full answer must await §4. In this section, however, the question will be shown to have an easy answer for a number of familiar congruence-distributive equational classes.

To focus the discussion, let us consider several specific problems to which the methods of this section are amenable.

2.1. *Problem.* Let **H** be the equational class of Heyting algebras, i.e., relatively pseudo-complemented distributive lattices with 0, or pseudo-Boolean algebras in the sense of Rasiowa and Sikorski [38, p. 58], as algebras $\langle A, \vee, \wedge, \to, 0, 1\rangle$. (Here $\to$ is relative pseudo-complementation.) Let $\mathbf{W}_n$ be the class of all Heyting algebras whose width as lattices is at most n, i.e., such that of any $n + 1$ elements, some two are comparable. Solve AP($\mathbf{W}_n$).

2.2. *Problem.* Let **RA** be the class of relation algebras (Tarski [40], Jónsson and Tarski [22]), i.e., algebras $\langle A, \vee, \wedge, \neg, 0, 1, ;, \breve{}, 1'\rangle$, subject to certain identities based on the properties of the algebra of binary relations on a set. Let $\mathbf{B}_n$ be the set of relation algebras which are finite with at most n atoms, or equivalently, which have breadth at most n (i.e., the join of any $n + 1$ elements is redundant.) Solve AP($\mathbf{B}_n$).

2.3. *Problem.* For $m > 1$, let AR(m) be the class of all rings satisfying the identity $x^m = x$. AR(m) is known to consist of commutative, arithmetic rings

(rings with distributive ideal lattices). For recent theory see [27], [34], and [43]. Let $\mathbf{C}_{n,m}$ be the class of all members of $\mathbf{AR}(m)$ which have at most n elements. Solve $\mathrm{AP}(\mathbf{C}_{n,m})$.

As an example of the approach to be used in solving such problems, the following proposition and solution are representative.

2.4. **Proposition** [4, **Theorem 2.1**]. *Let S be a UDE in the language of Heyting algebras, say*

$$S = (\forall x_1)\cdots(\forall x_m) \mathrm{f}_1(\mathrm{x}) = \mathrm{g}_1(\mathrm{x}) \ \bigvee\!\!\!\bigvee \cdots \bigvee\!\!\!\bigvee \ \mathrm{f}_N(\mathrm{x}) = \mathrm{g}_N(\mathrm{x}).$$

Let $\mathbf{K}$ *be the class of Heyting algebras defined by S. Then* $\mathbf{K}^e$ *consists of all Heyting algebras satisfying the identity*

$$(\forall x_1)\cdots(\forall x_m)[\mathrm{f}_1(\mathrm{x}) \leftrightarrow \mathrm{g}_1(\mathrm{x})] \vee \cdots \vee [\mathrm{f}_N(\mathrm{x}) \leftrightarrow \mathrm{g}_N(\mathrm{x})] = 1.$$

Here $a \leftrightarrow b$ is shorthand for $(a \rightarrow b) \wedge (b \rightarrow a)$.

Proof. For Heyting algebras, $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N) = \theta(a_1 \leftrightarrow b_1, 1) \cap \cdots \cap \theta(a_N \leftrightarrow b_N, 1) = \theta([a_1 \leftrightarrow b_1] \vee \cdots \vee [a_N \leftrightarrow b_N], 1)$. The Primitive Satisfaction Theorem 1.5 applies.

2.5. **Application.** Problem 2.1 above can be solved by noting that $\mathbf{W}_n$ is defined, relative to $\mathbf{H}$, by the UDE $\delta = (\forall x_1)\cdots(\forall x_{n+1}) \ \bigvee\!\!\!\bigvee_{i \neq j} (x_i \leq x_j)$, where $x \leq y$ is regarded as an abbreviation of $x \wedge y = x$. Thus a solution consists of the identity $(\forall x_1)\cdots(\forall x_{n+1})[\bigvee_{i \neq j} (x_i \wedge x_j \leftrightarrow x_i) = 1]$, together with defining identities for the class $\mathbf{H}$ of Heyting algebras. In particular, the solution is finite. (The solution can be slightly simplified by means of the Heyting algebra identity $(x \wedge y) \leftrightarrow x = x \rightarrow y$.)

More generally, for a congruence-distributive equational class $\mathbf{E}$, primitive satisfaction is easily re-expressible with identities if $\mathbf{E}$ happens to possess, for each $N \geq 2$, a pair of polynomial symbols $\mathbf{D}_0^{(N)}$, $\mathbf{D}_1^{(N)}$ in $2N$ variables $y_1, z_1, \cdots, y_N, z_N$ such that

$$\begin{aligned} &\text{for any } \mathfrak{A} \in \mathbf{E} \text{ and } a_1, b_1 \cdots, a_N, b_N \in A, \\ (2.6) \qquad &\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N) = \theta(d_0, d_1) \text{ in } \Theta(\mathfrak{A}), \\ &\text{where } d_i = D_i^{(N)}(a_1, b_1 \cdots, a_N, b_N) \quad \text{for } i = 0, 1. \end{aligned}$$

In the case of Heyting algebras, for example, $\mathbf{D}_0^{(N)}(y_1, z_1, \cdots, y_N, z_N) = (y_1 \leftrightarrow z_1) \vee \cdots \vee (y_N \leftrightarrow z_N)$, while $\mathbf{D}_1^{(N)}$ is the constant polynomial 1.

For convenience, let us call such $\mathbf{D}_0^{(N)}$, $\mathbf{D}_1^{(N)}$ a *pair of intersection polynomials in $2N$ variables.* Thus the Primitive Satisfaction Theorem 1.5 has the following practical consequence.

2.7. **Theorem.** *Suppose the congruence-distributive equational class* $\mathbf{E}$ *possesses, for each* N, *a pair of principal intersection polynomials* $\mathbf{D}_0^{(N)}, \mathbf{D}_1^{(N)}$ *in* $2N$ *variables. If* $\mathbf{K}$ *is the class of all algebras in* $\mathbf{E}$ *satisfying a given UDE* $S = (\forall \mathbf{x}) \bigvee_{i=1}^{N} \mathbf{f}_i(\mathbf{x}) = \mathbf{g}_i(\mathbf{x})$, *then* $\mathbf{K}^e$ *consists of all algebras in* $\mathbf{E}$ *which satisfy the identity* $D[S]$ *given by*

$$D[S] = (\forall \mathbf{x})[D_0^{(N)}(\mathbf{f}_1(\mathbf{x}), \mathbf{g}_1(\mathbf{x}), \cdots, \mathbf{f}_N(\mathbf{x}), \mathbf{g}_N(\mathbf{x})) \\ = D_1^{(N)}(\mathbf{f}_1(\mathbf{x}), \mathbf{g}_1(\mathbf{x}), \cdots, \mathbf{f}_N(\mathbf{x}), \mathbf{g}_N(\mathbf{x}))].$$

More generally, if Σ *is a set of UDE's and* $\mathbf{K}$ *is the class of algebras of* $\mathbf{E}$ *which satisfy* Σ, *then one solution to* $\mathrm{AP}(\mathbf{K})$ *consists of the identities* $D[S]$ *for* $S \in \Sigma$, *together with a set of defining identities for* $\mathbf{E}$.

When we attempt to apply this theorem, several questions naturally arise: (1) If $\mathbf{E}$ has principal intersection polynomials in $2N$ variables for one value of $N \geq 2$, must $\mathbf{E}$ have such polynomials for all N? (2) Which equational classes $\mathbf{E}$ do have principal intersection polynomials? (3) For such $\mathbf{E}$, how can such polynomials be found explicitly?

