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ON STRUCTURE SPACES OF IDEALS IN RINGS OF CONTINUOUS FUNCTIONS

BY

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ABSTRACT. A ring of continuous functions is a ring of the form $C(X)$, the ring of all continuous real-valued functions on a completely regular Hausdorff space X.

With each ideal I of $C(X)$, we associate certain subalgebras of $C(X)$, and discuss their structure spaces.

We give necessary and sufficient conditions for two ideals in rings of continuous functions to have homeomorphic structure spaces.

Introduction. For a subset A of $C(X)$, we define rA to be $\{f+c \mid f \in A$ and $c \in R\}$. (R denotes the set of all real numbers, and we make the usual identification between the real number c and the function which maps every $x \in X$ onto c.) We denote by A^u the closure of A in the uniform topology.

With each ideal I in $C(X)$, we associate four subalgebras of $C(X)$, I itself, I^u, rI, and $r(I^u)$. (In [4], rI and $r(I^u)$ were denoted by (I) and (I^u), respectively.) In this paper, we characterize the maximal ideals of I^u, rI, and $r(I^u)$ (the maximal ideals of I were characterized in [4]) and then endow these sets of maximal ideals with the hull-kernel topology. We then investigate the resulting structure spaces.

In §1 we show that the prime and maximal ideals of rI are the intersections of rI respectively with the prime and maximal ideals of $C(X)$. This allows us to establish homeomorphisms between structure spaces of rI and modifications of structure spaces of $C(X)$. We also show that the structure space of rI is (homeomorphic to) the one-point compactification of the structure space of I.

In §2 we discuss the prime and maximal ideals of the algebras I^u and $r(I^u)$. We show that I and I^u have the same structure space and that rI and $r(I^u)$ do also. In view of the fact that $r(I^u)$ is (isomorphic to) a ring of continuous functions (see [4, 5.6]), it is thus established that every ideal in $C(X)$ is a real ideal in a subalgebra of $C(X)$ which has the same structure space as a ring of continuous functions. Results in §§1 and 2 generalize certain results in [5].

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The significance of homeomorphic structure spaces of two rings of continuous functions is well known; namely $C(X)$ and $C(Y)$ have homeomorphic structure spaces if and only if $C^*(X)$ and $C^*(Y)$ are isomorphic. (These are the subrings of bounded functions in $C(X)$ and $C(Y)$ respectively.) In §3 we establish necessary and sufficient conditions for two ideals in rings of continuous functions to have homeomorphic structure spaces.

In [2] and [3], the authors considered structure spaces of certain subalgebras of $C(X)$ called "algebras on X." In §4, we discuss the relationships between the properties of an algebra on X and the properties of the algebras considered in this paper.

Preliminaries and notations. For each $f \in C^*(X)$ there is a unique extension $\hat{f}$ from βX into R. For each $f \in C(X)$, there is a unique extension f^* from βX into the two-point compactification of R. (See [1, 7.5] and [4, 2.4] for more about f^*.) If $f \in C^*(X)$, then $\hat{f} = f^*$.

For any commutative ring, a collection of prime ideals endowed with the hull-kernel topology is called a *structure space* of the ring. The collection of all prime maximal ideals is referred to as *the* structure space of the ring. It is well known that βX is (homeomorphic to) the structure space of either $C(X)$ or $C^*(X)$ and that υX is the structure space of real maximal ideals of $C(X)$. (The reader is referred to 7.10 and 7.11 in [1] and [4, 2.3] for more about structure spaces.) The natural mapping $M \to M^*$, where $M^* = M^u \cap C^*(X)$, is a homeomorphism of the structure space of $C(X)$ onto the structure space of $C^*(X)$. (See [1, 7.11].)

Associated with each ideal I in $C(X)$ is a space $X(I)$ and a fixed maximal ideal in $C(X(I))$ denoted by $F(I)$ with the property that I^u is isomorphic to $F(I)$. (See [4, 5.6].)

If $f \in C(X)$, we shall let (f) denote the principal ideal generated by f, and we let $Z(f) = \{x \in X | f(x) = 0\}$. If $g \in C(X)$, then $f \vee g$ denotes the pointwise maximum of f and g.

