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A SIEGEL FORMULA FOR ORTHOGONAL GROUPS OVER A FUNCTION FIELD

BY

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ABSTRACT. We obtain a Siegel formula for a quadratic form over a function field, by establishing the convergence of the corresponding Eisenstein-Siegel series directly, then via the Hasse principle, that of the associated Poisson formula.

Introduction. In this paper, we obtain a Siegel formula, as recast by Weil [7], for a quadratic form over a function field. The difficulty is that there is no criterion to guarantee the convergence of the integral

$$\int_{G_A/G_k} \sum_{\xi \in X_k} \Phi(g \cdot \xi)\,|dg|_A,$$

which occurs in the formula (see §1 for the notation), as was the case for k a number field, cf. Weil [7], Igusa [2]. We establish convergence of the corresponding Siegel-Eisenstein series, then by the Hasse principle obtain the Siegel formula and the convergence of the above integral.

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1. Notation and the Siegel formula. Let k be a function field in one variable over a finite constant field, that is, a finitely generated extension of a finite prime field F_q, of degree of transcendence one over F_q. We shall assume that characteristic $(k) \neq 2$.

Let X be a vector space of dimension m and $q(x)$ a nondegenerate quadratic form on X, all defined over k. Take $G = SO(q)$ (a semisimple algebraic group, defined over k, for $m \geq 3$) to be the special orthogonal group of q. The Siegel formula is given for the standard representaion $\rho\colon G \longrightarrow \mathrm{Aut}(X)$; it reads

$$\int_{G_A/G_k} \left(\sum_{\xi \in X_k} \Phi(g \cdot \xi) \right) |dg|_A = 2 \sum_{i^* \in k} \int_{X_A} \Phi(x)\chi(q(x)i^*)\,|dx|_A + 2\Phi(0)$$

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where G_A, G_k are the adelisation, the k-rational points (respectively) of G; $\Phi \in \mathcal{S}(X_A)$ is a Schwartz function on the adelisation of X and χ is a fixed, nontrivial character of k_A, the adelisation of k, which is 1 on k.

2. **Orbits, stablisers.** To analyse the integral $\int_{G_A/G_k} \Sigma_{\xi \in X_k} \Phi(g \cdot \xi)|dg|_A$, we recall results established by Weil [7, §14–29]. The orbits of G in X which contain points of X_k are the sets $U(i) = \{x \in X | \ q(x) = i, \ x \neq 0\}$ where $i \in k$, $U(i)_k \neq \emptyset$, and $\{0\}$. This is precisely Witt's theorem. Further, two points $x, y \in X_K$, not zero, belong to the same orbit of G_K if and only if they belong to the same orbit of G. This for any $K \supset k$.

For the nonempty $U(i)_k$, fix $\xi_i \in U(i)_k$ and let H_i be the stabiliser of G at ξ_i, i.e., $H_i = \{g \in G | \ g \cdot \xi_i = \xi_i\}$, an algebraic group defined over k. Hence

$H_i = SO(m-1)$, of rank $m-1$, for $i \neq 0$;

$H_0 = SO(m-1) \cdot$ unipotent, a semidirect product;

and in all cases, the Tamagawa numbers for G, H_i are 2, Weil [5].

Furthermore, the mapping $g \longrightarrow g \cdot \xi_i$ of $G \longrightarrow U(i)$ induces an isomorphism of G/H_i onto $U(i)$. By Witt's theorem there is a generic section for this map. Also, as k is an infinite field and G is a reductive group, G_k is Zariski dense in G (Borel [1]). Whence, the mapping $g \longrightarrow g \cdot \xi_i$ induces the identification $G_A/(H_i)_A = U(i)_A$ of the adelisations.

