

GOLDIE-LIKE CONDITIONS ON JORDAN MATRIX RINGS⁽¹⁾

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ABSTRACT. In this paper Goldie-like conditions are put on a Jordan matrix ring $J = H(R_n, \gamma_a)$ which are necessary and sufficient for R to be a $*$ -prime Goldie ring or a Cayley-Dickson ring. Existing theory is then used to obtain a Jordan ring of quotients for J .

I. Introduction. Our basic reference for this paper is [5]. The Coordinatization Theorem [5, p. 137] states that any Jordan algebra over a field of characteristic $\neq 2$ with three or more connected idempotents which sum to 1 is isomorphic to a Jordan matrix algebra $H(R_n, \gamma_a)$ of symmetric elements with respect to a canonical involution γ_a . It is with this and Goldie's theorem for semiprime rings [4, p. 270] in mind that we approach the problem of quotient rings of Jordan rings and obtain a quotient ring which is of the type given in the Second Structure Theorem [5, p. 179].

II. Preliminaries and statement of main result. Let J be a Jordan ring whose multiplication is denoted by " \cdot ". For each $a \in J$ there is the U -operator given by $U_a(b) = 2a \cdot (a \cdot b) - a^2 \cdot b$. J is said to be prime provided that, if A and B are ideals of J and $U_A(B) = \{U_a(b) : a \in A, b \in B\} = 0$, then $A = 0$ or $B = 0$ [8]. $Q \subseteq J$ is said to be a quadratic ideal provided Q is an additive subgroup of $(J, +)$ and $U_Q(J) \subseteq Q$.

If we let R be a nonassociative ring with characteristic $\neq 2$ such that $\frac{1}{2} \in R$ and $*$ is an involution on R , then the set of symmetric elements under a canonical involution (see [5, p. 125]), $J = H(R_n, \gamma_a)$, is a Jordan ring for $n \geq 3$ with product $c \cdot b = \frac{1}{2}(cb + bc)$ if and only if either R is associative or $n = 3$ and R is alternative with its set of symmetric elements H contained in its nucleus N [5, p. 127].

Let $a = \text{diag}\{a_1, \dots, a_n\}$ and the a_i 's are symmetric invertible elements in the nucleus of R . The canonical involution γ_a acts on the $n \times n$ matrix X to give $a^{-1}X^{*t}a$ where X^{*t} denotes the matrix obtained by applying $*$ to each of the entries of X and then taking the transpose.

Received by the editors September 8, 1972.

AMS (MOS) subject classifications (1970). Primary 17A40; Secondary 16A58.

⁽¹⁾ Most of this paper appeared in the author's Ph.D. dissertation written under the direction of Professor F. Kosier at the University of Iowa. The author wishes to thank Professor Kosier for his help and encouragement.

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General assumption. In this paper: γ_a is a canonical involution; $J = (R_n, \gamma_a)$, $n \geq 2$, is a Jordan ring; $\frac{1}{2} \in R$ an alternative ring with involution $*$ such that $H \subseteq N$.

If $z \in R$ and $\{e_{ij}\}$ is the standard set of matrix units, then we shall write z_{ij} for $ze_{ij} + a_j^{-1}z^*a_i e_{ji}$, $i \neq j$, in J . Although we shall have occasion to talk about matrices over R of more general form, we shall attempt to eliminate the confusion by reserving the subscripts i and j for this purpose, unless the matrix is written out. One can easily see that the elements of J are merely sums of elements of the form z_{ij} , $i \neq j$, and elements of the form $ba_i e_{ii}$ where $b \in H$. We shall use J_{ij} , $i \neq j$, for $\{x_{ij} : x \in R\}$, and if $A \subseteq J$ then $A_{ij} = A \cap J_{ij}$. A quadratic ideal Q is said to be an *ij-quadratic ideal*, $i \neq j$, provided $Q_{ij} \neq 0$.

The next two definitions are basic for our Goldie-like conditions.

Definition. A nonempty set $\{Q_k\}$ of distinct nonzero quadratic ideals will be called a direct system provided that, if $\{Q_n : n \in I_1\}$ and $\{Q_l : l \in I_2\}$ are finite subsets of $\{Q_k\}$, where $I_1 \cap I_2 = \emptyset$, then the quadratic ideal generated by ΣQ_n intersected with the quadratic ideal generated by ΣQ_l is 0. The direct system is said to be infinite if $\{Q_k\}$ contains infinitely many quadratic ideals.

Definition. If Q is a quadratic ideal in J , then the quadratic ideal A generated by $\{z_{ij} \in J_{ij} : U_{z_{ij}}(Q_{ij}) = 0\}$ is defined to be the *ij-annihilators* of Q , $ij\text{-ann}(Q)$, provided $U_{A_{ij}}(Q_{ij}) = 0$, $i \neq j$; otherwise $ij\text{-ann}(Q) = 0$.

Our Jordan analogue for the associative ring of quotients is given as follows:

Definition. The Jordan ring J' is said to be a Jordan ring of quotients for the Jordan ring J provided:

- (i) there exists an isomorphism $f: J \rightarrow J'$ (we shall consider J as a Jordan subring of J');
- (ii) every regular element a in J (i.e., U_a is injective as it acts on J) is invertible in J' ;
- (iii) every element of J' is of the form $U_a^{-1}(b)$ for a, b in J with a regular in J .