The answer to the first question is affirmative. Indeed, $\mathbf{D}_0^{(N)}, \mathbf{D}_1^{(N)}$ for some $N \geq 2$ will yield $\mathbf{D}_0^{(2)}, \mathbf{D}_1^{(2)}$ by suitable repetition of variables; on the other hand, if $\mathbf{D}_0^{(2)}, \mathbf{D}_1^{(2)}$ are known, then $\mathbf{D}_i^{(N)}$ for $N > 2$ in variables y_j, z_j can be defined recursively by the composition $\mathbf{D}_i^{(N)} = \mathbf{D}_i^{(2)}(\mathbf{D}_0^{(N-1)}, \mathbf{D}_1^{(N-1)}, y_N, z_N)$.

A simple, practical answer to the second question starts from the trivial observation that if $\mathbf{E}$ has principal intersection polynomials, then $\mathbf{E}$ has the "principal intersection property" (PIP): For any algebra $\mathfrak{A}$ in $\mathbf{E}$, the intersection of two principal congruence relations on $\mathfrak{A}$ is again principal. Interestingly, for congruence-distributive equational classes, the converse of the observation is valid and provides the desired criterion:

2.8. **Theorem.** *Let* $\mathbf{E}$ *be a congruence-distributive equational class. Then the following conditions are equivalent*:

(1) $\mathbf{E}$ *has the PIP*;

(2) $\mathbf{E}$ *has a pair of principal intersection polynomials in* $2N$ *variables for some* $N \geq 2$;

(3) $\mathbf{E}$ *has such polynomials for all* $N \geq 2$.

For example, Heyting algebras, relation algebras, and cylindric algebras of finite dimension are known to share the PIP and so must have principal intersection polynomials.

Proof. We already know (2) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (1). For (1) $\Rightarrow$ (2), let $F = F(a_1, b_1, a_2, b_2, r_1, r_2, \cdots)$ be the free $\mathbf{E}$-algebra on the countably many generators indicated. By hypothesis, $\theta(a_1, b_1) \cap \theta(a_2, b_2)$ in F is principal, say equal to $\theta(c_0, c_1)$ for some $c_0, c_1 \in F$. Here c_0, c_1 are polynomial expressions in finitely many generators, say at most those from a_1 through r_k. We shall eliminate the dependency on any r_i. Let ϕ: $F \to F$ be an endomorphism which leaves a_1, b_1, a_2, b_2 fixed, which moves r_i to r_{i-k} for $i > k$, and which moves $r_1, \cdots, r_k$ into the subalgebra $F(a_1, b_1, a_2, b_2)$ of F. Since ϕ is surjective, Lemma 1.11 shows that $\theta(a_1, b_1) \cap \theta(a_2, b_2) = \theta(d_0, d_1)$ in F, where $d_i = \phi c_i \in F(a_1, b_1, a_2, b_2)$ for $i = 0, 1$. Thus d_i is a polynomial expression in a_1, b_1, a_2, b_2 alone, say $d_i = D_i(a_1, b_1, a_2, b_2)$. Write $\mathbf{D}_i^{(2)} = \mathbf{D}_i$. Then $\mathbf{D}_0^{(2)}, \mathbf{D}_1^{(2)}$ have the desired property (2.6) when a_1, b_1, a_2, b_2 are among the generators of a free algebra F on countably many generators. By the use of an appropriate surjection and Lemma 1.11, (2.6) can be verified for any elements a_1, b_1, a_2, b_2 in any countably generated algebra in $\mathbf{E}$. But by Mal'cev's internal description of principal congruence relations, quoted as Theorem 3.3 below, a principal congruence relation $\theta(a, b)$ on any algebra $\mathfrak{A}$ is the set-union of the corresponding principal congruence relations on those finitely generated subalgebras of $\mathfrak{A}$ which contain a and b. The condition (2.6) follows for arbitrary algebras in $\mathbf{E}$. Thus (2) holds with $N = 2$, and the proof is complete.

For the third question, that of finding suitable, explicit $\mathbf{D}_0^{(N)}, \mathbf{D}_1^{(N)}$ for $\mathbf{E}$, one answer has been provided by the proof just concluded: Compute $\theta(a_1, b_1) \cap \theta(a_2, b_2)$ in the free algebra $F(a_1, b_1, a_2, b_2)$, express it as a principal congruence relation to obtain $\mathbf{D}_0^{(2)}, \mathbf{D}_1^{(2)}$, and then compute $\mathbf{D}_0^{(N)}, \mathbf{D}_1^{(N)}$ by induction. In some cases this procedure is easy; in others, however, the free algebra is complex and the necessary computation of congruence relations is difficult. Instead, the following characterization can be used to invent the $\mathbf{D}_i^{(N)}$ on an ad hoc basis.

2.9. **Proposition.** *Let* $\mathbf{E}$ *be a congruence-distributive equational class. Then polynomial symbols* $\mathbf{D}_0^{(N)}, \mathbf{D}_1^{(N)}$ *in* $2N$ *variables are principal intersection polynomials if and only if*

(2.9*) *for any SI algebra* $\mathfrak{A} \in \mathbf{E}$ *and for any* $a_1, b_1, \cdots, a_N, b_N \in A$, $D_0^{(N)}(a_1, b_1, \cdots, a_N, b_N) = D_1^{(N)}(a_1, b_1, \cdots, a_N, b_N)$ *if and only if* $a_i = b_i$ *for some* i.

This condition is reminiscent of the defining property of an integral domain. In fact, if the SI algebras in $\mathbf{E}$ happen to *be* integral domains, then suitable $\mathbf{D}_0^{(N)}, \mathbf{D}_1^{(N)}$ are given by $\mathbf{D}_0^{(N)}(y_1, z_1, \cdots, y_N, z_N) = (y_1 - z_1)(y_2 - z_2) \cdots (y_N - z_N)$, $\mathbf{D}_1^{(N)} = 0$. Exactly this situation obtains for arithmetic rings; see 2.11 below.

Proof of 2.9. Suppose $\mathbf{D}_0^{(N)}, \mathbf{D}_1^{(N)}$ are principal intersection polynomials. If $\mathfrak{A}$ is SI, then 0 in $\Theta(\mathfrak{A})$ is meet-irreducible, so $D_0^{(N)}(a_1, b_1, \cdots, a_N, b_N) = D_1^{(N)}(a_1, b_1, \cdots, a_N, b_N)$ iff $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N) = 0$ iff $a_i = b_i$ for some i. Conversely, suppose (2.9*) holds; we must verify (2.6). Let any $\mathfrak{A} \in \mathbf{E}$ and any $a_1, b_1, \cdots, a_N, b_N \in A$ be given, and let $d_i = D_i^{(N)}(a_1, b_1, \cdots, a_N, b_N)$. Since congruence relations in the algebraic lattice $\Theta(\mathfrak{A})$ are intersections of completely meet-irreducible (c.m.i.) congruence relations π, it is enough to check that for each such π, $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N) \leq \pi$ if and only if $\theta(d_0, d_1) \leq \pi$. Upon using Lemma 1.11 to pass to $\mathfrak{A}/\pi$, with congruence classes denoted $\bar{a}_1$, etc., the desired equivalence becomes "$\theta(\bar{a}_1, \bar{b}_1) \cap \cdots \cap \theta(\bar{a}_N, \bar{b}_N) = 0$ if and only if $\theta(\bar{d}_0, \bar{d}_1) = 0$". Since $\mathfrak{A}/\pi$ is SI, this condition holds by (2.9*) applied to $\mathfrak{A}/\pi$ and the $\bar{a}_i, \bar{b}_i$.

As illustrative applications of the theory now developed, let us solve the remaining problems raised in the introduction of this section.