A subring A of $C(X)$ is called a *subalgebra* of $C(X)$ if A is closed under multiplication by constants. The structure space (of prime maximal ideals) of A will be denoted by μA.

The symbol $\cong$ is used to denote a homeomorphism between spaces, and $\approx$ is used to denote an isomorphism between rings.

For the convenience of the reader we now list the main symbols associated with an ideal I of $C(X)$.

I^u = the uniform closure of I;

$I^* = I^u \cap C^*(X)$;

$rI = \{f + c | f \in I \text{ and } c \in R\}$;

$r(I^u) = \{f + c | f \in I^u \text{ and } c \in R\}$;

μI = the structure space of I;
$\Delta I = \bigcap\{Z(f) | f \in I\}$;
$M(I) = \{M | M$ is a maximal ideal in $C(X)$ and $M \supseteq I\}$;
$P(I) = \{P | P$ is a prime ideal in $C(X)$ and $P \supseteq I\}$;
$X(I)$ = a space with the property that $C(X(I)) \approx r(I^u)$;
$F(I)$ = the real ideal in the space $X(I)$ isomorphic to I^u.

1. **The structure space of** rI. It is known [4, 3.6] that the maximal ideals of an ideal I are precisely the intersections of I with the maximal ideals of $C(X)$ not containing I. We now characterize the maximal ideals of the algebra rI.

1.1 **Remark.** Evidently, I itself is a real maximal ideal of rI.

1.2 **Lemma.** *If M is a maximal ideal in $C(X)$, then $M \cap rI$ is a maximal ideal in rI.*

Proof. It is easy to show that for a proper ideal J of $C(X)$, $J \supseteq I$ if and only if $J \cap rI = I$. Thus if $M \supseteq I$, $M \cap rI$ equals I and is a maximal ideal in rI. Now suppose $M \not\supseteq I$. Then there exist $m \in M$ and $i \in I$ with $m + i = 1$. Assume $M \cap rI \subsetneq K \subseteq rI$, for some ideal K of rI. Let $k \in K \setminus M$. Then there exist $m' \in M$ and $h \in C(X)$ so that $hk + m' = 1$. Thus $i = ihk + im' \in K$. But $m = -i + 1 \in M \cap rI \subseteq K$; whence $1 \in K$ and $K = rI$.

We now proceed to establish the converse of 1.2.

1.3 **Lemma.** *If K is a proper prime ideal in rI, then $K = P \cap rI$ for some prime ideal P in $C(X)$. Furthermore, P is unique if $K \neq I$.*

Proof. From the multiplicative semigroup $G = (rI)\setminus K$. From Zorn's lemma, it follows that there exists a prime ideal P in $C(X)$ so that $G \cap P = \emptyset$ and P is maximal with respect to this property. (See [1, 0.16].) We claim that $K = P \cap rI$. It is easy to see that $K \supseteq P \cap rI$. Now if $P \supseteq I$, then $K = I = P \cap rI$, so assume $P \not\supseteq I$. We must show that $K \subseteq P$. To this end, let $k \in K$ and assume $k \notin P$. Let J denote $P + k \cdot I$, an ideal in $C(X)$ which contains P properly. Hence $J \cap G \neq \emptyset$, and there exists $p + ki \in G$, where $p \in P$ and $i \in I$. It follows that $p \in rI$ and hence $p \in K$. But then $p + ki \in K$, a contradiction. The uniqueness is evident.

1.4 **Theorem.** *If K is a maximal ideal in rI, then $K = M \cap rI$ for some maximal ideal M in $C(X)$. Furthermore, if $K \neq I$, then M is unique.*

Proof. By [4, 3.3] K is prime, and hence $K = P \cap rI$, for some prime ideal P in $C(X)$. Letting M be the maximal ideal containing P, the result follows.

We now consider some natural mappings between structure spaces of rI and $C(X)$. We shall let $\mathcal{P}$ denote the space of prime ideals of $C(X)$, $\mathcal{P}(rI)$ the space of prime ideals of rI and $P(I)$ the set of all prime ideals in $C(X)$ which contain I. (Of course, $\mathcal{P}$ and $\mathcal{P}(rI)$ are endowed with the hull-kernel topology.)