Main Theorem. Let R be an alternative ring with involution $*$ such that $H \subseteq N$ and $\frac{1}{2} \in R$ and let $J = H(R_n, \gamma_a)$, $n \geq 2$, be a Jordan matrix ring. Then J is prime, satisfies ACC on *ij-annihilators*, and contains no infinite direct system of *ij-quadratic ideals* if and only if either R is a $*$ -prime Goldie ring or $n = 2, 3$ and R is a Cayley-Dickson ring. Moreover, in this case, the Jordan ring of quotients J' for J is $J' = H(R'_n, \gamma_a)$ where R' is the ring of quotients for R so that R' is a $*$ -simple Artinian ring or a Cayley-Dickson algebra.

III. Only if. Throughout this section we shall assume that J satisfies the conditions of the Main Theorem.

An ideal A of R is said to be a $*$ -ideal if $A^* = A$. R is said to be a $*$ -prime provided that, if A and B are $*$ -ideals and $AB = 0$, then either $A = 0$ or $B = 0$.

If A is a $*$ -ideal then $A \cap J$ is an ideal of J . Thus the primeness of J gives the $*$ -primeness of R so that by [1] R contains a prime ideal P such that $P \cap P^* = 0$ (we shall use the existence of this prime ideal throughout). Also by [1], this implies that R is a Cayley-Dickson ring or R is associative. Thus we assume that R is associative, so that $U_a(b) = aba$ for all $a, b \in J$.

Theorem 3.1. *If J is prime and contains no infinite direct system of ij -quadratic ideals and R is associative then R contains no infinite direct sum of one sided ideals.*

Proof. Suppose $\sum L_k$ is an infinite direct sum of left ideals in R and let P be a prime ideal such that $P \cap P^* = 0$. Then there are infinitely many L_k 's not contained in P or infinitely many L_k 's not contained in P^* . We assume $L_k \not\subseteq P$ for all k . Thus $(L_k^* + P^*)(L_k + P) \neq 0$ by the primeness of P so that $B_k = (L_k^* + P^*) \cap (L_k + P) \neq 0$. Let $0 = \sum b_k$ such that $b_k = 0$ for all but finitely many k 's and $b_k \in B_k$, so that $b_k = l_k + p_k = l_k^{**} + p_k^{**}$ where $l_k, l_k^* \in L_k$ and $p_k, p_k^* \in P$. If $P = 0$, then $b_k = 0$ for all k since $\sum L_k$ is a direct sum. If $P \neq 0$ then $0 = P^*0 = P^*\sum b_k$, so that $P^*l_k = 0$. Thus $l_k \in P$ and hence $b_k \in P$. By a similar argument $b_k \in P^*$ so that in any case $b_k = 0$. Therefore $\sum B_k$ is a direct sum of nonzero subgroups of $(R, +)$.

If we set $Q_k = \{ba_i e_{ii} + ba_j e_{ij} + b^* a_i e_{ji} + b^* a_j e_{jj} : b, b' \in H \cap B_k \text{ and } b \in B_k\}$, it is easy to check that Q_k is a quadratic ideal. Now let $\{Q_n\}_{n \in I_1}$ and $\{Q_m\}_{m \in I_2}$ be two finite disjoint subsets of $\{Q_k\}$. Let Q_{I_1} be the quadratic ideal generated by $\sum_{n \in I_1} Q_n$ and Q_{I_2} be the quadratic ideal generated by $\sum_{m \in I_2} Q_m$. Let $C = \sum_{n \in I_1} L_n$ and $D = \sum_{m \in I_2} L_m$ so that $C + D$ is a direct sum of left ideals, and we may perform similar constructions using C and D to obtain Q_C and Q_D , respectively, as we did to construct Q_k using L_k . We then have $Q_{I_1} \subseteq Q_C$ and $Q_{I_2} \subseteq Q_D$ and $Q_{I_1} \cap Q_{I_2} \subseteq Q_C \cap Q_D = 0$ since

$$[(C^* + P^*) \cap (C + P)] \cap [(D^* + P^*) \cap (D + P)] = 0.$$

Thus we see that $\{Q_k\}$ is an infinite direct system of ij -quadratic ideals.

We now turn our attention to considering ij -annihilators. For a nonempty subset S of R , we shall use $A_L(S), A_R(S)$ for the left, right annihilator of S , respectively. If $L = A_L(S)$ and $T = A_R(L)$, we see that $A_L(T)$ is L . We would like to use T and L to give some insight into ij -annihilators.