2.10. *Solution to Problem* 2.2. We must investigate principal congruence relations. Among the postulates for relation algebras is the requirement that $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ be a Boolean algebra. Therefore each congruence relation ψ is determined by the congruence class of 0, which is a lattice ideal: $a \equiv b \pmod{\psi}$ iff $a \,\Delta\, b \equiv 0 \pmod{\psi}$, where $a \,\Delta\, b$ denotes the symmetric difference $(\neg\, a \wedge b) \vee (\neg\, b \wedge a)$. It follows from [22, §4] that a principal congruence relation $\theta(a, b)$ (or equivalently, $\theta(a \,\Delta\, b, 0)$) corresponds to the principal lattice ideal $[1; (a \,\Delta\, b); 1]$, where $[c]$ denotes $\{v \colon v \leq c\}$. Equipped with this information, we see that $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N)$ corresponds to the lattice ideal $[1; (a_1 \,\Delta\, b_1); 1] \cap \cdots \cap [1; (a_N \,\Delta\, b_N); 1] = [\bigwedge_i (1; (a_i \,\Delta\, b_i); 1)]$ and so equals $\theta(0, d_1)$ where $d_1 = \bigwedge_i (1; (a_i \,\Delta\, b_i); 1)$. Thus **RA** has the PIP and $\mathbf{D}_0^{(N)}, \mathbf{D}_1^{(N)}$ fall out of the computation. The subclass $\mathbf{B}_n$ of **RA** is definable relative to **RA** by the UDE

$$(\forall x_1) \cdots (\forall x_{n+1}) \quad \bigvee\!\!\!\bigvee_i \left(\bigvee_j x_j = \bigvee_{j \neq i} x_j \right).$$

According to Theorem 2.7, then, a solution to AP $(\mathbf{B}_n)$ is given by the identity

$$(\forall x_1) \cdots (\forall x_{n+1}) 0 = \bigwedge_i \left(1; \left(\bigvee_j x_j \,\Delta\, \bigvee_{j \neq i} x_j\right); 1\right),$$

together with defining identities for **RA**.

2.11. *Solution to Problem* 2.3. It is known [27], [43] that every SI member of $\mathbf{AR}(m)$ is a finite field. Hence, as mentioned in remarks after 2.9, suitable $\mathbf{D}_0^{(N)}, \mathbf{D}_1^{(N)}$ are immediately found using products of differences. $\mathbf{C}_{n,m}$ is defined, relative to $\mathbf{AR}(m)$, by the UDE

$$(\forall x_1) \cdots (\forall x_{n+1}) \quad \bigvee\!\!\!\bigvee_{i<j} x_i = x_j.$$

Therefore a solution to AP$(\mathbf{C}_{n,m})$ consists of the identity

$$(\forall x_1) \cdots (\forall x_{n+1}) \prod_{i<j} (x_i - x_j) = 0.$$

The theorems of this section apply to other equational classes as well:

2.12. **Application.** Let $\mathbf{CA}(n)$ be the class of cylindric algebras of finite dimension n, as algebras $\langle A, \vee, \wedge, \neg, 0, 1, \mathrm{c}_i, \mathrm{d}_{ij}\rangle_{i,j<n}$, subject to certain identities [15, Definition 2.1, p. 97]. Here $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra, the d_{ij} are constants, and the c_i are unary operations with properties in imitation of those of existential quantifiers. As in 2.10, let Δ denote the symmetric difference. It is known [15, Theorem 2.5, p. 98] that the SI members of $\mathbf{CA}(n)$ are those in which $a \neq 0$ implies $c_0c_1 \cdots c_{n-1}a = 1$. From this fact, it is easy to invent a pair of polynomial symbols $\mathbf{D}_0^{(N)}, \mathbf{D}_1^{(N)}$ satisfying the criterion of Proposition 2.6: Let $\mathbf{D}_0^{(N)} = 0$ and let

$$\mathbf{D}_1^{(N)} = [\mathrm{c}_0\mathrm{c}_1 \cdots \mathrm{c}_{n-1}(y_1 \,\Delta\, z_1)] \wedge \cdots \wedge [\mathrm{c}_0\mathrm{c}_1 \cdots \mathrm{c}_{n-1}(y_N \,\Delta\, z_N)].$$

In particular, $\mathbf{CA}(n)$ has the PIP.

The situation for finite-dimensional polyadic algebras is similar.

2.13. **Application.** Let $\mathbf{VG}$ be the equational class of vector groups, i.e., lattice-ordered groups which are subdirect products of totally ordered groups [11, p. 88]. Let us use multiplicative notation with unit element 1 and absolute value function $|x| = x \vee x^{-1}$. Recall that $|x| \geq 1$ holds and that $|x| = 1$ iff $x = 1$. Since the SI members of $\mathbf{VG}$ are totally ordered, principal intersection polynomials are easily invented: $\mathbf{D}_0^{(N)}(y_1, z_1, \cdots, y_N, z_N) = |y_1z_1^{-1}| \wedge \cdots \wedge |y_Nz_N^{-1}|$, $\mathbf{D}_1^{(N)} = 1$. In spite of their simple order structure, the members of $\mathbf{VG}$ form many equational subclasses, such as those defined by nilpotence or solvability conditions. Abelian lattice-ordered groups constitute a least nontrivial equational subclass of $\mathbf{VG}$.

The case of F-rings (subdirect products of totally ordered rings) is similar.

Other equational classes with the PIP include vector lattices, distributive lattices, Boolean algebras, and n-valued Post algebras ([10], [42]). Since these classes have no nontrivial equational subclasses, their equational theories are not normally objects of study. However, the theory of this section does lead directly to the discovery of results such as the following.

2.14. **Proposition.** *Let L be a distributive lattice and let $a_1, b_1, a_2, b_2 \in L$. Then $\theta(a_1, b_1) \cap \theta(a_2, b_2) = \theta(m(a_1, a_2, b_1), m(a_1, b_2, b_1))$, where $m(x, y, z)$ is the lattice median $(x \vee y) \wedge (x \vee z) \wedge (y \vee z)$.*

Proof. The only SI distributive lattice is the two-element chain $2 = \{0, 1\}$. The condition (2.9*) is immediate.

In this connection, G. Grätzer had previously established that if $a_1 \leq b_1$ and $a_2 \leq b_2$ in a distributive lattice L, then $\theta(a_1, b_1) \cap \theta(a_2, b_2) =$

$\theta(a_1 \vee a_2, a_1 \vee a_2 \vee (b_1 \wedge b_2))$ [14]. Proposition 2.14 yields the alternate form $\theta((a_1 \vee a_2) \wedge b_1, (a_1 \vee b_2) \wedge b_1)$.

A number of interesting properties are direct consequences of the PIP.

2.15. **Theorem.** *Let* $\mathbf{E}$ *be a congruence-distributive equational class having the principal intersection property and definable by a finite list of identities. Then:*

(a) *In* $\mathbf{E}$, *the property of being finitely subdirectly irreducible (FSI) is strictly elementary.*

(b) *Any subalgebra of an SI or FSI algebra is FSI.*

(c) *If* $\mathbf{K}$ *is any strictly elementary positive universal subclass of* $\mathbf{E}$, *then* $\mathrm{AP}(\mathbf{K})$ *has a finite solution.*

(d) *If* $\mathfrak{A}$ *is any finite algebra in* $\mathbf{E}$, *then* $\mathfrak{A}$ *has a finite equational basis.*

(e) *In the lattice of equational subclasses of* $\mathbf{E}$, *the finitely based members form a sublattice.*

The proof will be given as 2.17 below.

2.16. *Notes on the theorem.* (a) By "strictly elementary" is meant "expressible by a single first-order sentence", or equivalently, by finitely many such sentences. The case of vector lattices [7, Chapter XV] shows that being SI (in place of FSI) need not be strictly elementary even if the hypotheses of the theorem are met: The reals form an SI vector lattice R, an ultrapower of R is a non-SI vector lattice $^*R$ of nonstandard reals, and yet ultrapowers preserve all strictly elementary properties.

(b) Of course, an SI algebra is FSI. Instructive examples are obtained from totally ordered Heyting algebras, all of which are FSI. Only those with a largest element less than 1 are SI.