1.5 **Lemma.** *$\mathcal{P}\setminus P(I)$ is homeomorphic to $\mathcal{P}(rI)\setminus\{I\}$.*

Proof. Consider the natural mapping $P \to P \cap rI$ and denote it by ϕ. Clearly ϕ is one-to-one and onto. Let $s \in rI$ and consider $E(s) = \{K \mid K \in \mathcal{P}(rI)\setminus\{I\}$ and $s \in K\}$, a basic closed set in $\mathcal{P}(rI)\setminus\{I\}$. Then $\phi^{-1}(E(s)) = \{P \in \mathcal{P}\setminus P(I) \mid s \in P\}$ a basic closed set in $\mathcal{P}\setminus P(I)$. On the other hand, if $f \in C(X)$, and $E(f) = \{P \in \mathcal{P}\setminus P(I) \mid f \in P\}$, a basic closed set in $\mathcal{P}\setminus P(I)$, then $\phi(E(f)) = \{P \cap rI \mid P \cap rI \supseteq f \cdot I\}$, a closed set in $\mathcal{P}(rI)\setminus\{I\}$.

We now extend the mapping ϕ to obtain a mapping onto all of $\mathcal{P}(rI)$. The preimage of I is $P(I)$ shrunk to a point. Specifically, we let $\mathcal{P}' = [\mathcal{P}\setminus P(I)] \cup \{\alpha\}$ where a neighborhood of α is a set of the form $\{\alpha\} \cup [W\setminus P(I)]$ where W is open in $\mathcal{P}$ and $W \supseteq P(I)$. (For $P \in \mathcal{P}\setminus P(I)$, a neighborhood of P is a neighborhood in the relative topology.)

1.6 **Theorem.** *Let $\psi: \mathcal{P}' \to \mathcal{P}(rI)$ be defined by $\psi(P) = P \cap rI$ if $P \not\supseteq I$, and $\psi(\alpha) = I$. Then ψ is a homeomorphism of $\mathcal{P}'$ onto $\mathcal{P}(rI)$.*

Proof. Clearly ψ is one-to-one and onto $\mathcal{P}(rI)$, and by virtue of 1.5, it suffices to show that ψ is continuous at α and ψ^{-1} is continuous at I. To this end, let V be a basic neighborhood of $\psi(\alpha)$ in the space $\mathcal{P}(rI)$. Then $V = \sim\{K \in \mathcal{P}(rI) \mid s \in K\}$ for some $s = i + c \in rI$. Since $I \in V$, $c \neq 0$. Let $W = \sim\{P \in \mathcal{P} \mid s \in P\}$, an open set in $\mathcal{P}$.

It is easily seen that $W \supseteq P(I)$, for if $P \supseteq I$, and $P \notin W$, it would follow that $s \in P$, a contradiction. Thus the set $\{\alpha\} \cup [W\setminus P(I)]$ is open in $\mathcal{P}'$, and it follows that its image under ψ is contained in V.

We now show that ψ^{-1} is continuous at I. Let $\{\alpha\} \cup [W\setminus P(I)]$ be a neighborhood of α in $\mathcal{P}'$, with $W = \sim\{P \in \mathcal{P} \mid P \supseteq J\}$ for some ideal J in $C(X)$. Since $W \supseteq P(I)$, it follows that $I + J = C(X)$, and hence there exist $i \in I$ and $j \in J$ so that $i + j = 1$. Letting V denote the open set $\sim\{K \in \mathcal{P}(rI) \mid j \in K\}$ $(j \in rI)$ it follows that $\psi^{-1}(V) \subseteq \{\alpha\} \cup [W\setminus P(I)]$.

1.7 **Remark.** If we shrink $M(I)$ to a point in the space of maximal ideals in $C(X)$, then the resulting space will be homeomorphic to $\mu(rI)$. (The argument would be the same as 1.5 and 1.6 above.) It then follows that $\mu(rI)$ is homeomorphic to $(\mu I)^*$, the one-point compactification of μI. (See [4, 3.9].)