Take $\Phi \in \mathcal{S}(X_A)$; then for $i \in k$ so that $U(i)_k \neq \emptyset$,

$$\int_{G_A/G_K} \sum_{\xi \in U(i)_k} \Phi(g \cdot \xi)\,|dg|_A = \tau(H_i) \int_{U(i)_A} \Phi |D_i|_A, \tag{1}$$

where $\tau(H_i)$ is the Tamagawa number of H_i, $|D_i|_A$ is the Tamagawa measure derived from $D_i = dg/dh_i$, dg, dh_i invariant differential forms of maximal degree without zeros or poles for G, H_i, respectively, defined over k. The convergence factors may be taken to be 1, from the explicit nature of the stabilisers H_i.

By the Hasse principle for quadratic forms, $U(i)_k = \emptyset$ implies that $U(i)_A = \emptyset$. Thus we see that (1) is valid for all $i \in k$.

3. **Asymptotic estimates.** Let v be a valuation on k, which is trivial on the field of constants, with k_v as the completion. Then k_v is nonarchimedean and denote by $\mathcal{O}_v$, p_v and q_v the maximal compact subring of k_v, the ideal of nonunits of $\mathcal{O}_v$ and the number of elements of $\mathcal{O}_v/p_v$ (resp.). Let X_v, X_v^0 be the k_v, $\mathcal{O}_v$ (resp.)-rational points of X and $|di|_v$, $|dx|_v$ be autodual measures on k_v and X_v.

For χ_v a nontrivial character of k_v, we identify X_v with its dual by $(x, x') \longrightarrow \chi_v(x\,{}^t x')$, where we write the elements of X_v as row vectors, with respect to some k-basis. For $\Phi \in \mathcal{S}(X_v)$, the Schwartz-Bruhat space, the Fourier transform is defined by $\Phi^*(x^*) = \int_{X_v} \Phi(x)\langle x, x^*\rangle |dx|_v$, where $\langle x, x^*\rangle = x^*(x)$. We choose as before $|dx|_v$ to be the autodual measure on X_v.

For $\Phi \in \mathcal{S}(X_v)$, we consider the function for $i^* \in k_v$ defined by

$$F_\Phi^*(i^*) = \int_{X_v} \Phi(x)\chi_v(q(x)i^*)\,|dx|_v.$$

The first sections of Weil [7] are devoted to proving general properties of such functions, in actually a more general setting. Namely, for X, Y locally compact abelian groups and $f\colon X \to Y$ a continuous mapping, the principal result concerns the decomposition of the measure dx on X, when f satisfies a "condition (A)". If $\Lambda(X)$ denotes the subspace of $\mathfrak{L}^1(X)$ consisting of those continuous functions Φ with $\Phi^* \in \mathfrak{L}^1(X^*)$, then Fourier transformation gives a bijection of $\Lambda(X)$ with $\Lambda(X^*)$, so that $(\Phi^*)^*(x) = \Phi(-x)$ for every $x \in X$. Among other things, Weil proves that if f satisfies "condition (A)", i.e.,

$$F_\Phi^*(y^*) = \int_X \Phi(x)\langle f(x), y^*\rangle\,dx$$

is integrable on Y^*, uniformly so in Φ when Φ is restricted to a compact subset of $\mathcal{S}(X)$, then

(i) F_Φ^* belongs to $\Lambda(Y^*)$, and

(ii) there exists a unique family of measures $d\mu_y$ on X, each $d\mu_y$ being the image measure under $f^{-1}(y) \to X$, of a measure on $f^{-1}(y)$, such that F_Φ^* becomes the Fourier transform of $F_\Phi(y) = \int_X \Phi(x)\,d\mu_y(x)$.

We shall show that in the local and global cases, $f = q$, the quadratic form satisfies "condition (A)".