(c) By a positive universal class let us mean one which can be defined by positive universal sentences, or equivalently, by UDE's. The condition (c) is equivalent to the statement that for a positive universal class $\mathbf{K}$ in $\mathbf{E}$, if $\mathbf{K}$ is strictly elementary then so is $\mathbf{K}^e$. A typical application is the conclusion that if $\mathbf{K}$ consists of all Heyting algebras with at most n elements, then $\mathbf{K}^e$ is finitely based.

(d) An equational basis for A means a set of defining identities for $\{A\}^e$. By a careful argument, to appear elsewhere, it can be shown that the conclusion (d) holds under the sole hypothesis that $\mathbf{E}$ be congruence-distributive. Under the hypotheses of Theorem 2.15, however, the proof of (d) is immediate. In particular, the theorem unifies the cases of Heyting algebras [25], cylindric algebras of finite dimension ([29], [26]), and relation algebras [26], for which (d) was known.

(e) Here closure under intersection is automatic; the condition (e) is thus equivalent to the statement that the join of two finitely based equational subclasses is again such.

If the hypothesis that $\mathbf{E}$ be finitely definable is deleted, then the conclusions and proof of the theorem remain valid, provided that strict elementarity/finite basis statements are interpreted relative to $\mathbf{E}$.

2.17. **Proof of Theorem 2.15.** (a) An algebra $\mathfrak{A}$ is FSI when 0 is finitely meet-irreducible in the congruence lattice $\Theta(\mathfrak{A})$. Since every member of $\Theta(\mathfrak{A})$ is a join of principal congruence relations, an equivalent requirement is that $\theta(a, b) \cap \theta(c, d) = 0$ imply $\theta(a, b) = 0$ or $\theta(c. d) = 0$. Since $\mathbf{E}$ has principal intersection polynomials $\mathbf{D}_0^{(2)}$, $\mathbf{D}_1^{(2)}$, we conclude that $\mathfrak{A} \in \mathbf{E}$ is FSI if and only if $\mathfrak{A}$ satisfies the condition

$$(2.17^*)\quad (\forall y_1)(\forall z_1)(\forall y_2)(\forall z_2)[\mathbf{D}_0^{(2)}(y_1, z_1, y_2, z_2) = \mathbf{D}_1^{(2)}(y_1, z_1, y_2, z_2) \rightarrow (y_1 = z_1 \bigvee y_2 = z_2)].$$

(b) A universal sentence, such as (2.17*), is preserved under passage to subalgebras.

(c) A strictly elementary positive universal class is definable by finitely many UDE's, in terms of which Theorem 2.7 gives a finite solution.

(d) The class $\mathbf{K}$ of homomorphic images of subalgebras of $\mathfrak{A}$ is a positive universal class [12, p. 275]; $\mathbf{K}$ is strictly elementary since it contains only a finite number of algebras, up to isomorphism. By (c), $\mathbf{K}^e = \{\mathfrak{A}\}^e$ is finitely based.

(e) Let us prove the equivalent form given under 2.16(e). If $\mathbf{K}_1$ and $\mathbf{K}_2$ are equational subclasses of $\mathbf{E}$, then $\mathbf{K}_1 \vee \mathbf{K}_2$ is simply $(K_1 \cup K_2)^e$. If K_i is defined by a finite set Σ_i of identities for $i = 1, 2$, then $K_1 \cup K_2$ is determined by a positive universal sentence: the conjunction of the finitely many positive universal sentences of the form $S_1 \bigvee S_2$, where $S_1 \in \Sigma_1$, $S_2 \in \Sigma_2$. Thus (c) applies.

Remark. The conclusions of Theorem 2.15 remain valid if the PIP is replaced by the "compact intersection property" (CIP): The intersection of two compact congruence relations is compact. This property is actually equivalent to the conjunction of (a) and (b) and is almost equivalent to each of (c) and (e). Details will appear elsewhere.

3. **Intersections of principal congruence relations.** An intersection $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N)$, like all other congruence relations, is a join of principal congruence relations [12, p. 52]. Which ones? The purpose of this section is to answer this question (Theorem 3.4). In the next section, this answer will be used to find defining identities for classes $\mathbf{K}^e$.

Let us start simply by looking for any obvious principal congruence relations contained in an intersection $\theta(a_1, b_1) \cap \theta(a_2, b_2)$. In a lattice L, if $a_1 \le b_1$ and $a_2 \le b_2$, one example is easy to find: $\theta(c, d)$, where c and d are the images of a_2 and b_2 under the familiar map $u \mapsto (u \vee a_1) \wedge b_1$ which retracts L onto the interval $[a_1, b_1]$; i.e., $c = (a_2 \vee a_1) \wedge b_1$, $d = (b_2 \vee a_1) \wedge b_1$. This construction, it will be recalled, is the core of the lattice-theoretic version of the Jordan-Hölder theorem [32]. Let us regard the pair $\langle c, d\rangle$ as the result of a kind of operation of $\langle a_1, b_1\rangle$ upon $\langle a_2, b_2\rangle$ and write $\langle c, d\rangle = \langle a_1, b_1\rangle * \langle a_2, b_2\rangle$.

Now consider more generally the case of any algebra $\mathfrak{A}$ which possesses a median polynomial, i.e., a polynomial $m(x, y, z)$ satisfying the identities $m(x, x, y) = x$, $m(x, y, x) = x$, $m(y, x, x) = x$. (Grätzer suggests the term "democratic polynomial" for such a polynomial, since its value is determined by a majority vote. See also Pixley [35].) In this case, for pairs $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in A \times A$, the map $x \mapsto m(a_1, x, b_1)$ yields an operation $\langle a_1, b_1\rangle * \langle a_2, b_2\rangle = \langle m(a_1, a_2, b_1), m(a_1, b_2, b_1)\rangle$. Again, we obtain $\theta(\langle a_1, b_1\rangle * \langle a_2, b_2\rangle) \subseteq \theta(a_1, b_1) \cap \theta(a_2, b_2)$. Observe that for the case of lattices, where a suitable median polynomial is $m(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$, the new median-induced operation on pairs does reduce to the operation originally defined, when $a_1 \le b_1$.

This idea can be generalized further to congruence-distributive equational classes, by using the following characterization due to Jónsson:

3.1. **Theorem** (Jónsson [20]). *An equational class* $\mathbf{E}$ *is congruence-distributive if and only if there exists a finite list of polynomial symbols* $\mathbf{t}_0, \cdots, \mathbf{t}_n$ $(n \ge 2)$ *in three variables such that the following identities hold for all algebras in* $\mathbf{E}$:

(3.1a) $$t_k(x, y, x) = x \qquad (k = 0, 1, \cdots, n);$$

(3.1b) $$t_k(x, x, z) = t_{k+1}(x, x, z) \quad \text{if } k \text{ is even},$$
$$t_k(x, z, z) = t_{k+1}(x, z, z) \quad \text{if } k \text{ is odd};$$

(3.1c) $$t_0(x, y, z) = x, \qquad t_n(x, y, z) = z.$$

Observe that $\mathbf{t}_0$ and $\mathbf{t}_n$ are included merely as a notational convenience.

In the simplest case, where $n = 2$, t_1 is a median polynomial. Thus a natural generalization of the $*$ operation defined above is to define a sequence of operations $*_1, \cdots, *_{n-1}$ on pairs as follows: If $\mathfrak{A} \in \mathbf{E}$ and $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in A \times A$, define $\langle a_1, b_1\rangle *_k \langle a_2, b_2\rangle = \langle t_k(a_1, a_2, b_1), t_k(a_1, b_2, b_1)\rangle$ for $k = 1, \cdots, n - 1$. By (3.1a), we have again

3.2. **Observation.** $\theta(\langle a_1, b_1\rangle *_k \langle a_2, b_2\rangle) \subseteq \theta(a_1, b_1) \cap \theta(a_2, b_2)$ *for* $k = 1, \cdots, n-1$.