1.8 **Remark.** If we denote the set of all real maximal ideals containing I by $R(I)$ (possibly $R(I) = \emptyset$), then the structure space of real ideals of rI is homeomorphic to $(\upsilon X\setminus R(I)) \cup \{\alpha\}$ where a neighborhood of α is a set of the form $\{\alpha\} \cup [W\setminus R(I)]$ for some W open in υX with $W \supseteq R(I)$.

Of course, the albebra rI will not in general be (isomorphic to) a ring of continuous functions. Indeed since I is a real ideal in rI, it follows from [4, 5.7] that rI is a ring of continuous functions if and only if I is uniformly closed.

In view of the fact that $(f_1 + c_1) \vee (f_2 + c_2) = (f_1 \vee f_2) + (c_1 \vee c_2)$ where $f_j \in C(X)$ and c_j are real numbers, it follows that rI is a Φ-algebra (see [5]) if and only if $f \vee g \in I$ whenever f and g are members of I. In particular, rI is a Φ-algebra whenever I is absolutely convex in $C(X)$. (An ideal I is said to be *absolutely convex* in $C(X)$ if $f \in I$ and $g \in C(X)$ with $|g| \leq |f|$ imply that $g \in I$.)

2. The albebras I^u and $r(I^u)$. We now proceed to characterize the maximal ideals of I^u and $r(I^u)$. We recall that in [4, 5.6] it was shown that $r(I^u)$ is (isomorphic to) a ring of continuous functions. (Indeed, $r(I^u)$ is in a sense the smallest ring of continuous functions in which I is an ideal.)

2.1 **Remark.** I^u is a real maximal ideal in $r(I^u)$.

2.2 **Lemma.** *If $s \in r(I^u)$ with $s(x) \geq \delta > 0$ for all $x \in X$, then $(1/s) \in r(I^u)$.*

Proof. Suppose $s = f + c$ where $f \in I^u$ and $c \in R$, and $s(x) \geq \delta > 0$ for all $x \in X$. It is easily seen that $c \neq 0$, and we then have $1/s = -f/c(s) + 1/c$. We claim that $f/(f + c) \in I^u$. To see this, let $\epsilon > 0$ be given, and consider $\epsilon' = \delta \cdot \epsilon$. Then there exists $i \in I$ with $|f - i| < \epsilon'$, whence

$$\left| \frac{f}{f+c} - \frac{i}{f+c} \right| < \frac{\epsilon'}{|f+c|} \leq \frac{\epsilon'}{\delta} = \epsilon.$$

2.3 **Lemma.** *If M is a maximal ideal in $C(X)$ with $M \not\supseteq I$, then $M \cap r(I^u)$ is a maximal ideal in $r(I^u)$.*

Proof. The argument is the same as 1.2.

2.4 **Remark.** If $M \supseteq I$, it is possible that $M \cap r(I^u) \subsetneq I^u$, and hence $M \cap r(I^u)$ is not maximal (for example, if I is a hyperreal maximal ideal and $M = I$).

2.5 **Remark.** It is possible to have prime ideals in $r(I^u)$ which are not of the form $P \cap r(I^u)$ for P prime in $C(X)$. As a simple example, the ideal I^u itself may not be of this form. We can say, however

2.6 **Lemma.** *If K is a prime ideal in $r(I^u)$ and $K \not\supseteq I$, then $K = P \cap r(I^u)$ for a unique prime ideal P in $C(X)$.*

Proof. The argument is essentially the same as 1.3.

2.7 **Theorem.** *Let K be a maximal ideal of $r(I^u)$. If $K \not\supseteq I$, then $K = M \cap r(I^u)$ for a unique maximal ideal M in $C(X)$. If $K \supseteq I$, then $K = I^u$.*

Proof. The first part of the theorem is evident. For the second part, suppose $K \supseteq I$, and let $k \in K$, say $k = f + c$ where $f \in I^u$ and $c \in R$. Assume $c > 0$. Then for some $i \in I$, $|f - i| < c/3$. Since $i \in K$, it follows that $f + c - i \in K$. But $(f + c - i)(x) > 2c/3$ for all $x \in X$, and this contradicts 2.2. Similarly, if $c < 0$ we arrive at a contradiction, so we must have $c = 0$ and $K \subseteq I^u$.