If V is either a left or right ideal of R then

$$(1) \quad Q_{ji}(V) = \{x e_{ii} + a_i^{-1} v^* a_j e_{ij} + v e_{ji} + y e_{jj} : v \in V, x \in a_i^{-1}(V \cap H), \text{ and } y \in H a_j\},$$

and

$$(2) \quad Q'_{ij}(V) = \{xe_{ii} + ve_{ij} + a_j^{-1}v^*a_i e_{ji} + ye_{jj}; \\ v \in V, x \in (V \cap H)a_i, \text{ and } y \in a_j^{-1}H\}$$

are quadratic ideals of J . One should notice that $xe_{ii} = a^{-1}be_{ii}$, $b \in V \cap H$, is in J_{ii} as described above since $x = (a_i^{-1}ba_i^{-1})a_i$ and $a_i^{-1}ba_i^{-1} \in H$. Also, if $A_L(T) = L$ and $A_R(L) = T$ then

$$U_{[Q'_{ij}(L)]_{ij}}([Q_{ji}(T)]_{ij}) = 0.$$

Is $Q'_{ij}(L)$ the ij -annihilator of $Q_{ji}(T)$?

Lemma 3.2. *Let R be a prime associative ring and W an additive subgroup of $(R, +)$. Then $A(W) = \{a \in R : awRwa = 0 \text{ for all } w \in W\} = A_L(W) \cup A_R(W)$.*

Proof. Let $a \in A(W)$ and suppose $wa \neq 0$ for some $w \in W$, so that $aw = 0$. Suppose $aw' \neq 0$ for some $w' \in W$, but $a(w + w') = 0$ or $(w + w')a = 0$, both of which lead to a contradiction.

Thus in the $*$ -prime case, we see that $A(W) = \{a \in R : awRwa = 0 \text{ for all } w \in W\}$ where W is an additive subgroup of $(R, +)$ is equal to $\bigcup_{i=1}^4 K_i(W)$ where

$$K_1(W) = \{a \in R : aW \subseteq P \text{ and } aW \subseteq P^*\} \\ K_2(W) = \{a \in R : aW \subseteq P \text{ and } Wa \subseteq P^*\} \\ K_3(W) = \{a \in R : Wa \subseteq P \text{ and } Wa \subseteq P^*\} \\ K_4(W) = \{a \in R : Wa \subseteq P \text{ and } aW \subseteq P^*\}.$$

Lemma 3.3. *Let R be a $*$ -prime associative ring and $J = H(R_n, \gamma_a)$. Let B be a quadratic ideal of J and W be the set of elements of R which occur as ji -components of elements of B_{ij} (using the standard matrix units). A necessary and sufficient condition that B has a nonzero ij -annihilator is that the quadratic ideal Q generated by $[A(W)]_{ij} = \{x_{ij} : x \in \bigcup K_i(W)\}$ is nonzero and $Q_{ij} = [A(W)]_{ij}$. Moreover, if this condition is satisfied then $Q = ij\text{-ann}(B)$.*

Proof. By the definition of ij -annihilators, it suffices to show that $[A(W)]_{ij} = \{x_{ij} \in J_{ij} : U_{x_{ij}}(B_{ij}) = 0\}$.

Let B and W be as stated above so that $B_{ij} = \{w_{ji} : w \in W\}$.

Suppose $U_{x_{ij}}(B_{ij}) = 0$ and $x_{ij} = xe_{ij} + a_j^{-1}x^*a_i e_{ji}$ so that $0 = U_{x_{ij}}U_{w_{ji}}(y_{ij}) = xwywx_{ij} + a_j^{-1}(xwywx)^*a_i e_{ji}$ for all $y \in R, w \in W$ and hence $x \in \bigcup K_i(W)$.

Now suppose $x \in \bigcup K_i(W)$ and $x_{ij} = xe_{ij} + a_j^{-1}x^*a_i e_{ji}$. $U_{x_{ij}}(B_{ij}) = \{xwx_{ij} + a_j^{-1}(xwx)^*a_i e_{ji} : w \in W\}$. Since $x \in \bigcup K_i(W)$, one of the following is the case:

$$xw, w^*x^* \in P; \quad xw, x^*w^* \in P; \quad wx, x^*w^* \in P; \quad wx, w^*x^* \in P.$$

In any case, $xwx \in P \cap P^* = 0$ so that $U_{x,ij}(B_{ij}) = 0$.

Remark. Lemma 3.3 tells us that a necessary condition for B to have a non-zero ij -annihilator is that $\bigcup K_i(W)$ be an additive subgroup of $(R, +)$, but in this case one can show that $\bigcup K_i(W) = K_q(W)$ for some $1 \leq q \leq 4$ by showing that if σ is the permutation (1 2 3 4) then

- (a) $\bigcup K_i(W) = K_j(W) \bigcup K_{j\sigma}(W) \cup K_{j\sigma^2}(W)$, for one $j = 1$ or 3 , and
- (b) $K_j(W) \subseteq K_{j\sigma}(W)$ or $K_{j\sigma}(W) \subseteq K_j(W)$ for $1 \leq j \leq 4$.

The proof of this Remark is straightforward.

Lemma 3.4. *Let T be a nonzero right ideal of R ; $L = A_L(T)$; and $T' = TP + TP^*$. Then*

- (i) if R is prime then $[ij\text{-ann}(Q_{ji}(T))]_{ij} = [Q'_{ij}(L)]_{ij}$ and
- (ii) if R is $*$ -prime (not prime) then $[ij\text{-ann}(Q_{ji}(T'))]_{ij} = [Q'_{ij}(L)]_{ij}$.

