TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 190, 1974

FINITE EXTENSIONS OF MINIMAL TRANSFORMATION GROUPS[1]

BY

ROBERT J. SACKER AND GEORGE R. SELL

ABSTRACT. In this paper we shall study homomorphisms $p: W \to Y$ on minimal transformation groups. We shall prove, in the case that W and Y are metrizable, that W is a finite (N-to-1) extension of Y if and only if W is an N-fold covering space of Y and p is a covering map. This result places no further restrictions on the acting group. We shall then use this characterization to investigate the question of lifting an equicontinuous structure from Y to W. We show that, under very weak restrictions on the acting group, this lifting is always possible when W is a finite extension of Y.

1. Introduction. Let W be a compact metric space and T a topological group. Then (W, T, π) is said to be a compact transformation group if $\pi: W \times T \to W$ is a continuous map satisfying $\pi(w, e) = w$ and $\pi(\pi(w, s), t) = \pi(w, st)$, where e is the identity element of T.

If (Y, T, σ) is another transformation group then a mapping $p: W \to Y$ is said to be a *homomorphism* if p is a continuous map of W onto Y and $\sigma(p(w), t) = p(\pi(w, t))$ for $(w, t) \in W \times T$. Hereafter $\pi(w, t)$ or $\sigma(y, t)$ will be denoted by $w \cdot t$ or $y \cdot t$, and for $E \subset W$ and $A \subset T$, $E \cdot A$ denotes $\bigcup\{w \cdot t: w \in E, t \in A\}$. Recall that (W, T, π) is minimal if $\overline{w \cdot T} = W$ for all $w \in W$. The transformation group (W, T, π) is said to be a *finite extension* of (Y, T, σ) if there exists a homomorphism $p: W \to Y$ and an integer $0 < N < \infty$ such that $\operatorname{card} p^{-1}(y) = N$ for all $y \in Y$. (Some authors refer to a finite extension as an N-to-1 extension to emphasize that $\operatorname{card} p^{-1}(y)$ is constant.)

Our first theorem is a structure theorem which characterizes a finite extension of a compact minimal transformation group as a compact transformation group with p a covering projection. This characterization is valid with no further restriction on T. In particular it does not involve the particular topology on T.

Received by the editors April 27, 1973.

AMS (MOS) subject classifications (1970). Primary 34C35, 54H15, 54H20, 57E99; Secondary 34C25.

Key words and phrases. Transformation groups, finite extensions, covering space, equicontinuity, distal.

(1) This research was begun while visiting at the Istituto di Matematica dell' Università di Firenze under auspices of the Italian Research Council (C. N. R.); the first author was partially supported by U. S. Army Grant DA-ARO-D-31-124-71-G176; the second author was partially supported by NSF Grant No. GP-38955.

The next question we study is whether an equicontinuous structure on (Y, T, σ) can be lifted to (W, T, π) whenever the latter is a finite extension of the former. We given an affirmative answer, in our second theorem, provided that T has the property that there exists a compact subset $K \subset T$ such that T can be generated by any open neighborhood of K. The collection $\mathcal{T}$ of such topological groups is quite large and contains, in particular, all connected groups, all compactly generated groups, all groups generated by an arbitrary neighborhood of the identity, and furthermore $\mathcal{T}$ is closed under the formation of arbitrary products of its elements using the standard product topology.

It is important to note that the second theorem places no further restriction on the spaces W and Y, and therefore partially solves a problem posed by R. Ellis [1, p. 56]. It is stated in [1] that if W is assumed to be locally connected then the lifting of the equicontinuity can be obtained without any restrictions on T. We include a proof of this fact (Theorem 3) for the sake of completeness.

Our investigation was motivated by the study of almost periodic ordinary differential equations where the space W is a subset of a product $X \times Y$ and the homomorphism p becomes the natural projection onto Y. Application of the results presented here will be carried out in a separate paper [2] where it will also be shown that the metric structure on Y can be replaced by a uniform structure.

Some questions concerning the lifting of dynamical properties are discussed in [3], [4] and [5].

Throughout, we denote by d and d_Y the metrics on W and Y respectively.

We would like to thank Professor Leonard Shapiro for some helpful comments and suggestions.