We now consider the maximal ideals of the algebra I^u. We shall make use of the fact that $r(I^u)$ is isomorphic to $C(X(I))$ and that the isomorphism takes I^u onto $F(I)$.

2.8 **Theorem.** *Let K be a maximal ideal in I^u. Then $K = M \cap I^u$ for some unique maximal ideal M in $C(X)$ with $M \not\supseteq I$.*

Proof. Let ξ denote the isomorphism of $r(I^u)$ onto $C(X(I))$. Since K is a maximal ideal in I^u, it follows that $\xi(K) = F(I) \cap M'$ where M' is maximal in $C(X(I))$. But then $M' = \xi[M \cap r(I^u)]$ (clearly $M' \neq \xi(I^u)$) and it follows that $K = M \cap I^u$. Now, assume $M \supseteq I$, and let $f \in I^u \backslash K$. From the maximality of K, it follows that $K + f \cdot I^u = I^u$, whence $f = k + fg$ for some $k \in M \cap I^u$ and $g \in I^u$. But this implies that $k = f(1 - g) \in M$, and hence $1 - g \in M$. Since $M \supseteq I$, $g \in M^u$, a contradiction.

2.9 **Corollary.** *The ideal I in I^u is not contained in a maximal ideal of I^u.*

Proof. Follows directly from 2.8 above.

2.10 **Lemma.** *If M is maximal in $C(X)$ and $M \not\supseteq I$, then $M \cap I^u$ is a maximal ideal in I^u. If $M \supseteq I$, then $M \cap I^u$ is not a maximal ideal in I^u.*

Proof. The argument for the first part is the same as 1.2. The second part follows from 2.9.

2.11 **Corollary.** *μI is homeomorphic to $\mu(I^u)$.*

Proof. The natural mapping is a homeomorphism.

2.12 **Remark.** It is easily seen that the natural mapping $M \cap rI \to M \cap r(I^u)$, for $M \not\supseteq I$, and $I \to I^u$ is a homeomorphism of $\mu(rI)$ onto $\mu(r(I^u))$, and $\mu(r(I^u))$ is the structure space of a ring of continuous functions. We thus have that every ideal in a ring of continuous functions is a real ideal in an algebra whose structure space is homeomorphic to a structure space of a ring of continuous functions.

2.13 **Remark.** $(rI)^u = r(I^u)$.

Proof. Let $s \in (rI)^u$. Then there is a sequence $\langle s_n \rangle$ in rI, say $s_n = i_n + c_n$ where $i_n \in I$ and $c_n \in R$, which converges to s. We claim that the sequence $\langle c_n \rangle$ is a Cauchy sequence. To see this, consider $\epsilon > 0$. There exists a positive integer N, so that $|s(x) - (i_n(x) + c_n)| < \epsilon/2$ for all $n > N$ and all $x \in X$. Let n and m be any positive integers greater than N, and let $t \in X$ so that $i_n(t) = i_m(t) = 0$. We then have $|s(t) - c_n| < \epsilon/2$ and $|s(t) - c_m| < \epsilon/2$, from which it follows that $|c_m - c_n| < \epsilon$. Let c denote $\lim_{n\to\infty} c_n$. It then follows that the sequence $\langle i_n \rangle$ converges to a function $f \in I^u$, and $s = f + c$. Conversely, if $s = f + c \in r(I^u)$ then, given $\epsilon > 0$, there exists $i \in I$ so that $|f - i| < \epsilon$. But $|f - i| = |(f + c) - (i + c)|$, and $i + c \in rI$.

The above remark generalizes [5, 3.8].

2.14 **Remark.** The algebras rI and $r(I^u)$ cannot be (ring-) isomorphic unless they are identical. To see this, it suffices to observe that any isomorphism would take I onto I^u which would imply their equality by [4, 4.8].

3. **Structure spaces of ideals.** The structure space of an ideal is not usually a compact space. Indeed

3.1 **Remark.** For any ideal I, the following are equivalent.

(1) μI is compact.