Proof. (i) Since R is prime, the right annihilator of T is zero. Thus $\bigcup K_i(T) = K_1(T) = L$, and, by Lemma 3.3, $[Q'_{ij}(L)]_{ij} = [ij\text{-ann}(Q_{ji}(T))]_{ij}$.

(ii) Since R is $*$ -prime (not prime), $P \neq 0$ so that $T' \neq 0$.

We shall show $\bigcup K_i(T') = L$. Clearly $L \subseteq \bigcup K_i(T')$. Let $a \in K_i(T')$ so that $aTPa = 0 = aTP^*a$. $aTPa = 0$ implies $a \in P^*$ or $aT \subseteq P^*$ and $aTP^*a = 0$ implies $a \in P$ or $aT \subseteq P$. Thus $a \in L$ and $\bigcup K_i(T') = L$, so that, by Lemma 3.3, $[ij\text{-ann}(Q_{ji}(T'))]_{ij} = [Q'_{ij}(L)]_{ij}$.

Theorem 3.5. *If J satisfies ACC on ij -annihilators and R is a $*$ -prime associative ring then R satisfies ACC on left annihilator ideals.*

Proof. In Lemma 3.4, we showed that if $L_1 \subseteq L_2 \subseteq L_3 \subseteq \dots$ is an ascending chain on left annihilator ideals then we may form an ascending chain, $B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots$ of ij -annihilators such that $(B_k)_{ij} = [Q'_{ij}(L_k)]_{ij}$. Since the chain of ij -annihilators terminates and B_k is generated by $(B_k)_{ij}$, we see that the chain of $[Q'_{ij}(L_k)]_{ij}$ terminates so that the chain $L_1 \subseteq L_2 \subseteq L_3 \subseteq \dots$ terminates.

Summarizing results to this point we have:

Theorem 3.6. *Let $J = H(R_n, \gamma_a)$, $n \geq 2$, be a prime Jordan ring containing no infinite direct system of ij -quadratic ideals and satisfying ACC on ij -annihilators and let R be an alternative ring with characteristic $\neq 2$, $\frac{1}{2} \in R$, such that the set of symmetric elements, H , of R is contained in the nucleus N of R . Then R is a $*$ -prime associative Goldie ring or n is either 2 or 3 and R is a Cayley-Dickson ring.*

By definition of a Cayley-Dickson ring, if Z is the center of R (i.e., $Z = \{z \in N : zx = xz \text{ for all } x \in R\}$) and Z' is the field of quotients of Z , then

$R' = Z' \otimes_Z R$ is a Cayley-Dickson algebra. E. Kleinfeld [6] has shown that if R is a prime alternative (not associative) ring, then $N = Z$. If R is a Cayley-Dickson ring with involution $*$ such that $H \subseteq N$ then $H = Z$. This is due to the fact that, if we let R' be the Cayley-Dickson algebra associated with R and extend $*$ on R to $*$ on R' by $(z^{-1} \otimes r)^* = z^{-1} \otimes r^*$ for $z \in Z$ and $r \in R$, then R' is a Cayley-Dickson algebra with its symmetric elements invertible in its nucleus, so that $*$ is a standard involution (see [5]). This gives us that $x^*x = xx^*$ for all $x \in R$, and that $H = Z = N$.

IV. If. We first consider the case when R is a Cayley-Dickson ring. Let R' be the associated Cayley-Dickson algebra and consider R as a subring of R' and $*$ extended to R' . We shall write $z^{-1}r$ for $z^{-1} \otimes r$. Thus $R' = Z'R$ where Z' is as above. If $J' = H(R'_n, \gamma'_a)$ then J' is a Z' algebra and $J \subseteq J'$.

Let $J = H(R_n, \gamma_a)$, $n = 2$ or 3 where R is a Cayley-Dickson ring with $H \subseteq N$ and $\frac{1}{2} \in R$. Let b_{ij} and x_{ji} be elements in J_{ij} , $i \neq j$, where $b_{ij} = be_{ij} + b^*a_j^{-1}a_i e_{ji}$ and $x_{ji} = x^*a_j a_i^{-1}e_{ij} + xe_{ji}$. Then since a_i 's are in the center $Z(R)$, we have that $U_{b_{ij}}(x_{ji}) = bxb_{ij} + b^*x^*b^*a_j^{-1}a_i e_{ji}$. From this it is clear that $U_{b_{ij}}(Z'x_{ji})$ is a one dimensional subspace of J' provided $bxb \neq 0$. But if $R' = Z'R$ then $bR'b \neq 0$ for $b \neq 0$, so that $bRb \neq 0$ for $b \neq 0$. Thus $U_{b_{ij}}(J'_{ij})$ is a subspace of J' of dimension greater than or equal to one for $b_{ij} \neq 0$.