Unfortunately, the operations $*_k$ are not in themselves sufficient to describe $\theta(a_1, b_1) \cap \theta(a_2, b_2)$. (One exception is the case of distributive lattices, where it has been shown (Corollary 2.14) that indeed $\theta(a_1, b_1) \cap \theta(a_2, b_2) = \theta(\langle a_1, b_1\rangle * \langle a_2, b_2\rangle)$.) We are thus forced to examine in more detail the structure of principal congruence relations. This structure is given by a theorem said by Grätzer [12, p. 54] to be implicit in Mal'cev [24]. The key ingredient is the concept of a "unary algebraic function": a function ϕ on an algebra to itself obtained by freezing all entries except one in some polynomial, i.e., $\phi(u) = p(c_1, \cdots, c_{i-1}, u, c_{i+1}, \cdots, c_l)$ for some polynomial function p on the algebra [12, p. 45]. For example, the function $u \mapsto m(a_1, u, b_1)$ used above is a unary algebraic function. Let $U_{\mathfrak{A}}$ be the set of all unary algebraic functions on $\mathfrak{A}$.

3.3. **Theorem** (Mal'cev [24], [12, p. 54]). *Let $\mathfrak{A}$ be an algebra, and let $a, b, c, d \in A$. Then $c \equiv d \bmod \theta(a, b)$ if and only if there exists a finite sequence $c = c_0, c_1, \cdots, c_r = d$ in A and $\phi_1, \cdots, \phi_r \in U_{\mathfrak{A}}$ such that, for each $i = 1, \cdots, r$, $\{c_{i-1}, c_i\} = \{\phi_i(a), \phi_i(b)\}$. [Note the unordered pairs.]*

Thus $\theta(a, b)$ is the smallest equivalence relation containing the smallest "$U_{\mathfrak{A}}$-invariant" subset of $A \times A$ which contains $\langle a, b\rangle$, namely the subset $\{\langle \phi a, \phi b\rangle : \phi \in U_{\mathfrak{A}}\}$.

The questions asked at the beginning of this section can now be answered.

3.4. **Theorem.** *Let $\mathbf{E}$ be a congruence-distributive equational class, let $\mathbf{t}_0, \cdots, \mathbf{t}_n$ be Jónsson's associated polynomial symbols, let $\mathfrak{A} \in \mathbf{E}$, and let $*_1, \cdots, *_{n-1}$ be the induced operations on $A \times A$. Then for any $\langle a_1, b_1\rangle, \cdots, \langle a_N, b_N\rangle \in A \times A$, $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N) = \bigvee \theta(\langle \phi_1 a_1, \phi_1 b_1\rangle *_{k(2)} \cdots *_{k(N)} \langle \phi_N a_N, \phi_N b_N\rangle)$, where $\phi_1, \cdots, \phi_N$ range through $U_{\mathfrak{A}}$, where $k(2), \cdots, k(N)$ range independently from 1 to $n-1$, and where the products $*_{k(i)}$ are understood to be associated from the left.*

The bulk of the proof will be broken into a series of lemmas for later reference; the balance of the proof will be completed under 3.9 below. The proof is a mixture of the theorems of Jónsson and Mal'cev.

Let $\mathbf{E}$, n, the polynomial symbols $\mathbf{t}_k$, $\mathfrak{A}$, the operations $*_k$, and their manner of association be as in Theorem 3.4.

3.5. **Lemma.** *For any sequence $c = c_0, c_1, \cdots, c_r = d$, $\theta(c, d) = \bigvee_{k,i} \theta(\langle c, d\rangle *_k \langle c_i, c_{i+1}\rangle)$, where k ranges from 1 to $n-1$ and i ranges from 0 to $r-1$.*

Proof. For fixed k consider the sequence $t_k(c, c, d) = t_k(c, c_0, d)$, $t_k(c, c_1, d), \cdots, t_k(c, c_{r-1}, d), t_k(c, c_r, d) = t_k(c, d, d)$. Since any pair of adjacent terms is of the form $\langle c, d\rangle *_k \langle c_i, c_{i+1}\rangle$, the congruence $t_k(c, c, d) \equiv t_k(c, d, d)$ mod $\bigvee_i \theta(\langle c, d\rangle *_k \langle c_i, c_{i+1}\rangle)$ must hold. Now let k vary by considering the sequence of elements $c = t_1(c, c, d)$, $t_1(c, d, d) = t_2(c, d, d)$, $t_2(c, c, d) = t_3(c, c, d)$, $\cdots, d$, where the equalities follow from (3.1b) and (3.1c) and the terminal d is in the guise of either $t_{n-1}(c, c, d)$ or $t_{n-1}(c, d, d)$, depending on the parity of n. Since any two adjacent terms are connected by the congruence described above, we have $c \equiv d$ mod $\bigvee_k \bigvee_i \theta(\langle c, d\rangle *_k \langle c_i, c_{i+1}\rangle)$, or equivalently, $\theta(c, d) \subseteq \bigvee_{k,i} \theta(\langle c, d\rangle *_k \langle c_i, c_{i+1}\rangle)$. For the opposite inclusion, it suffices to observe that, for each k and i, $\theta(\langle c, d\rangle *_k \langle c_i, c_{i+1}\rangle) \subseteq \theta(c, d) \cap \theta(c_i, c_{i+1})$ [by 3.2] $\subseteq \theta(c, d)$.

Remark. For $r = 2$, the proof just concluded is essentially contained in the original reasoning of Jónsson [20, proof of Theorem 2.1].

3.6. **Lemma.** *If* $\theta(c, d) \subseteq \theta(a, b)$, *then* $\theta(c, d) = \bigvee_{k,\phi} \theta(\langle c, d\rangle *_k \langle \phi a, \phi b\rangle)$, *where* k *ranges from* 1 *to* $n-1$ *and* ϕ *ranges in* $U_{\mathfrak{A}}$.

Proof. By Mal'cev's theorem, there is a sequence $c = c_0, c_1, \cdots, c_r = d$ such that each pair $\{c_i, c_{i+1}\}$ is of the form $\{\phi a, \phi b\}$ for some $\phi \in U_{\mathfrak{A}}$ depending on i. Then by the preceding lemma, $\theta(c, d) = \bigvee_{k,i} \theta(\langle c, d\rangle *_k \langle c_i, c_{i+1}\rangle) \subseteq \bigvee_{k,\phi} \theta(\langle c, d\rangle *_k \langle \phi a, \phi b\rangle) \subseteq \bigvee_{k,\phi} [\theta(c, d) \cap \theta(\phi a, \phi b)] \subseteq \theta(c, d)$, so that the asserted equality must hold. Observe that the possibility $\langle c_i, c_{i+1}\rangle = \langle \phi b, \phi a\rangle$ in Mal'cev's theorem results merely in a harmless reversal of a pair $\langle c, d\rangle *_k \langle \phi a, \phi b\rangle$ used to generate a principal congruence relation.

3.7. **Lemma.** *If* $\theta(c, d) \subseteq \theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N)$, *then* $\theta(c, d) = \bigvee \theta(\langle c, d\rangle *_{k(1)} \langle \phi_1 a_1, \phi_1 b_1\rangle *_{k(2)} \cdots *_{k(N)} \langle \phi_N a_N, \phi_N b_N\rangle)$, *where* $k(1), \cdots, k(N)$ *range in* $\{1, \cdots, n-1\}$ *and* $\phi_1, \cdots, \phi_N$ *range in* $U_{\mathfrak{A}}$.