2. Finite extensions. A homomorphism $p: W \to Y$ is said to be of *distal type* if whenever $w_1, w_2 \in p^{-1}(y)$ and $w_1 \neq w_2$, then there is an $\alpha = \alpha(w_1, w_2) > 0$ such that $d(w_1 \cdot t, w_2 \cdot t) \geq \alpha$ *for all* $t \in T$. If $(T, \succ)$ is a directed set [6] then we say that p is of *positive distal type* if the previous statement holds with "for all $t \succ e$" where e is the identity of T. If $p: W \to Y$ and $0 < N < \infty$ then W is said to be an *N-fold covering* of Y with covering projection p if $\operatorname{card} p^{-1}(y) = N$ for all $y \in Y$ and for each $y \in Y$ there is an open neighborhood V of y such that $p^{-1}(V)$ consists of N disjoint open sets U_i and $(p \mid U_i): U_i \to V$ is a homeomorphism, $i = 1, \cdots, N$.

Theorem 1 (Structure theorem). *Let (W, T, π) and (Y, T, σ) be transformation groups where W and Y are compact metric spaces. Assume (Y, T, σ) is minimal and $p: W \to Y$ a homomorphism. Then the following three statements are equivalent*:

(A) *p is of distal type and for some $y_0 \in Y$*, $\operatorname{card} p^{-1}(y_0) = N$, $0 < N < \infty$.

(B) card $p^{-1}(y) = N$ *for all* $y \in Y$ *for some* N, $0 < N < \infty$.

(C) W *is an* N*-fold covering of* Y.

Moreover, if $(T, \succ)$ *is a directed set then* (A) *may be replaced by*

(A′) p *is of positive distal type and, for some* $y_0 \in Y$, card $p^{-1}(y_0) = N$, $0 < N < \infty$, *that is,* (A) *and* (A′) *are equivalent.*

Finally if any of these hold then W *can be expressed as the disjoint union* $W = W_1 \cup \cdots \cup W_k$ *of compact minimal sets, where each* W_i *is an* n_i*-fold covering of* Y *and* $n_1 + \cdots + n_k = N$.

Proof. (C) $\Rightarrow$ (B) is obvious.

(B) $\Rightarrow$ (A) For each $\alpha > 0$ define

$$J_\alpha = \{y \in Y: \text{if } w_1, w_2 \in p^{-1}(y) \text{ and } w_1 \neq w_2 \text{ then } d(w_1, w_2) \geq \alpha\}.$$

We claim J_α is closed in Y. Indeed if $y_n \to y$, $y_n \in J_\alpha$ and if $w_{1n}, w_{2n} \in p^{-1}(y_n)$ with $w_{1n} \neq w_{2n}$ then $d(w_{1n}, w_{2n}) \geq \alpha$. By choosing a subsequence we may assume that $w_{1n} \to \hat{w}_1$ and $w_{2n} \to \hat{w}_2$. By continuity one has $d(\hat{w}_1, \hat{w}_2) \geq \alpha$ and clearly $\hat{w}_1, \hat{w}_2 \in p^{-1}(y)$. It follows then that the N sequences of points in the fibers over $p^{-1}(y_n)$ (assuming some fixed labeling of points in each fiber) contain subsequences that converge to N distinct points $\{\hat{w}_1, \cdots, \hat{w}_N\} \subset p^{-1}(y)$ with $d(\hat{w}_i, \hat{w}_j) \geq \alpha$ for $i \neq j$. Since these N points exhaust the fiber $p^{-1}(y)$ we have $y \in J_\alpha$, that is, J_α is closed.

Let $\alpha_n > 0$ be any sequence with $\alpha_n \to 0$, and note that $\alpha \leq \alpha'$ implies $J_\alpha \supset J_{\alpha'}$. Clearly then $\bigcup_{\alpha > 0} J_\alpha = \bigcup_{n=1}^\infty J_{\alpha_n} = Y$. Since Y is complete, Baire's theorem asserts that some J_α contains a nonempty open subset $U \subset Y$. We then claim that for some β, $0 < \beta < \alpha$, we have $J_\beta = Y$. To see this, fix a $\tau \in T$ and note that the homeomorphism $\pi_{\tau^{-1}}: W \to W$ given by $w \to w \cdot \tau^{-1}$ is uniformly continuous, i.e., for each $\xi > 0$ there is an $\eta = \eta(\xi, \tau) > 0$ such that if $d(w_1, w_2) \geq \xi$ then $d(w_1 \cdot \tau, w_2 \cdot \tau) \geq \eta$. If we set $\xi = \alpha$ and restrict w_1, w_2 to $p^{-1}(y)$ where $y \in U$ then we see that if $\tilde{y} \in U \cdot \tau$ then $\tilde{y} \in J_{\eta(\alpha,\tau)}$. Since Y is minimal the sets $\{U \cdot \tau: \tau \in T\}$ form an open cover and by compactness we can express $Y = U \cdot \tau_1 \cup \cdots \cup U \cdot \tau_k$. If we set $\beta = \min\{\eta(\alpha, \tau_1), \cdots, \eta(\alpha, \tau_k)\}$ then it follows that $J_\beta \supset J_{\eta(\alpha,\tau_j)} \supset U \cdot \tau_j$; $j = 1, \cdots, k$, that is, $J_\beta = Y$.