Theorem 4.1. *Let $J = H(R_n, \gamma_a)$, $n = 2$ or 3 , be a Jordan matrix ring. If R is a Cayley-Dickson ring with involution such that $H \subseteq N$ then J contains no infinite direct system of ij -quadratic ideals, $i \neq j$.*

Proof. Suppose the theorem is false. That is, J contains an infinite direct system of ij -quadratic ideals, $i \neq j$, say $\{Q_k\}$. Picking nonzero $q_k \in Q_k$ for each k , we obtain the system $\{U_{q_k}(J'_{ij})\}$ of nonzero Z' subspace of J' . By the finite dimensionality of J' over Z^k , we see for some choice of q_k and q_m 's not equal to q_k that

$$U_{q_k}(J'_{ij}) \cap \left[\sum_m U_{q_m}(J'_{ij}) \right] \neq 0.$$

Thus

$$0 \neq U_{q_k}(z^{-1}y_{ij}) = \sum_m U_{q_m}(z_m^{-1}y_{ij}^{(m)})$$

for some choice of $z, z_m \in Z$ and $y_{ij}, y_{ij}^{(m)} \in J_{ij}$. Therefore setting π equal to the product of the z_m 's we have

$$0 \neq U_{q_k}(\pi y_{ij}) = \sum_m (U_{q_m}(\pi z_m^{-1}y_{ij}^{(m)})).$$

However, $\pi z_m^{-1} y_{ij}^{(m)}$ and πy_{ij} are elements in J . This contradicts the assumption that $\{Q_k\}$ is an infinite direct system.

Theorem 4.2. *Let $J = H(R_n, \gamma_a)$, $n = 2$ or 3 , be a Jordan matrix ring. If R is a Cayley-Dickson ring with involution such that $H \subseteq N$ then J satisfies ACC on ij -annihilators, $i \neq j$.*

Proof. Let $Q_i = ij\text{-ann}(B_i)$ and $0 \neq Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \dots$ be an ascending chain of ij -annihilators in J . Let $Q'_i = Z'Q_i$. The chain $Q'_1 \subseteq Q'_2 \subseteq Q'_3 \subseteq \dots$ terminates since it is a chain of subspaces in a finite dimensional space. Thus there exists a $Q'_m = \bigcup Q'_i$.

Now, we show that $Q_m = \bigcup Q_i$. It suffices to show $Q_m \cap J_{ij}$ contains $(\bigcup Q_i) \cap J_{ij}$ since Q_i is generated by $Q_i \cap J_{ij}$. Let $x \in (\bigcup Q_i) \cap J_{ij}$ so that x is in Q'_m . That is, for some s and t in Z and y in $Q_m \cap J_{ij}$ $x = s^{-1}ty$. Thus $U_x(B_m \cap J_{ij}) = U_{s^{-1}ty}(B_m \cap J_{ij}) = (s^{-1}t)^2 U_y(B_m \cap J_{ij}) = 0$ and we see that x is an element in J_{ij} which annihilates $B_m \cap J_{ij}$ so that x is in $ij\text{-ann}(B_m) = Q_m$.

The primeness of $J = H(R_n, \gamma_a)$, $n = 2$ or 3 , when R is a Cayley-Dickson ring follows from the fact that $H(R'_n, \gamma_a) = Z'J$.

We now turn our attention to the case when R is a $*$ -prime associative Goldie ring. The primeness of J follows from the involution primeness of R_n under γ_a [5, p. 129].

Until stated otherwise we will make the following assumption which will lead us to a contradiction. We shall assume that J contains an infinite direct system, $\{Q_k\}$, of ij -quadratic ideals for some $i \neq j$, and R is a $*$ -prime Goldie ring.

Since R is a $*$ -prime Goldie ring with involution R is both left and right Goldie so that R/P and R/P^* are each both left and right Goldie [4, p. 268]. Thus R/P is a left and right order in the complete matrix ring, D_w , over a division ring D .

Let $\{f_{bb'}\}$ be the standard set of matrix units in D_w . Every element in D_w may be written as $\sum a_{bb'} f_{bb'}$ where $a_{bb'}$ is an element in the centralizer of $\{f_{bb'}\}$. We shall consider the coefficients, $a_{bb'}$, to be lexicographically ordered according to their subscripts. We shall say that $(a_{bb'}) \in D_w$ has l -zeros if the first l coefficients are zero and if $l \neq w^2$ then the $(l + 1)$ st one is not zero.

We shall consider R as being a subring of the direct sum $R/P + R/P^*$. Let g and g_* be the projections

$$g: R/P + R/P^* \rightarrow R/P, \quad g_*: R/P + R/P^* \rightarrow R/P^*.$$

Also let

$$M_{ij}(Q_k) = \{g(x) : x \in R \text{ and } x_{ij} \in Q_k\} \quad \text{and} \quad M_{ij}^*(Q_k) = \{g_*(x) : x \in R \text{ and } x_{ij} \in Q_k\}.$$

Since R/P and R/P^* are prime Goldie rings we may consider each as a subring

of a matrix ring over a division ring, so that $M_{ij}(Q_k)$ and $M_{ij}^*(Q_k)$ are sets of matrices and it makes sense to talk about the zeros of their elements. It should be pointed out that in light of this situation the elements of J are matrices whose entries are ordered pairs of matrices since $R \subseteq R/P + R/P^*$.