(2) $M(I)$ is open (and closed) in βX.

(3) I is the principle ideal generated by an idempotent.

Proof. The equivalence of (1) and (2) follows from the fact that $\mu I \cong \beta X \setminus M(I)$ (see [4, 3.9]).

(2) $\Rightarrow$ (3) If $M(I)$ is open, then $\sim M(I) = \{M | M \supseteq J\}$ for some ideal J in $C(X)$. It follows that $I \cap J = \{0\}$ and $I + J = C(X)$, and hence there exist $i \in I$ and $j \in J$ with $i + j = 1$. Thus $I = (i)$ and $i^2 = i$.

(3) $\Rightarrow$ (2) If $I = (i)$ and $i^2 = i$, let $J = (i - 1)$. Then $\sim M(I) = \{M | M \supseteq J\}$.

However, the structure space of I is always locally compact, and its one-point compactification is homeomorphic to $\mu(r(I^u))$, a Stone-Čech compactification $(\beta[X(I)])$. (It is also true that any βX is the one-point compactification of a μI; simply take I to be a maximal ideal in $C(X)$.) Thus μI is in a sense a large space, in that it lacks only one point from being a Stone-Čech compactification.

We now wish to show to what extent the structure space of I determines I; specifically, when are μI and μJ homeomorphic?

We begin with a mapping from the set of ideals in $C(X)$ into the set of ideals of $C^*(X)$. For an ideal I in $C(X)$, we denote $I^u \cap C^*(X)$ by I^*. Note that this * mapping is the usual one from the set of maximal ideals in $C(X)$ onto the set of maximal ideals in $C^*(X)$. (See [4, 2.4].)

3.2 **Lemma.** $I^* = \{f \in C^*(X) | Z(\hat{f}) \supseteq M(I)\}$.

Proof. Follows from [4, 5.1] and the fact that $f^* = \hat{f}$ for bounded functions.

3.3 **Remark.** The * mapping is onto the set of full ideals of $C(\beta X)$. (An ideal is said to be *full* in $C(Y)$ if it is of the form $\{f \in C(Y) | Z(f) \supseteq G\}$ for some closed set G in Y.)

3.4 **Lemma.** *μI is homeomorphic to $\mu(I^*)$.*

Proof. The mapping * is a homeomorphism of the structure space of $C(X)$ onto the structure space of $C^*(X)$ [1, 7.11], and clearly $M(I)$ will be mapped onto the set of maximal ideals in $C^*(X)$ which contain I^*.

3.5 **Lemma.** *I^* is isomorphic to the set of continuous real-valued functions on $\mu(rI)$ which vanish at I.*

Proof. Let M_I denote $\{g \in C(\mu(rI)) | g(I) = 0\}$ and for each $f \in I^*$, let $\bar{f}$ be the function on $\mu(rI)$ defined by $\bar{f}(M \cap rI) = \hat{f}(M)$ for $M \not\supseteq I$ and $\bar{f}(I) = 0$. By virtue of the homeomorphism of $\mu(rI)$ into βX, it follows that $\bar{f}$ is continuous at $M \cap rI$ for any $M \not\supseteq I$. To see that $\bar{f}$ is continuous at I, consider $\epsilon > 0$. Let $W = \hat{f}^{-1}(-\epsilon, \epsilon)$, an open set in βX, and hence $W = \sim\{M | M \supseteq J\}$ for some ideal J in $C(X)$. For any $M \supseteq I$, $\hat{f}(M) = f^*(M) = 0$ by [4, 5.1], from which it follows that there exist $i \in I$ and $j \in J$ with $i + j = 1$. Consider $U = \sim\{K \in \mu(rI) | j \in K\}$, a neighborhood of I in $\mu(rI)$. Clearly, for any $M \cap (rI) \in U$, $M \in W$, and hence $\bar{f}(M) \in (-\epsilon, \epsilon)$.

For any $g \in M_I$, define $\hat{f}: \beta X \to R$ by $\hat{f}(M) = g(M \cap rI)$ for $M \not\supseteq I$ and $\hat{f}(M) = 0$ for $M \supseteq I$. Then $f \in I^*$ and $\bar{f} = g$. (f is the restriction of $\hat{f}$ to X.)