Remark 1. We have shown that if (B) holds then for all $y \in Y$ and any $w_1, w_2 \in p^{-1}(y)$ with $w_1 \neq w_2$ one has $d(w_1 \cdot t, w_2 \cdot t) \geq \beta$ for all t, i.e., that p is of distal type and moreover, in the definition of distal type $\alpha(w_1, w_2)$ can be chosen uniformly equal to β.

Remark 2. We have also shown that if (B) holds then for $y \in Y$ and $y_n \to y$, one has $\overline{\lim}_{n\to\infty} p^{-1}(y_n) = p^{-1}(y)$, that is, p is an open mapping.

(B) $\Rightarrow$ (C) Using Remark 1, let $\rho > 0$ be chosen so that $d(w_1, w_2) \geq 3\rho$ for $w_1, w_2 \in p^{-1}(y)$ where $y \in Y$ is arbitrary and $w_1 \neq w_2$. Let $\delta = \delta(\epsilon)$ be the modulus of uniform continuity of p and assume, without loss of generality, that $\delta(\epsilon) \leq \rho$ for all $\epsilon > 0$. Then for any $\epsilon > 0$ and any $y \in Y$ the open sets $B_\delta(w_i)$, $i = 1, \cdots, N$, where $\{w_1, \cdots, w_N\} = p^{-1}(y)$ are disjoint and $p(B_\delta(w_i)) \subset B_\epsilon(y)$. By our choice of ρ the restriction mapping $p_i = p \mid \overline{B_\delta}(w_i)$ is one-to-one and hence p_i: $B_\delta(w_i) \to \text{Image}\, p_i$ is a homeomorphism. Since p is open (Remark 2) it follows that for each i there is an open subset V_i of Y such that $y \in V_i \subset \text{Image}\, p_i$.

Define $V = \bigcap_{i=1}^{N} V_i$ and $U_i = p^{-1}(V) \cap B_\delta(w_i)$. Then clearly $p^{-1}(V)$ is the disjoint union of the U_i and $(p \mid U_i) = (p_i \mid U_i)$: $U_i \to V$ is a homeomorphism thus establishing that W is an N-fold covering of Y.

(A) $\Rightarrow$ (B) Let $y \in Y$. We will show that $\text{card}\, p^{-1}(y) \geq N$. Since Y is minimal $y_0 \cdot T$ is dense in Y and therefore there is a sequence t_n such that $y_0 \cdot t_n \to y$. Let $p^{-1}(y_0) = \{w_1, \cdots, w_N\}$. By the distal property there is an $\alpha > 0$ such that $d(w_i \cdot t, w_j \cdot t) \geq \alpha$ for all $t \in T$ provided $i \neq j$. By choosing subsequences, we may assume that each sequence $w_1 \cdot t_n, \cdots, w_N \cdot t_n$ converges to limits $\hat{w}_1, \cdots, \hat{w}_N$ in $p^{-1}(y)$. By continuity one has $d(\hat{w}_i, \hat{w}_j) \geq \alpha$ for $i \neq j$ and therefore $\text{card}\, p^{-1}(y) \geq N$. Now if $\text{card}\, p^{-1}(y) \geq N + 1$ we use the fact that $y \cdot T$ is dense in Y and the same argument to show that $\text{card}\, p^{-1}(y_0) \geq N + 1$, a contradiction. Therefore $\text{card}\, p^{-1}(y) = N$.

We have thus proved the equivalence of (A), (B) and (C). Now clearly (A) $\Rightarrow$ (A$'$). To prove the reverse implication we need only show that (A$'$) $\Rightarrow$ (B). Examining the proof of (A) $\Rightarrow$ (B) we see that it suffices to show that for all $y \in Y$, the semitrajectory $y \cdot S_e$ is dense in Y where, for $\lambda \in T$, $S_\lambda = \{t \in T: t \succ \lambda\}$. We define $\omega_y = \bigcap_{\alpha > \alpha_0} \overline{y \cdot S_\alpha}$. It can easily be shown that ω_y does not depend on the choice of α_0 and further that ω_y is invariant; $\omega_y \cdot T = \omega_y$. From this and the minimality of Y it follows that $\omega_y = Y$. However $\overline{y \cdot S_e} \supset \omega_y$, that is, $y \cdot S_e$ is dense in Y.