Lemma 4.3. *If each $Q_k \cap J_{ij}$ contains a nonzero element $x^{(k)} = x_k a_j e_{ij} + x_k^* a_i e_{ji}$ such that $g(x_k a_j)$ has l -zeros, $l \neq w^2$ where viewed as a matrix in R/P , then J contains an infinite direct system of ij -quadratic ideals $\{A_p\}$ such that each $A_p \cap J_{ij}$ contains a nonzero element $y^{(p)} = y_p a_j e_{ij} + y_p^* a_i e_{ji}$ such that $g(y_p a_j)$ has at least $(l + 1)$ -zeros. Moreover, if the $x^{(k)}$'s have the property that $g_*(x_k a_j) = 0$ then the $y^{(p)}$'s have the property that $g_*(y_p a_j) = 0$.*

Proof. If we let A_p be the quadratic ideal generated by $Q_p + Q_{p+1}$ for odd integers then $\{A_p\}$ is an infinite direct system of ij -quadratic ideals of J . We shall use the existence of $x^{(p)}$ and $x^{(p+1)}$ to construct $y^{(p)}$ in A_p .

Let p be an odd integer and let $x^{(p)}$ and $x^{(p+1)}$ be as in the statement of the lemma. Suppose the $(l + 1)$ st position is the one corresponding to the pair (r, s) so that

$$g(x_p a_j) = (a_{bb'}) = \begin{cases} 0 & \text{if } b < r \text{ or if } b = r \text{ but } b' < s, \\ a_{rs} \neq 0, & \\ ? & \text{otherwise,} \end{cases}$$

and

$$g(x_{p+1} a_j) = (b_{bb'}) = \begin{cases} 0 & \text{if } b < r \text{ or if } b = r \text{ but } b' < s, \\ b_{rs} \neq 0, & \\ ? & \text{otherwise.} \end{cases}$$

By the Faith-Utumi theorem, the centralizer D of $\{f_{bb'}\}$ contains a left and right order I such that $\sum I f_{bb'} \subseteq R/P$. Thus every element in D may be written as $c^{-1}d = uv^{-1}$ for some $c, d, u, v \in I$ so that $a_{rs} = c^{-1}d = uv^{-1}$ for c, d, u, v in I and $b_{rs} = c_0^{-1}d_0 = u_0 v_0^{-1}$ for c_0, d_0, u_0, v_0 in I . Since I is a left and right order in D , by Ore's theorem [4, p. 262] there exist nonzero elements x, x_0, y, y_0 in I such that $xd = x_0 d_0$ and $uy = u_0 y_0$. Now, let m and q be elements of R such that

$$g(m) = (m_{bb'}) = \begin{cases} 0 & \text{if } b \neq s \text{ or } b' \neq r, \\ v y x c & \text{if } b = s \text{ and } b' = r, \end{cases}$$

and

$$g(q) = (q_{bb'}) = \begin{cases} 0 & \text{if } b \neq s \text{ or } b' \neq r, \\ v_0 y_0 x_0 c_0 & \text{if } b = s \text{ and } b' = r. \end{cases}$$

Since $g(R) = R/P$, such an m and q exist in R .

Let $m_{ji} = a_i^{-1} m^* a_j e_{ij} + m e_{ji}$ and $q_{ji} = a_i^{-1} q^* a_j e_{ij} + q e_{ji}$. Here e_{ij} and e_{ji} are elements in the set of matrix units in R_n . Thus

$$\begin{aligned} U_{x^{(p)}}(m_{ji}) &= (x_p a_j e_{ij} + x_p^* a_i e_{ji})(a_i^{-1} m^* a_j e_{ij} + m e_{ji})(x_p a_j e_{ij} + x_p^* a_i e_{ji}) \\ &= (x_p a_j) m (x_p a_j) e_{ij} + x_p^* m^* a_j x_p^* a_i e_{ji}. \end{aligned}$$

But

$$(x_p a_j) m (x_p a_j) = g(x_p a_j) g(m) g(x_p a_j) + g_*(x_p a_j) g_*(m) g_*(x_p a_j)$$

and

$$\begin{aligned} g(x_p a_j) g(m) g(x_p a_j) &= (a_{bb'}) (m_{bb'}) (a_{bb'}) = (a_{bb'}) \left(\sum_r m_{sr} a_{rb'} \right)_{sb'} \\ &= \left(\sum_s a_{bs} \left(\sum_r m_{sr} a_{rb'} \right) \right)_{bb'} = \left(\sum_s \sum_r a_{bs} m_{sr} a_{rb'} \right)_{bb'} \\ &= \begin{cases} 0 & \text{if } b < r \text{ or if } b = r \text{ but } b' < s \\ u y x d & \text{if } b = r \text{ and } b' = s \\ ? & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly

$$U_{x^{(p+1)}}(q_{ji}) = (x_{p+1} a_j) q (x_{p+1} a_j) e_{ij} + x_{p+1}^* q^* a_j x_{p+1}^* a_i e_{ji}$$

where

$$(x_{p+1} a_j) q (x_{p+1} a_j) = g(x_{p+1} a_j) g(q) g(x_{p+1} a_j) + g_*(x_{p+1} a_j) g_*(q) g_*(x_{p+1} a_j)$$

and

$$g(x_{p+1} a_j) g(q) g(x_{p+1} a_j) = \begin{cases} 0 & \text{if } b < r \text{ or } b = r \text{ and } b' < s, \\ u_0 y_0 x_0 d_0 & \text{if } b = r \text{ and } b' = s, \\ ? & \text{otherwise.} \end{cases}$$