Proof by induction on N. For $N = 0$, let this assertion be interpreted to mean that, for any c and d, $\theta(c, d) \subseteq \theta(c, d)$, a triviality. For $N > 0$, the case $N-1$ is turned into the case N simply by applying Lemma 3.6 to each inclusion $\theta(\langle c, d\rangle *_{k(1)} \langle \phi_1 a_1, \phi_1 b_1\rangle *_{k(2)} \cdots *_{k(N-1)} \langle \phi_{N-1} a_{N-1}, \phi_{N-1} b_{N-1}\rangle) \subseteq \theta(a_N, b_N)$. This inclusion does hold since, by 3.2, $\theta(\langle c, d\rangle *_{k(1)} \langle \phi_1 a_1, \phi_1 b_1\rangle *_{k(2)} \cdots) \subseteq \theta(c, d) \subseteq \theta(a_N, b_N)$. Thus the induction is valid.

3.8. **Lemma.** $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N) = \bigvee \theta(\langle c, d\rangle *_{k(1)} \langle \phi_1 a_1, \phi_1 b_1\rangle *_{k(2)} \cdots *_{k(N)} \langle \phi_N a_N, \phi_N b_N\rangle)$, *where* c, d *range in* A, *where* $k(1), \cdots, k(N)$ *range in* $\{1, \cdots, n-1\}$, *and where* $\phi_1, \cdots, \phi_N$ *range in* $U_{\mathfrak{A}}$.

Proof. As a congruence relation, $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N)$ is the join of those principal congruence relations $\theta(c, d)$ which it contains. Thus Lemma 3.7

yields the equation of the present lemma, with the exception that c, d range subject to $\theta(c, d) \subseteq \theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N)$. To let c, d range further afield cannot increase the right-hand side, however, since by Observation 3.2 each additional principal congruence relation thus obtained is still contained in the left-hand side.

3.9. **Proof of Theorem 3.4.** It is required to show that in Lemma 3.8, the portion $\langle c, d\rangle *_{k(1)}$ can be erased. Observe that $\langle c, d\rangle *_{k(1)} \langle \phi_1 a_1, \phi_1 b_1\rangle = \langle \phi_1' a_1, \phi_1' b_1\rangle$, where ϕ_1' is the composition $\phi_1'(u) = t_{k(1)}(c, \phi_1(u), d)$. Since ϕ_1 already ranges over all unary algebraic functions, the contemplated erasure increases the right-hand side, if anything. However, as before, by Observation 3.2 the right-hand side remains bounded by the left-hand side, so cannot increase.

3.10. **Remark.** In Mal'cev's theorem (Theorem 3.3), it is not always necessary to use the full set $U_{\mathfrak{A}}$ of *all* unary algebraic functions; a smaller set U' may suffice. For convenience, let us call such a U' a *Mal'cev* family of unary algebraic functions. For example, in the case of lattices, compositions of unary algebraic functions of the forms $u \mapsto u \vee c$ and $u \mapsto u \wedge c$ form a Mal'cev family. More generally, let an "operational" unary algebraic function on $\mathfrak{A}$ mean a unary algebraic function obtained by freezing all entries except one in an *operation* of $\mathfrak{A}$; the compositions of all such operational unary algebraic functions form a Mal'cev family. An examination of the proofs of Lemmas 3.6, 3.7, and 3.8 above shows that they remain valid if $U_{\mathfrak{A}}$ is replaced by a Mal'cev family U'. (The proof 3.9 does not, since the composition ϕ_1' may lie outside U'.) Thus:

3.11. **Observation.** *The expression for* $\theta(a_1, b_1) \cap \cdots \cap \theta(a_N, b_N)$ *given in Lemma* 3.8 *remains valid if* $\phi_1, \cdots, \phi_N$ *range only over some Mal'cev family* U' *for* $\mathfrak{A}$.

3.12. **Remark.** The operation $*$ has some interesting properties: (1) For lattices, regard $*$ as an operation on $Q(L)$, the set of quotients of L. Then $*$ is associative if and only if L is distributive. (2) In the same context, L is modular if and only if $*$ is "near-associative," i.e., three quotients associate when two are equal. (3) For an algebra $\mathfrak{A}$ with a median polynomial m, the associated operation $*$ is at least idempotent, and is associative if and only if the related algebra $\langle A, m\rangle$ is a subdirect power of $\langle 2, m\rangle$, the unique two-element algebra with one ternary operation obeying the majority-vote rule. The operation $*$ might bear further investigation.

4. **Solution of equational axiom problems.** Let $\mathbf{K}$ be a class of algebras of the same type. Usually, $\mathbf{K}$ will be a subclass of some known, tractable equational class $\mathbf{E}$. In this case, it is natural to solve the equational axiom problem $\mathrm{AP}(\mathbf{K})$ by adding defining identities for $\mathbf{E}$ to a set of identities which determines $\mathbf{K}^e$ relative to $\mathbf{E}$. The problem of finding this second set of identities will be

called the *relative* equational axiom problem AP(K rel E). Thus, the equational axiom problems solved in [2] were of the form AP(K rel $\mathfrak{L}$) for the class $\mathfrak{L}$ of all lattices.

In this section, AP(K rel E), and thereby AP(K), will be solved for any UDE-defined subclass K of any congruence-distributive equational class E. Representative examples of such E and K were given in the introduction.

The basic method to be used is a straightforward combination of the results of §3 with those of §1. In its first incarnation, the resulting general solution of AP(K rel E) will be admittedly inefficient (Theorem 4.2). It will next be shown how economical solutions can be constructed by cautious modification of the first solution. Whether this economical solution is finite or infinite will depend on the class E, but not on K. (Thus on occasion there may exist a finite solution to AP(K) which is not revealed even by the economical general method.)

4.1. *The basic construction.* Let E be a congruence-distributive equational class, and let K be a subclass of E defined, relative to E, by a set Δ of UDE's, assumed to be explicitly known. Let $t_0, \cdots, t_n$ be a choice of Jónsson's polynomial symbols for E [20]. The t_i will also be assumed to be explicitly known, a condition invariably fulfilled in practice.

The desired identities will be formed by wholesale substitution of polynomial symbols in place of a_i, b_i and ϕ_i in the appropriate expression of Theorem 3.4; auxiliary variables w^i_j will replace the frozen constant entries of the ϕ_i. For convenience, if F and G are polynomial symbols, let equ$\langle$F, G$\rangle$ denote the atomic formula F = G. Thus, in an algebra with median operation m, an atomic formula $m(f_1(x), f_2(x), g_1(x)) = m(f_1(x), g_2(x), g_1(x))$ could be rewritten as equ$(\langle f_1(x), g_1(x)\rangle * \langle f_2(x), g_2(x)\rangle)$. For a UDE $D = (\forall x_1) \cdots (\forall x_m) f_1(x) = g_1(x) \vee \cdots \vee f_N(x) = g_N(x)$, let $I[D]$ be the set of all identities of the form $(\forall x)(\forall W)\,\mathrm{equ}(\langle F_1, G_1\rangle *_{k(2)} \cdots *_{k(N)} \langle F_N, G_N\rangle)$, where for each i, $F_i = p_i(f_i(x), w^i_1, \cdots, w^i_{l(i)})$ and $G_i = p_i(g_i(x), w^i_1, \cdots, w^i_{l(i)})$ for some polynomial symbol p_i, where $k(2), \cdots, k(N) \in \{1, \cdots, n-1\}$, and where W denotes the list of all variables w^i_j used. (As in §3, the operations $*_{k(i)}$ are understood to be associated from the left; these operations can be regarded formally as being performed in the Cartesian square of the word algebra of polynomial symbols in appropriate variables.)

4.2. **Theorem.** *Let* E *be a congruence-distributive equational class with specified Jónsson polynomial symbols* $t_0, \cdots, t_n$. *Let* K *be a subclass of* E *defined, relative to* E, *by a set* Δ *of UDE's. Then a solution to* AP(K rel E) *is the set of identities* $I[\Delta] = \bigcup_{D \in \Delta} I[D]$. *The union of* $I[\Delta]$ *and a set of defining identities for* E *constitutes a solution to* AP(K).