To prove the last statement of the theorem we express $W = W_1 \cup \cdots \cup W_k \cup E$ where the W_j are all the minimal subsets and $E \cap W_j = \emptyset$. To see that the number of minimal sets is finite in number we define $p_j = p \mid W_j$ and note that each p_j: $W_j \to Y$ maps W_j onto Y, and therefore $\text{card}\, p_j^{-1}(y) \geq 1$ for each $y \in Y$. Hence $k \leq N$. Now let $y_0 \in Y$ and define $n_j = \text{card}\, p_j^{-1}(y_0)$. Then $n_j \leq N$ and applying the first part of the theorem to the transformation groups (W_j, T, π), (Y, T, σ) we see that (A) is satisfied and therefore W_j is an n_j-fold covering of Y with covering map p_j.

If $E = \emptyset$ then we see immediately that $n_1 + \cdots + n_k = N$ and the theorem is completely proved. Arguing negatively, let $w \in E$. Since $\overline{w \cdot T}$ is nonempty and invariant we see that $\overline{w \cdot T} \supset W_j$ for some fixed j, $1 \leq j \leq k$. Let $t_n \in T$ be a sequence such that $w \cdot t_n \to w_0 \in W_j$. Let $y_0 = p(w_0)$ and $y = p(w)$, then $y \cdot t_n \to y_0$. Define $p_j = p \mid W_j$. Let $\beta > 0$ be given by Remark 1 and choose $U \subset W$ to be open such that $w_0 \in U$, diam $U < \beta$ and

$$p_j\colon U \cap W_j \to p_j(U \cap W_j) = V$$

is a homeomorphism. Let n be fixed and sufficiently large so that $y \cdot t_n \in V$ and $w' = w \cdot t_n \in U$. Define $x = p_j^{-1} \circ p(w')$. Then $x \in U \cap W_j$ and therefore $d(x, w') < \beta$. Since E and W_j are invariant, $x \neq w'$. But x and w' both lie in $p^{-1}(y \cdot t_n)$ and Remark 1 implies that $d(x, w') \geq \beta$, a contradiction. Thus the theorem is proved.

Remark 3. The fact that the space W is metrizable was used in a crucial way for the implications (B) $\Rightarrow$ (C) and (B) $\Rightarrow$ (A). The other implications would be valid when both W and Y are compact Hausdorff (hence uniform) spaces. Example 4 in [7] shows that the implication (B) $\Rightarrow$ (A) is false when W is not metrizable. The notion of distal type is also discussed by J. Auslander [8] where, in fact, it is shown that if p is of distal type, then p is open and $p^{-1}(y)$ has the same cardinality for all y.

3. **Equicontinuous transformation groups.** A transformation group (W, T, π) is *equicontinuous* if for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that $d(w_1 \cdot t, w_2 \cdot t) < \epsilon$ for all $t \in T$ whenever $d(w_1, w_2) < \delta$. This is equivalent to *uniformly distal* which means that for each $\xi > 0$ there is an $\eta = \eta(\xi) > 0$ such that $d(w_1 \cdot t, w_2 \cdot t) \geq \eta$ for all $t \in T$ whenever $d(w_1, w_2) \geq \xi$. We shall call $\eta(\xi)$ the *modulus of expansion* of the equicontinuous transformation group (W, T, π).

In this section we consider two compact transformation groups (W, T, π) and (Y, π, σ) and a mapping $p\colon W \to Y$ which is an N-fold covering homomorphism. Recall that from Remark 1 there exists a $\rho > 0$ such that for all $w_1, w_2 \in p^{-1}(y)$, where $y \in Y$ is arbitrary, one has $d(w_1, w_2) \geq 3\rho$ whenever $w_1 \neq w_2$. We fix ρ for the remainder of the discussion. For any $\alpha > 0$ and $w \in W$ we define $U_\alpha(w)$ to be the α-neighborhood of the fiber $p^{-1} \circ p(w)$. Clearly then $U_\alpha(w) = U_\alpha(\hat{w})$ whenever w and $\hat{w}$ belong to the same fiber. By $U_\alpha^c(w)$ we denote the complement in W. If $\alpha \leq \rho$ then $U_\alpha(w)$ consists of N disjoint balls $U_\alpha(w) = B_\alpha(w_1) \cup \cdots \cup B_\alpha(w_N)$ where $w \in \{w_1, \cdots, w_N\} = p^{-1} \circ p(w)$.

The assumptions and notation described in the above paragraph will be a standing hypothesis throughout the remainder of this section.