$U_{x^{(p)}}(m_{ji})$ is a nonzero element in Q_p and $U_{x^{(p+1)}}(q_{ji})$ is a nonzero element in Q_{p+1} so that $U_{x^{(p)}}(m_{ji})$ minus $U_{x^{(p+1)}}(q_{ji})$ is not equal to zero since $Q_p \cap Q_{p+1} = 0$. $g(x_p a_j) g(m) g(x_p a_j) - g(x_{p+1} a_j) g(q) g(x_{p+1} a_j)$ is an element of D_w with at least $(l+1)$ -zeros, since $u y x d = u_0 y_0 x_0 d_0$. Let

$$y^{(p)} = U_{x^{(p)}}(m_{ji}) - U_{x^{(p+1)}}(q_{ji}).$$

$y^{(p)}$ is a nonzero element in $A_p \cap J_{ij}$ such that $g(y_p a_j)$ has at least $(l+1)$ -zeros, where y_p is taken to be $x_p a_j m x_p - x_{p+1} a_j q x_{p+1}$ so that $y^{(p)} = y_p a_j e_{ij} + y_p^* a_j e_{ji}$

This completes the proof of the lemma, since the last statement of the lemma is clear from the construction.

We use Lemma 4.3 to show that there is a contradiction built in the assumption stated above.

Theorem 4.4. *Let $J = H(R_n, \gamma_a)$ where R is associative and $n \geq 2$. If R is *-prime Goldie then J does not contain an infinite direct system of ij -quadratic ideals for all $i \neq j$.*

Proof. Suppose that the theorem is false. We may assume J contains an infinite direct system $\{Q_k\}$ such that each Q_k contains an $x^{(k)} = x_k a_j e_{ij} + x_k^* a_i e_{ji}$ such that $g(x_k a_j)$ has l -zeros, $l \neq w^2$, for all integral values of k , since this or the corresponding statement using $g_*(x_k a_j)$ is true for some infinite subset of $\{Q_k\}$.

By Lemma 4.3, J contains an infinite direct system $\{Q_k^{(1)}\}$ such that each $Q_k^{(1)} \cap J_{ij}$ contains a nonzero element $y^{(k)} = y_k a_j e_{ij} + y_k^* a_i e_{ji}$ such that $g(y_k a_j)$ has m -zeros where $m > l$.

Continuing by induction and Lemma 4.3, J contains an infinite direct system $\{Q_k^{(r)}\}$, $r \leq w^2$, such that $Q_k^{(r)} \cap J_{ij}$ contains an element $c^{(k)} = c_k a_j e_{ij} + c_k^* a_i e_{ji} \neq 0$ such that $g(c_k a_j) = 0$. But since $c^{(k)} \neq 0$, it must be the case that $g_*(c_k a_j) \neq 0$.

Similar to what we did above, we may assume that $g_*(c_k a_j)$ has m -zeros for each k . Then we may go through an argument similar to the one just completed to obtain an infinite direct system $\{Q_k^{(r)}\}$, $r \leq 2w^2$ such that $Q_k^{(r)} \cap J_{ij}$ contains an element $d^{(k)} = d_k a_j e_{ij} + d_k^* a_i e_{ji} \neq 0$ such that $g_*(d_k a_j) = 0$. But the last sentence of Lemma 4.3 tells us that this construction may be done so that $g(d_k a_j) = 0$. This is impossible.

We now show that R *-prime Goldie implies that J satisfies ACC on ij -annihilators.

Lemma 4.5. *If $K_q(W_1) \subseteq K_q(W_2) \subseteq \dots \subseteq K_q(W_m) \subseteq \dots$ while $q = 1, 2, 3, 4$ and the W_m 's are additive subgroups of $(R, +)$, then $K_q(W_i) = K_q(\sum_{j \geq i} W_j)$. (The notation used here is that which was introduced in the proof of Lemma 3.2.)*

Proof. If $q = 1$ or 3 then we are talking about left or right annihilators in R , and hence the lemma is true for $q = 1$ or 3 . Also, a slight adaptation of the following proof yields a proof for these cases.

Suppose $q = 2$. Let a be an element in $K_2(W_i)$ so that $aW_j \subseteq P$ and $W_j a \subseteq P^*$ for all $j \geq i$, since $K_2(W_i) \subseteq K_2(W_j)$ for $j \geq i$. Thus $a(\sum_{j \geq i} W_j) \subseteq P$ and $(\sum_{j \geq i} W_j)a \subseteq P^*$ so that a is an element in $K_2(\sum_{j \geq i} W_j)$. Therefore $K_2(W_i)$ is contained in $K_2(\sum_{j \geq i} W_j)$. Clearly if a is an element in $K^2(\sum_{j \geq i} W_j)$ then a is an element in $K_2(W_i)$ since $aW_i \subseteq a(\sum_{j \geq i} W_j) \subseteq P$ and $W_i a \subseteq (\sum_{j \geq i} W_j)a \subseteq P$ and $W_i a \subseteq (\sum_{j \geq i} W_j)a \subseteq P^*$. Thus $K_2(W_i) = K_2(\sum_{j \geq i} W_j)$.