A fact which will play an important role is that if $\psi\colon k_v^n \to \mathbf{T}$ is a nondegenerate second degree character of k_v^n, i.e., ψ is continuous and satisfies $\psi(x+y) = \psi(x)\cdot\psi(y)\cdot\langle x, yb\rangle$ for some bicontinuous isomorphism $b\colon k_v^n \to (k_v^n)^*$, then its Fourier transform is given by

$$\psi^*(x^*) = \gamma(\psi)\,|b|^{-1/2}\,\psi(x^*b^{-1})^{-1},$$

where $\gamma(\psi) \in \mathbf{T}$, a complex number of absolute value 1, and $|b|$ is the modulus of b (Weil [6, p. 161]). Hence

$$\left|\int_{k_v^n} \Phi(z)\psi(z)\,|dz|_v\right| \le \|\Phi^*\|_1\,|\det b|_v^{-1/2}. \tag{2}$$

For our case, take $\psi(x) = \chi_v(q(x))$, so that (2) reads $|F_\Phi^*(i^*)| \leq \|\Phi^*\|_1 |i^*|_v^{-m/2}$. Since, trivially, $|F_\Phi^*(i^*)| \leq \|\Phi\|_1$, we have

$$(3) \qquad |F_\Phi^*(i^*)| \leq \max(\|\Phi\|_1, \|\Phi^*\|_1) \cdot \max(1, |i^*|_v)^{-m/2}.$$

Therefore, we have proved:

Lemma 1. *Let C be a compact subset of $\mathcal{S}(X_v)$. Then, there exists a positive constant c, such that*

$$|F_\Phi^*(i^*)| \leq c \max(1, |i^*|_v)^{-m/2}$$

for all $\Phi \in C$, $i^* \in k_v$.

It is easy to check that, for $t \in k_v^\times$,

$$\int_{k_v} \max(|t|_v, |i|_v)^{-\sigma} |di|_v = \text{const}\, |t|_v^{1-\sigma}.$$

This, combined with Lemma 1, shows that q: $X_v \to k_v$ satisfies "condition (A)". Therefore, there exists a uniquely determined family of positive measures $\{\mu_i |\ i \in k_v\}$ on X_v, such that

(i) support $(\mu_i) \subset \{x \in X_v |\ q(x) = i\}$;

(ii) for any coutinuous function Φ with compact support on X_v, the function $F_\Phi(i) = \int_{X_v} \Phi(x) d\mu_i(x)$ defined on k_v is continuous and satisfies

$$\int_{k_v} F_\Phi |di|_v = \int_X \Phi(x) |dx|_v.$$

Moreover,

(iii) if $\Phi \in \mathcal{S}(X_v)$, F_Φ is continuous, integrable and has as its Fourier transform

$$F_\Phi^*(i^*) = \int_{X_v} \Phi(x) \chi_v(q(x) i^*) |dx|_v \qquad (i^* \in k_v).$$

As the sets $U_v(i) = \{x \in X_v |\ q(x) = i,\ x \neq 0\}$ are in fact the fibres, for $i \neq 0$, these sets carry the measure μ_i. But the same is true for $i = 0$. To see this, use $\Phi(tx)$ in place of $\Phi(x)$, for $t \in k_v^\times$. The uniqueness of the measures implies that $\mu_0(tx) = |t|_v^{m-2} \mu_0(x)$, so that no part of the measure μ_0 is carried by the set $\{0\}$.

To identify the measures μ_i, consider the gauge form $D_{v,i}(x) = (dx/dq(x))_i$ on $U_v(i)$. As q is submersive on $X_v - \{0\}$, this is well defined and satisfies

$$\int_{X_v - \{0\}} \Phi |dx|_v = \int_{k_v} |di|_v \int_{U_v(i)} \Phi |D_{v,i}|_v,$$

where $|D_{v,i}|_v$ is the measure on $U_v(i)$ determined by $D_{v,i}$. This holds for all continuous functions Φ with compact support contained in $X_v - \{0\}$. But $\{0\}$ has measure zero for $|dx|_v$, so we can extend the above equality to:

$$\int_{X_v} \Phi |dx|_v = \int_{k_v} |di|_v \int_{U_v(i)} \Phi |D_{v,i}|_v,$$

whence by the uniqueness of the family $\{\mu_i\}$, we have $\mu_i = |D_{v,i}|_v$ $(i \in k_v)$.