Proof. According to Theorem 1.5, if $\mathfrak{A} \in \mathrm{E}$ then $\mathfrak{A} \in \mathrm{K}^e$ if and only if for all $D \in \Delta$ and for all $c_1, \cdots, c_m \in \mathfrak{A}$, $\theta(f_1(\mathrm{c}), g_1(\mathrm{c})) \cap \cdots \cap \theta(f_N(\mathrm{c}), g_N(\mathrm{c})) = 0$, where $D = (\forall x_1) \cdots (\forall x_m) f_1(\mathrm{x}) = g_1(\mathrm{x}) \bigvee\!\!\!\bigvee \cdots \bigvee\!\!\!\bigvee f_N(\mathrm{x}) = g_N(\mathrm{x})$. According to Theorem 3.4, for each D and $c_1, \cdots, c_m$, this condition is equivalent to the condition that, for all $k(2), \cdots, k(N) \in \{1, \cdots, n-1\}$ and $\phi_1, \cdots, \phi_N \in U_{\mathfrak{A}}$, the two coordinates of the pair $\langle \phi_1 f_1(\mathrm{c}), \phi_1 g_1(\mathrm{c})\rangle *_{k(2)} \cdots *_{k(N)} \langle \phi_N f_N(\mathrm{c}), \phi_N g_N(\mathrm{c})\rangle$ are equal. But this is exactly what the identities in $I[\Delta]$ assert, since as p_i ranges over all polynomial symbols and the auxiliary variables w_j^i take on all values d_j^i in $\mathfrak{A}$, the resulting unary algebraic functions $\phi_i(u) = \mathrm{p}_i(u, d_1^i, \cdots, d_{e(i)}^i)$ range over $U_{\mathfrak{A}}$. Thus $I[\Delta]$ does determine K^e, relative to E. The second assertion of the theorem then follows.

4.3. *A more efficient solution.* It is apparent from the preceding proof that to have each p_i range over all polynomial symbols was an unnecessary extravagance, for two reasons: (1) Many different polynomials can produce the same unary algebraic function, and (2) as indicated in Remark 3.10, it is not necessary to use all unary algebraic functions in the first place. A Mal'cev family of unary algebraic functions will suffice if we are willing to use the expression for an intersection of principal congruence relations given in Lemma 3.8. The identities used to solve AP(K rel E) can be adjusted accordingly.

If ϕ is a unary algebraic function for $\mathfrak{A}$, let us say that a polynomial symbol p in variables $x_0, \cdots, x_l$ *induces* ϕ if there exist $d_1, \cdots, d_l \in A$ such that $\phi(a) = p(a, d_1, \cdots, d_l)$ for all $a \in A$. The import of the above discussion is that the construction of solution identities can involve only a set $\mathcal{P}$ of polynomials sufficient to induce a Mal'cev family of unary algebraic functions on each $\mathfrak{A} \in \mathrm{E}$, for example, the compositions of operational unary algebraic functions (Remark 3.10). Thus, for lattices, one choice of $\mathcal{P}$ would be $\mathcal{P} = \{(\cdots((x_0 \vee x_1) \wedge x_2) \vee \cdots) \wedge x_l \colon l = 0, 2, 4, \cdots\}$.

However, one additional refinement is possible. In the proof of the preceding theorem, it would have been sufficient to check the equivalence $\mathfrak{A} \in \mathrm{K}^e$ iff $\mathfrak{A} \models I[\Delta]$ only in the case where $\mathfrak{A}$ is SI, since an equational class is uniquely determined by its SI members. Thus, it will be sufficient to require that $\mathcal{P}$ induce a Mal'cev family of unary algebraic functions on each SI algebra $\mathfrak{A} \in \mathrm{E}$. For example, if E is the equational class generated by [lattices of flats of] projective planes, the SI members of E are the projective planes and certain degenerate planes; a principal congruence relation can be computed in these lattices by using at most three-step projectivities; a calculation then shows that a suitable choice of $\mathcal{P}$ is to have $\mathcal{P} = \{((x_0 \vee x_1) \wedge x_2) \vee x_3, ((x_0 \wedge x_1) \vee x_2) \wedge x_3\}$.

To summarize: For a UDE $D = (\forall x_1) \cdots (\forall x_m) \mathrm{f}_1(\mathrm{x}) = \mathrm{g}_1(\mathrm{x}) \bigvee\!\!\!\bigvee \cdots \bigvee\!\!\!\bigvee \mathrm{f}_N(\mathrm{x}) = \mathrm{g}_N(\mathrm{x})$, and for a set $\mathcal{P}$ of polynomial symbols, let $I[D, \mathcal{P}]$ be the set of all identities of the form

$$(\forall x)(\forall u)(\forall v)(\forall W)\ \mathrm{equ}\,(\langle u, v\rangle *_{k(1)} \langle F_1, G_1\rangle *_{k(2)} \cdots *_{k(N)} \langle F_n, G_N\rangle),$$

where for each i, F_i denotes $p_i(f_i(x), w_1^i, \cdots, w_{l(i)}^i)$ and G_i denotes $p_i(g_i(x), w_1^i, \cdots, w_{l(i)}^i)$ for some $p_i \in \mathcal{P}$; where $k(1), \cdots, k(N) \in \{1, \cdots, n-1\}$; and where W denotes the set of all variables w_j^i used. Here $*_1, \cdots, *_{n-1}$ are based on a chosen set of ternary polynomial symbols $t_1, \cdots, t_{n-1}$. Association of the operations $*_k$, as before, is from the left.

4.4. **Theorem.** *Let* E *be a congruence-distributive equational class with specified Jónsson polynomial symbols* $t_0, \cdots, t_n$. *Let* $\mathcal{P}$ *be a set of polynomial symbols sufficient to induce a Mal'cev family of unary algebraic functions on each SI algebra* $\mathfrak{A} \in$ E. *Let* K *be a positive universal subclass of* E *defined, relative to* E, *by a set* Δ *of UDE's. Then a solution to* AP(K rel E) *is the set of identities* $I[\Delta, \mathcal{P}] = \bigcup_{D\in\Delta} I[D, \mathcal{P}]$. *The union of* $I[\Delta, \mathcal{P}]$ *and a set of defining identities for* E *constitutes a solution to* AP(K).

4.5. *A typical application.*

Problem. Let LG be the class of all lattice-ordered groups (l-groups), and let K_1, K_2 be equational subclasses where each K_i is defined, relative to LG, by a single identity $(\forall x)f_i(x) = g_i(x)$ $(i = 1, 2)$. Find an identity or identities which determine (relative to LG) the join $K_1 \vee K_2$ in the lattice of all equational subclasses of LG.