The proof for $q = 4$ is similar and therefore it is omitted.

Theorem 4.6. *Let $J = H(R_n, \gamma_a)$, $n \geq 2$, where R is associative and $*$ -prime. If R is Goldie then J satisfies ACC on ij -annihilators.*

Proof. From the Remark of §III, it suffices to show R satisfies ACC on sets of the form $\bigcup K_i(T)$ where T is an additive subgroup of $(R, +)$. Thus it suffices to show R satisfies ACC on sets of the form $K_q(W)$ where W is an additive subgroup of $(R, +)$.

Suppose we have such a chain $K_q(W_1) \subseteq K_q(W_2) \subseteq \dots \subseteq K_q(W_m) \subseteq \dots$. By Lemma 4.5, we may assume that $W_1 \supseteq W_2 \supseteq \dots \supseteq W_m \supseteq \dots$.

By ACC on left and right annihilators in R/P , we have the following chains in R/P terminating

$$A_R(W_i/P) \subseteq A_R(W_{i+1}/P), \quad A_L(W_i/P) \subseteq A_L(W_{i+1}/P)$$

and correspondingly in R/P^*

$$A_R(W_i/P^*) \subseteq A_R(W_{i+1}/P^*), \quad A_L(W_i/P^*) \subseteq A_L(W_{i+1}/P^*).$$

Hence the following chains terminate in R :

$$\begin{aligned} \{a \in R: a + P \in A_R(W_i/P)\} &\subseteq \{a \in R: a + P \in A_R(W_{i+1}/P)\}, \\ \{a \in R: a + P \in A_L(W_i/P)\} &\subseteq \{a \in R: a + P \in A_L(W_{i+1}/P)\}, \\ \{a \in R: a + P^* \in A_R(W_i/P^*)\} &\subseteq \{a \in R: a + P^* \in A_R(W_{i+1}/P^*)\}, \\ \{a \in R: a + P^* \in A_L(W_i/P^*)\} &\subseteq \{a \in R: a + P^* \in A_L(W_{i+1}/P^*)\}, \end{aligned}$$

But the chain $K_q(W_i) \subseteq K_q(W_{i+1})$ is the intersection of corresponding terms of two chains which terminate. Hence we have that the chain $K_q(W_i) \subseteq K_q(W_{i+1})$ terminates.

V. Quotients. In order to complete the proof of the Main Theorem, the last statement is all that needs to be shown.

In the case when R is a $*$ -prime Goldie ring this follows from [2].

Let R be a Cayley-Dickson ring and let R' be the Cayley-Dickson algebra associated with R . Extend the involution on R to the standard involution on R' and extend γ_a on R_n to γ_a on R' . We need to show that every element in $J' = H(R'_n, \gamma_a)$, $n = 2, 3$, has the form $U_a^{-1}(b)$ for some a and b in J with a regular in J , and that regular elements in J are invertible in J' .

Since every element in R' is of the form $z^{-1}b$ for $0 \neq z$ in the center $Z = Z(R)$ and b in R , given an element t in $H(R'_n, \gamma_a)$ we may express it as an $n \times n$ matrix $(c_{kk'})$ where each entry $c_{kk'}$ is of the form $z_{kk'}^{-1}b_{kk'}$ for $z_{kk'} \neq 0$ in $Z(R) = H(R)$ and $b_{kk'}$ in R . We shall exhibit a y and a w such that $t = U_w^{-1}(y)$. Let π be the product of $z_{kk'}$'s so that $\pi^2 z_{kk'}^{-1}$ is in $H(R)$ and let w be the diagonal

matrix $\pi e_{11} + \cdots + \pi e_{nn}$. Let $y = (\pi^2 z_{kk}^{-1}, b_{kk'})$. Then y is in $H(R_n, \gamma_a)$ since

$$\gamma_a(y) = \gamma_a(\pi^2 z_{kk}^{-1}, b_{kk'}) = \pi^2 \gamma_a(z_{kk}^{-1}, b_{kk'}) = \pi^2 (z_{kk}^{-1}, b_{kk'}) = (\pi^2 z_{kk}^{-1}, b_{kk'}) = y.$$

Clearly y and w have the desired property.

Finally, let x be a regular element of J . Let t be an arbitrary element in J' . By what was shown above, $t = U_{w^{-1}}(y)$ for w in $Z(R)$ and y in J . Here we are considering J' as a Z' -algebra where Z' is the field of quotients for $Z(R)$ and identifying z^{-1} with $z^{-1}1$ for 1 in J' . Hence

$$U_x(t) = U_x U_{w^{-1}}(y) = U_x(w^{-2}y) = w^{-2}U_x(y) \neq 0$$

since x is regular in J . Thus U_x is 1-1 on the finite dimensional Z' -algebra J' . Recall R' is 8-dimensional over Z' . Therefore, U_x is a 1-1 linear transformation on a finite dimensional vector space and hence U_x is onto so that 1 is in $U_x(J')$ and x is invertible.

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