It is convenient at this time to mention that the gauge form $D_i(x) = (dx/dq(x))_i$ on $U(i)$, for $i \in k$. is also defined and is invariant under G, so it differs from the earlier dg/dh_i by a factor of $k^\times$. Thus the measures given by $D_i(x)$ and dg/dh_i are the same, since the product formula is valid for $k^\times$.

Note that in the estimate (3), for Φ, Φ^* the characteristic functions of X_v^0, X_v^{0*} we have $|F_\Phi^*(i^*)| \leq \max(1, |i^*|_v)^{-m/2}$.

4. **A dominant series.** We shall now prove the convergence of the Siegel-Eisenstein series. The method of proof is based on the following lemma and the methods used in [3], due to Igusa.

As always k denotes a function field of transcendence degree one over a finite field k_0. We may assume that k_0 is algebraically closed in k. Put $q = \mathrm{card}(k_0)$ and let g denote the genus of k. Choose a prime divisor P_∞ of k such that $d = \deg(P_\infty) \geq 2g + 1$, whence $l(P_\infty) = d + 1 - g \geq g + 2$. So, there exists $x \in k$ with $(x)_\infty = P_\infty$.

Denote by $\mathcal{O}$ the k-normalization of $k_0[x]$. The group of units of $\mathcal{O} = k_0^\times$, hence finite. Also, every $b \neq 0 \in \mathcal{O}$ has $|b|_\infty \geq 1$.

Lemma 2. *Let* λ, α *denote real numbers*, $\lambda \geq 1$, $\alpha > 1$. *Then*

$$\sum_{a \in \mathcal{O}} \max(\lambda, |a|_\infty)^{-\alpha} \leq c\lambda^{1-\alpha}$$

where c *is independent of* λ.

Proof. We have

$$\sum_{a \in \mathcal{O}} \max(\lambda, |a|_\infty)^{-\alpha} = \sum_{e=0}^{\infty} \mathrm{card}(L(P_\infty^e) - L(P_\infty^{e-1})) \max(\lambda, q^{de})^{-\alpha}.$$

Write $\lambda = q^{d\delta}$, so that $0 \leq [\delta] \leq \delta < [\delta] + 1$. So

$$\sum_{a\in\mathcal{O}} \max(\lambda, |a|_\infty)^{-\alpha} = \begin{cases} A & \text{if } [\delta] \geq 1, \\ B & \text{if } [\delta] = 0, \end{cases}$$

where

$$A = \operatorname{card}(L(P_\infty^{[\delta]}))\lambda^{-\alpha} + \sum_{e=[\delta]+1}^{\infty} (q^{de+1-g} - q^{d(e-1)+1-g})\, q^{-de\alpha},$$

$$B = q\lambda^{-\alpha} + (q^{d+1-g} - q)q^{-d\alpha} + \sum_{e=2}^{\infty} (q^{de+1-g} - q^{d(e-1)+1-g})q^{-de\alpha}.$$

So, setting $\langle\delta\rangle = \delta - [\delta]$,

$$A = \lambda^{1-\alpha}\left\{q^{1-g-d\langle\delta\rangle} + \frac{q^{1-g}(1-q^{-d})q^{-(\alpha-1)d(1-\langle\delta\rangle)}}{1-q^{-(\alpha-1)d}}\right\},$$

$$B = \lambda^{1-\alpha}\left\{q^{1-d\langle\delta\rangle} + q^{-(\alpha-1)d\langle\delta\rangle}\left[(q^{d+1-g} - q)q^{-d\alpha} + \frac{q^{1-g}(1-q^{-d})q^{-2(\alpha-1)d}}{1-q^{-(\alpha-1)d}}\right]\right\}.$$

Fix this choice of generator x. The ideal class group of k for this $\mathcal{O}$ is finite and let $\mathfrak{r}_1, \cdots, \mathfrak{r}_b$ be coset representatives, which may be taken to be integral ideals. Set $S_\infty = \{P_\infty\}$.