Using the fact that $\hat{}$ is an isomorphism of $C^*(X)$ onto $C(\beta X)$, it follows easily that the mapping $f \to \bar{f}$ is an isomorphism of I^* onto M_I.

3.6 **Theorem.** *$\mu I \cong \mu J$ if and only if $I^* \approx J^*$.*

Proof. If $\mu I \cong \mu J$, then their one-point compactifications are homeomorphic. Thus, $\mu(rI) \cong \mu(rJ)$, and this homeomorphism takes the point I onto the point J. It follows that M_I and M_J (as in the notation of the proof of 3.5) are isomorphic; whence $I^* \approx J^*$ by 3.5.

Conversely if $I^* \approx J^*$, then $\mu(I^*) \cong \mu(J^*)$, which, by 3.4, yields the required result.

3.7 **Corollary.** *If $I^u \approx J^u$, then $I^* \approx J^*$.*

Proof. By 2.11, the hypothesis implies that $\mu I \cong \mu J$. The result then follows from 3.6 above.

3.8 **Corollary.** *$\mu I \cong \mu J$ if and only if $[F(I)]^* \approx [F(J)]^*$.*

Proof. Evidently the mapping $I^u \to F(I)$ described in §5 of [4] preserves bounded functions, and so does its inverse. Thus $I^* \approx [F(I)]^*$ and the result follows by 3.6.

3.9 **Remark.** It is possible for $\mu I \cong \mu J$ without $I^u \approx J^u$. For example, let X be any realcompact space which is not β-fixed (see [4, 6.12]) and let H be a homeomorphi of βX onto itself which takes a real ideal M_1 onto a hyperreal ideal M_2. Then $[\beta X \setminus \{M_1\}] \cong [\beta X \setminus \{M_2\}]$, i.e. $\mu M_1 \cong \mu M_2$, but certainly $(M_1)^u$ and $(M_2)^u$ are not isomorphic .

3.10 **Remark.** Theorem 3.6 tells us that the structure space of I in general tells us very little about the algebraic properties of the ideal I, since vastly different ideals can have the same uniform closures. Even with real ideals, homeomorphism of structure spaces does not necessarily imply isomorphism of the ideals. For example, consider $M = \{f \in C(N) | f(1) = 0\}$ and $M' = \{f \in C(\Sigma) | f(1) = 0\}$ where N is the space of natural numbers and Σ is the space of Exercise 4M in [1]. Then μM and $\mu M'$ are homeomorphic (since $\beta N \cong \beta\Sigma$) but certainly M and M' cannot be isomorphic (since $C(N)$ and $C(\Sigma)$ are not isomorphic).

Homeomorphism of real structure spaces of ideals does not seem to tell much about the algebraic structure of the ideals either. For example, let Q denote the space of rationals and let X denote the rationals with $\{0\}$ open. Then $M = \{f \in C(Q) | f(0) = 0\}$ and $M' = \{f \in C(X) | f(0) = 0\}$ have the same structure space of real ideals.

4. **Algebras on X.** In [2], the author defines an *algebra on* X to be a subalgebra A of $C(X)$ with the following properties: (i) A contains the constant functions. (ii) A is uniformly closed. (iii) If F is closed in X and $x \in X \setminus F$, then there exists $f \in A$ so that $f(x) \neq 0$ and $f(F) = 0$. (A separates points and closed sets.) (iv) If $f \in A$ with $Z(f) = \emptyset$, then $1/f \in A$. (A is closed under inversion.)

Of the four algebras considered here (I, I^u, rI, rI^u), none will be an algebra on X in general. We now discuss in a series of remarks and examples these algebras with respect to properties (iii) and (iv) above, since the first two properties involve only trivial considerations.

4.1 **Remark.** rI is closed under inversion in $C(X)$.

Proof. Let $f + c \in rI$ ($f \in I$ and $c \in R$) with $Z(f + c) = \emptyset$. Clearly $c \neq 0$, and we have $1/(f + c) = g + 1/c$ where $g = ((-1/c)/(f + c)) \cdot f \in I$.