Solution. Observe first that $K_1 \vee K_2 = (K_1 \cup K_2)^e$, so that the problem is the same as AP($K_1 \cup K_2$ rel LG). $K_1 \cup K_2$ is defined, relative to LG, by the positive universal sentence $[(\forall x)f_1(x) = g_1(x)]$ W $[(\forall x)f_2(x) = g_2(x)]$, or equivalently, by the UDE $D = (\forall x)(\forall y)f_1(x) = g_1(x)$ W $f_2(y) = g_2(y)$. Next, we must look for a Mal'cev family of unary algebraic functions. The operational unary algebraic functions on an l-group G are of the forms $u \mapsto c_1u$, $u \mapsto uc_2$, $u \mapsto u^{-1}$, $u \mapsto u \vee c_3$, $u \mapsto u \wedge c_4$. By using various l-group identities [7, Chapter XIII] such as lattice distributivity, any composition of these functions can be put in the form $\phi(u) = ((d_1ud_2) \vee d_3) \wedge d_4$ or $\psi(u) = ((d_1u^{-1}d_2) \vee d_3) \wedge d_4$. Thus a set of polynomial symbols sufficient to induce a Mal'cev family is simply $\mathcal{P} = \{p_1, p_2\}$, where $p_1 = ((x_1x_0x_2) \vee x_3) \wedge x_4$, $p_2 = ((x_1x_0^{-1}x_2) \vee x_3) \wedge x_4$. Since lattice operations are present, the only nontrivial Jónsson polynomial symbol necessary is t_1, a lattice median. According to Theorem 4.4, an answer to the original problem is $I[D, \mathcal{P}]$, which consists of the four identities of the form

$$(\forall x, y)(\forall u)(\forall v)(\forall W)\ \mathrm{equ}[(\langle u, v\rangle * \langle p_{i(1)}(f_1(x), w_1^1, \cdots, w_4^1), p_{i(1)}(g_1(x), w_1^1, \cdots, w_4^1)\rangle) * \langle p_{i(2)}(f_2(y), w_1^2, \cdots, w_4^2),\ p_{i(2)}(g_2(y), w_1^2, \cdots, w_4^2)\rangle],$$

for $i(1), i(2) \in \{1, 2\}$.

Here the disjoint sets of variables x, y together fill the role of x. By careful reasoning, it is possible to show that the map $u \longrightarrow u^{-1}$, and hence ψ, can be omitted, resulting in one identity in place of four. An even simpler identity can be found, but by an ad hoc method.

In this example, observe that since $\mathcal{P}$ is finite, any problem AP(**K** rel **LG**) will have a finite answer whenever **K** is definable by finitely many positive universal sentences.

4.6. **Remark.** For the class $\mathfrak{L}$ of lattices, with $\mathcal{P}$ as indicated in the discussion following 4.3, the identities in $I[D, \mathcal{P}]$ are not strictly the same as the corresponding identities σ_k constructed in [2], even after making allowance for the inequalities $f_i(x) \le g_i(x)$ assumed in that paper. The only real difference, interestingly, is that the composition of $*$ operations occurs in the reverse order in σ_k, with $*\langle u, v\rangle$ at the right-hand end (innermost composed entries). Similar "right-hand" identities can be found in replacement of those of Theorem 4.4, but calculations more delicate than those of §3 are required in support of the proof.

5. **Reduction to a single identity.** For the class $\mathfrak{L}$ of lattices, it is well known that any finite set of lattice identities is equivalent (relative to $\mathfrak{L}$) to some single lattice identity. More generally, the following fact can be shown.

5.1. **Theorem.** *Let* E *be any congruence-distributive equational class. Then, relative to* E, *any finite set of identities is equivalent to a single identity.*

Proof. Let us say that a polynomial symbol p in variables $x_1, \cdots, x_m$ is *dominated*, relative to E, by $x_{i(1)}, \cdots, x_{i(k)}$ if in all algebras of E, each polynomial function p induced by p has value c whenever $x_{i(1)}, \cdots, x_{i(k)}$ are given the common value c. For example, Jónsson's $t_0, \cdots, t_n$ in variables x, y, z are dominated by x, z. Further, let us say that an identity $(\forall x)p(x) = q(x)$ is *dominated* if p and q are dominated, relative to E, by the same nonempty set of variables. The theorem follows immediately from the following two claims:

Claim 1. Relative to E, any identity is equivalent to a finite set of dominated identities.

Claim 2. Any finite set of dominated identities, relative to E, is equivalent to a single identity.

Using these claims, any finite set of identities can be replaced by a finite set of dominated identities and then reduced to a single identity.

To prove the first claim, let us verify that an arbitrary identity $(\forall \mathrm{x})\mathrm{f}(\mathrm{x}) = \mathrm{g}(\mathrm{x})$ can be replaced by the set of identities $(\forall \mathrm{x})(\forall u, v)t_k(u, \mathrm{f}(\mathrm{x}), v) = t_k(u, \mathrm{g}(\mathrm{x}), v)$ $(k = 1, \cdots, n-1)$, or more compactly, $(\forall \mathrm{x})(\forall u, v)\,\mathrm{equ}(\langle u, v\rangle *_k \langle \mathrm{f}(\mathrm{x}), \mathrm{g}(\mathrm{x})\rangle)$. Here $\mathrm{t}_1, \cdots, \mathrm{t}_{n-1}$ are chosen Jónsson polynomial symbols for E. Evidently the new identities are implied by the old and are dominated by u, v. Conversely, by substitution of $f(\mathrm{x}), g(\mathrm{x})$ for u, v the new identities imply the weaker set of identities

$$(\forall x)\ \mathrm{equ}(\langle \mathrm{f}(\mathrm{x}), \mathrm{g}(\mathrm{x})\rangle *_k \langle \mathrm{f}(\mathrm{x}), \mathrm{g}(\mathrm{x})\rangle) \quad (k = 1, \cdots, n-1)$$

which imply the old identity, since by Lemma 3.5 with $r = 1$, the rule $\theta(c, d) = \bigvee_k \theta(\langle c, d\rangle *_k \langle c, d\rangle)$ holds for all elements c, d of algebras in E.

To prove the second claim, it suffices to show that two dominated identities $(\forall \mathrm{x})\mathrm{f}_1(\mathrm{x}) = \mathrm{f}_2(\mathrm{x})$ and $(\forall \mathrm{y})\mathrm{g}_1(\mathrm{y}) = \mathrm{g}_2(\mathrm{y})$ $(\mathrm{x} = x_1, \cdots, x_m, \mathrm{y} = y_1, \cdots, y_s)$ can be replaced by a single dominated identity. Without loss of generality, we may assume that the dominating variables are $x_1, \cdots, x_l$ for the first identity, and $y_1, \cdots, y_r$ in the second. Let u^i_j $(1 \leq i \leq l,\ 1 \leq j \leq r)$ be new variables, and for $1 \leq i \leq l$ and $k = 1, 2$, let $\mathrm{g}_k(u^i, \mathrm{y})$ denote $\mathrm{g}_k(u^i_1, \cdots, u^i_r, y_{r+1}, \cdots, y_s)$. Consider the single identity $(\forall \mathrm{x})(\forall \mathrm{y})(\forall \mathrm{u})\mathrm{h}_1 = \mathrm{h}_2$, where for $k = 1, 2$, $\mathrm{h}_k(\mathrm{u}, \mathrm{x}, \mathrm{y}) = f_k(g_k(u^1, \mathrm{y}), \cdots, g_k(u^l, \mathrm{y}), x_{l+1}, \cdots, x_m)$. This identity is dominated by the variables u^i_j and is implied by the original pair of identities. On the other hand, if $u^i_1, \cdots, u^i_r$ are all replaced by x_i for each i, then this single identity reduces to $(\forall \mathrm{x})\mathrm{f}_1(\mathrm{x}) = \mathrm{f}_2(\mathrm{x})$; if $u^1_j, \cdots, u^l_j$ are all replaced by y_j for each j, then the single identity reduces to $(\forall \mathrm{y})\mathrm{g}_1(\mathrm{y}) = \mathrm{g}_2(\mathrm{y})$.

5.2. **Corollary.** *If* E *is a congruence-distributive equational class with at most countably many operations, then any equational subclass of* E *can be defined, relative to* E, *by a sequence of identities of increasing strength.*

R. Padmanabhan [37] has established that the defining conditions for a median polynomial can be expressed by a single, more complicated identity. R. Quackenbush has noted that Theorem 5.1 then has the following consequence.

5.3. **Corollary.** *If* E *is a finitely based equational class with a median polynomial, then* E *is 2-based, i.e.,* E *can be defined by two identities.*

Note added in proof. R. Padmanabhan and R. Quackenbush have now generalized 5.3: In the presence of Jónsson's $\mathrm{t}_0, \cdots, \mathrm{t}_n$, n identities suffice.

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