Proposition 1. *Let n be a given integer > 0, and $\epsilon > 0$ be fixed. Suppose that for each valuation v on k, σ_v is a given real number, such that $\sigma_v > n$, for all v, $\sigma_v \geq n + 1 + \epsilon$, for almost all v. Then*

$$\sum_{i=(i_1,\cdots,i_n)\in k^n} \prod_v \max(1, |i_1|_v, \cdots, |i_n|_v)^{-\sigma_v}$$

is convergent.

Proof. The convergence is clear for $n = 0$, so suppose $n \geq 1$ and use induction. Let $E \subset \{1, 2, \cdots, n\}$ be a subset and

$$k_E = \{i \in k^n \mid i_p \neq 0 \ \text{ for } p \in E,\ i_p = 0 \ \text{ for } p \notin E\}.$$

Then we have the disjoint union $k^n = \bigcup_E k_E$. By induction, the partial sums over k_E are convergent for every $E \neq \{1, 2, \cdots, n\}$. So it remains to show that the partial sum over $(k^\times)^n$ is convergent.

By hypothesis, there is a finite set of valuations S on k, $S \supset S_\infty$ such that $\sigma_v > n$ for all v and $\sigma_v \geq \beta = 1 + n + \epsilon$ for all $v \notin S$. We can enlarge S without changing β, so suppose S contains all the prime factors of $\mathfrak{r}_1, \cdots, \mathfrak{r}_b$. Further, as a function of σ, $\max(1, |i_1|_v, \cdots, |i_n|_v)^{-\sigma}$ is monotone decreasing, so it suffices to prove convergence when $\sigma_v = \alpha > n$, $v \in S$, $\sigma_v = \beta$, $v \notin S$.

Let $i = (i_1, \cdots, i_n) \in (k^\times)^n$. Then $i_p\mathcal{O} = \mathfrak{a}_p/\mathfrak{b}$ for integral ideals $\mathfrak{b}, \mathfrak{a}_1, \cdots, \mathfrak{a}_n$. Choosing them to be relatively prime, this set is uniquely determined by i. Moreover, there is a unique index j so that $\mathfrak{b}\mathfrak{r}_j = \mathcal{O}b$, for some $b \neq 0 \in \mathcal{O}$. Setting $a_p = bi_p$ we have $a_p\mathcal{O} = \mathfrak{b}\mathfrak{r}_j i_p = \mathfrak{a}_p\mathfrak{r}_j \subset \mathcal{O}$ so $a_p \neq 0 \in \mathcal{O}$. By the choice of S

$$\prod_v \max(1, |i_1|_v, \cdots, |i_n|_v)^{-\sigma_v} = \prod_{v \in S} |b|_v^\alpha \max(|b|_v, |a_1|_v, \cdots, |a_n|_v)^{-\alpha}$$

$$\times \prod_{v \notin S} |b|_v^\beta \max(|b|_v, |a_1|_v, \cdots, |a_n|_v)^{-\beta}.$$

But, as the prime factors of the $\mathfrak{r}_j$ are in S, $\max_{v \notin S}(|b|_v, |a_1|_v, \cdots, |a_n|_v) = 1$. Hence, applying the product formula for $b \in \mathcal{O}$, the above becomes

$$\prod_{v \in S} |b|_v^{\alpha-\beta} \max(|b|_v, |a_1|_v, \cdots, |a_n|_v)^{-\alpha}.$$