We note that the above remark generalizes [5, 3.2].

In general, $r(I^u)$ is not closed under inversion in $C(X)$. This may seem somewhat surprising in view of the fact that $r(I^u)$ is (isomorphic to) a ring of continuous functions.

4.2 **Example.** Let M be a hyperreal maximal ideal in $C(X)$. Then there exists a unit $b \in M^u$ [1, 7.9(b)]. If $1/b \in r(M^u)$, we would have $1/b = m + c$, $m \in M^u$ and $c \in R$, whence $1 = b \cdot m + b \cdot c \in M^u$, a contradiction.

4.3 **Remark.** An ideal I determines the topology on X (separates points and closed sets) if and only if I is a free ideal.

Proof. Assume I is a free ideal, and let K be closed in X with $x \in X\backslash K$. There exists $f \in I$ with $f(x) \neq 0$ and there exists $g \in C(X)$ with $g(x) \neq 0$ and $g(K) = 0$. Thus $f \cdot g \in I$ which separates x from K. If I is a fixed ideal, then no point in ΔI can be separated from a closed set not containing that point.

The algebra rI clearly cannot determine the topology on X if ΔI has at least two points, and rI does determine the topology on X if $\Delta I = \emptyset$. If ΔI consists of precisely one point, then rI may or may not determine the topology.

4.4 **Example.** Let X be the real line and $I = \{f \in C(X) | f(0) = 0$ and f is eventually zero$\}$. Let K be the closed set $\{1, 2, 3, 4, \cdots\}$. Then $0 \notin K$, but no function in rI can separate 0 from K.

4.5 **Remark.** The isomorphic image $r\overline{I}$ of rI under the natural isomorphism of $r(I^u)$ onto $C(X(I))$ (see [4, 5.4]) determines the topology on $X(I)$.

Proof. Since $\Delta\overline{I} = \{F\}$, we have the ambiguous case described above. Let $x \in X(I)$ and U an open set in $X(I)$, with $x \in U$. If x is different from the point F, then $x \in X\backslash(F \cap X)$, and hence there is an $f \in I$ with $f(x) \neq 0$. Let $g \in C(X(I))$ with $g(x) \neq 0$ and $g(\sim U) = 0$. Then $\overline{f} \cdot g$ separates x from U. If x is F itself, then $U = \{F\} \cup [V \cap (X\backslash F)]$ where V is open in βX and $V \supseteq F$. By the normality of βX, we can find an open set W and a function $\hat{h} \in C(\beta X)$ so that $F \subseteq W \subseteq \mathrm{cl}_{\beta X} W \subseteq V$ and $\hat{h}(W) = 0$ and $\hat{h}(\sim V) = 1$. Also, there exists $\hat{k} \in C(\beta X)$ with $\hat{k}(F) = 0$ and $\hat{k}(W) = 1$, and hence $\hat{h} = \hat{h} \cdot \hat{k}$. As usual we denote the restrictions of $\hat{h}$ and $\hat{k}$ to X by h and k respectively. We consider the functions $\overline{h}$, $\overline{k}$ on the space $X(I)$. It is easily seen that $\overline{h} = \overline{h} \cdot \overline{k}$ and that $\overline{k}(F) = 0$. Hence $\overline{h} \in mF(I)$, whence $h \in I$. Since $\overline{h}(F) = 0$ and $\overline{h}(\sim U) = 1$, the result is established.

4.6 **Remark.** $r\overline{I}$ is closed under inversion in $C(X(I))$. (The proof is the same as Remark 4.1.) If $r\overline{I}$ is uniformly closed, then $r\overline{I} = C(X(I))$ by 2.13, and only in this case can $r\overline{I}$ be an algebra on $X(I)$.

If A is an algebra on X, then the structure space of maximal ideals μA (denoted by $H(A^*)$ in [2] and [3]) is a compactification of X. Among its properties are

(i) $A \cap C^*(X) = C(\mu A)$;

(ii) $A = E(\mu A)$ = the ring of continuous functions from μA into the two-point compactifications of the reals. (See [2, 1.2].)

Of course, results such as the above do not hold in general for the algebras considered here.

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