For $v \in S - S_\infty$,

$$\mathrm{ord}_\mathfrak{p}(b) = \mathrm{ord}_\mathfrak{p}(\mathfrak{b}) + \mathrm{ord}_\mathfrak{p}(\mathfrak{r}_j), \qquad \mathrm{ord}_\mathfrak{p}(a_i) = \mathrm{ord}_\mathfrak{p}(\mathfrak{a}_i) + \mathrm{ord}_\mathfrak{p}(\mathfrak{r}_j)$$

whence $\max_{v \in S-S_\infty}(|b|_v, |a_1|_v, \cdots, |a_n|_v) = N\mathfrak{p}^{-\mathrm{ord}_\mathfrak{p}(\mathfrak{r}_j)}$, since $\mathfrak{b}, \mathfrak{a}_1, \cdots, \mathfrak{a}_n$ are relatively prime. Here $N\mathfrak{p} = \mathrm{card}(\mathcal{O}/\mathfrak{p})$. Setting $c_\mathfrak{p} = \max\{\mathrm{ord}_\mathfrak{p}(\mathfrak{r}_j), 1 \leq j \leq h\}$, c' $(\Pi_{v \in S-S_\infty} N\mathfrak{p}^{c_\mathfrak{p}})^\alpha$, we find

$$\prod_v \max(1, |i_1|_v, \cdots, |i_n|_v)^{-\sigma_v} \leq c'\left(\prod_{v \in S} |b|_v\right)^{\alpha-\beta} \max(|b|_\infty, |a_1|_\infty, \cdots, |a_n|_\infty)^{-\alpha}.$$

Therefore, it suffices to show that the sum on the right, for $(a_1, \cdots, a_n) \in \mathcal{O}^n$ and $\mathcal{O}b$ over the set of principal ideals, $\neq 0$ of $\mathcal{O}$, is convergent.

Since $|b|_\infty \geq 1$ for $b \neq 0 \in \mathcal{O}$ and $\alpha > n$, we can apply Lemma 2 repeatedly, to show

$$\sum_{(a_1, \cdots, a_n) \in \mathcal{O}^n} \max(|b|_\infty, |a_1|_\infty, \cdots, |a_n|_\infty)^{-\alpha} \leq c^n |b|_\infty^{n-\alpha}$$

where c is a fixed constant, independent of b. Hence, it suffices to show that the series $\Sigma_{\mathcal{O}b}(\Pi_{v \in S}|b|_v)^{\alpha-\beta}|b|_\infty^{n-\alpha}$ is convergent. By the product formula, this is

$$\sum_{\mathcal{O}b}\left(\prod_{v \in S-S_\infty} |b|_v\right)^{\alpha-n}\left(\prod_{v \notin S} |b|_v\right)^{\beta-n}$$

$$= \sum_{\mathcal{O}b}\left(\prod_{\mathfrak{p} \in S-S_\infty} (N\mathfrak{p})^{-\mathrm{ord}_\mathfrak{p}(b)}\right)^{\alpha-n}\left(\prod_{\mathfrak{p} \notin S} N\mathfrak{p}^{-\mathrm{ord}_\mathfrak{p}(b)}\right)^{\beta-n}$$

$$< \sum_{\mathfrak{A} \neq 0,\ \text{all integral ideals}}\left(\prod_{\mathfrak{p} \in S-S_\infty} N\mathfrak{p}^{-\mathrm{ord}_\mathfrak{p}(\mathfrak{A})}\right)^{\alpha-n}\left(\prod_{\mathfrak{p} \notin S} N\mathfrak{p}^{-\mathrm{ord}_\mathfrak{p}(\mathfrak{A})}\right)^{\beta-n}.$$

But, by the Euler product, this differs by only an elementary factor from $\Sigma_{\mathfrak{A}\neq 0,\text{integral}} (N\mathfrak{A})^{-\sigma}$. But for $\sigma = \beta - n > 1$ this is convergent.

5. **The Siegel formula.** The character χ of k_A puts it into duality with k_A^* by $(i, i^*) \mapsto \chi(ii^*)$, for $i, i^* \in k_A$. Identifying X_A with its dual by $(x, y) \mapsto \chi(x\, {}^t y)$, for $x, y \in X_A$, the autodual measure $|dx|_A$ on X_A is then the Haar measure for which X_A/X_k has measure 1.

For every $\Phi \in \mathcal{S}(X_A)$, define

$$F_\Phi^*(i^*) = \int_{X_A} \Phi(x)\chi(q(x)i^*)\,|dx|_A,$$

for $i^* \in k_A$.

For almost all v, the usual Haar measure on k_v is autodual, $\mathcal{O}_v$ is the kernel of χ_v and $m(X_v^0) = 1$. Recall that X_A is the inductive limit of $X_s = X_0^0 \times X_1$, where $X_0^0 = \Pi_{v \notin S} X_v^0$, $X_1 = \Pi_{v \in S} X_v$, for S running over the family of finite sets of valuations on k. Therefore, for every compact subset C of $\mathcal{S}(X_A)$, there exist an S and a compact subset C_1 of $\mathcal{S}(X_1)$, such that every $\Phi \in C$ is of the form $\Phi_0 \otimes \Phi_1$, where Φ_0 is the characteristic function of X_0^0, Φ_1 is in C_1.

Put $\sigma_v = m/2$ for all v. Then, by Lemma 1 and Fubini's theorem, there is a positive constant c such that

$$\sum_{i^* \in k} |F_\Phi^*(i^*)| \leq c \sum_{i^* \in k} \prod_v \max(1, |i^*|_v)^{-\sigma_v}$$

for every $\Phi \in C$. By Proposition 1, the right-hand side is convergent for $m \geq 5$.

Also, the mapping

$$\begin{array}{ccc} (X_A) \times k_A & \longrightarrow & \mathcal{S}(X_A) \\ \cup\!\!\!\!\!\!\; & & \cup\!\!\!\!\!\!\; \\ (\Phi, i^*) & \longmapsto & \Phi_{i^*} \end{array}$$

where $\Phi_{i^*}(x) = \Phi(x)\chi(q(x)i^*)$ is continuous. Hence, by Weil's criterion [7, p. 8], the continuous mapping $q\colon X_A \to k_A$ satisfies "condition (A)" and the following Poisson formula:

$$\sum_{i^* \in k} F_\Phi^*(i^*) = \sum_{i \in k} (F_\Phi^*)^*(i). \tag{4}$$

Here $(F_\Phi^*)^*(i) = F_\Phi(-\,i)$ for every $i \in k_A$.

Lemma 3. $F_\Phi(i) = \int_{U(i)_A} \Phi|D_i|_A$, *for every* $i \in k_A$.

Proof. It suffices to show this for Φ restricted to a subset of $\mathcal{S}(X_A)$ which spans a dense subspace of $\mathcal{S}(X_A)$. Take $\Phi = \Pi_v \Phi_v$, where $\Phi_v \in \mathcal{S}(X_v)$ for every

v and Φ_v = the characteristic function of X_v^0, for all but finitely many v. Then F_Φ decomposes into the product of F_{Φ_v}, defined by $F_{\Phi_v}(i_v) = (F^*_{\Phi_v})^*(-i_v)$, whence, by the results of §3, $F_{\Phi_v}(i_v) = \int_{U_v(i_v)} \Phi_v \cdot |D_{v,i}|_v$, for every $i_v \in k_v$. This implies the desired result. Therefore, (4) now reads

$$\sum_{i^* \in k} \int_{X_A} \Phi(x)\chi(q(x)i^*)\,|dx|_A = \sum_{i \in k} \int_{U(i)_A} \Phi\,|D_i|_A.$$

Combining this with (1) and the exceptional orbit $\{0\}$, we obtain the Siegel formula,

Theorem.

$$\int_{G_A/G_k} \left(\sum_{\xi \in X_k} \Phi(g \cdot \xi) \right) |dg|_A = 2 \sum_{i^* \in k} \int_{X_A} \Phi(x)\chi(q(x)i^*)\,|dx|_A + 2\Phi(0),$$

which is valid for $m \geq 5$. Here G is the special orthogonal group, acting on X, of dimension m.

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