Lemma 1. *If* $w_1, w_2 \in W$ *and* $0 < \alpha \leq d(w_1, w_2) \leq \rho$ *then* $w_2 \in U_\alpha^c(w_1)$.

Proof. We must show that for all $\hat{w} \in p^{-1} \circ p(w_1)$ one has $d(\hat{w}, w_2) \geq \alpha$. If $\hat{w} = w_1$, it follows from the hypotheses. If $\hat{w} \neq w_1$ then $d(\hat{w}, w_1) \geq 3\rho$ and therefore $d(w_2, \hat{w}) \geq d(w_1, \hat{w}) - d(w_1, w_2) \geq 2\rho \geq \alpha$.

Lemma 2. *For any $\alpha > 0$ there is a $\beta = \beta(\alpha) > 0$ such that for any $w \in W$ one has $d_Y(p(z), p(w)) > \beta$ whenever $z \in U_\alpha^c(w)$. (We will call $\beta(\alpha)$ the modulus of expansion of p.)*

Proof. This follows from the continuity of p. If the assertion were false, then there is an $\alpha > 0$ and sequences w_n and z_n such that $z_n \in U_\alpha^c(w_n)$ and $d_Y(p(z_n), p(w_n)) \leq 1/n$. By extracting subsequences we may assume $w_n \to w_0$ and $z_n \to z_0$. But then $z_0 \in U_\alpha^c(w_0)$ and $d_Y(p(w_0), p(z_0)) = 0$. Hence $z_0 \in p^{-1} \circ p(w_0)$, a contradiction.

Lemma 3. *Let $K \subset T$ be compact and $\mu > 0$. Then there exists $\nu = \nu(\mu, K) > 0$ and an open subset $V_\mu \subset T$ with $K \subset V_\mu$ such that $d(w_1 \cdot t, w_2 \cdot t) < \mu$ whenever $d(w_1, w_2) < \nu$ and $t \in V_\mu$. (We call $\nu(\mu, K)$ the modulus of equicontinuity of K.)*

Proof. The proof is elementary and follows from the compactness of W and K and the joint continuity of π. We omit the details.

Recall that $\mathfrak{T}$ is the collection of all topological groups T with the property that there is a compact set $K \subset T$ such that T is generated by any open neighborhood of K.

Lemma 4. *Let $T \in \mathfrak{T}$ and $K \subset T$ be a compact subset such that any open neighborhood of K generates T. Let $\nu(\mu, K)$ be the modulus of equicontinuity of K and set $\nu_0 = \nu(\rho, K)$. If $w_1, w_2 \in W$ are such that $d(w_1, w_2) < \nu$ for some ν, $0 < \nu \leq \nu_0$, then one of the following must hold:*

(A) $$d(w_1 \cdot t, w_2 \cdot t) < \nu \quad \text{for all } t \text{ in } T, \quad \text{or}$$

(B) $$w_2 \cdot t \in U_\nu^c(w_1 \cdot t) \quad \text{for some } t \text{ in } T.$$

Proof. If (A) fails then for some $t \in T$, $d(w_1 \cdot t, w_2 \cdot t) \geq \nu$. Let $V_\rho \subset T$ be the open subset guaranteed by Lemma 3. Then V_ρ generates T and thus we may express $t = \tau_1 \tau_2 \cdots \tau_k$ where $\tau_i \in V_\rho$. Define t_n, $n = 0, \cdots, k$, by $t_0 = e$, $t_n = \tau_1 \tau_2 \cdots \tau_n$. The t_n are ordered by their subscripts and we can assume without loss of generality that $d(w_1 \cdot t_n, w_2 \cdot t_n) < \nu \leq \nu_0$, for $n = 0, 1, \cdots, k-1$. Since $w \cdot t_k = (w \cdot t_{k-1}) \cdot \tau_k$ it follows from the definition of the modulus of continuity of K that $d(w_1 \cdot t_k, w_2 \cdot t_k) < \rho$. Since $d(w_1 \cdot t_k, w_2 \cdot t_k) \geq \nu$, by assumption, it follows from Lemma 1 that $w_2 \cdot t_k \in U_\nu^c(w_1 \cdot t_k)$, and the proof is complete.

Theorem 2 (**Equicontinuous lifting theorem**). *Let* (W, T, π) *and* (Y, T, σ) *be compact transformation groups satisfying the following properties:*

(A) *There is a homomorphism* $p: W \to Y$ *of* W *onto* Y.

(B) W *is an* N*-fold covering of* Y *with covering map* p.

(C) $T \in \mathcal{T}$.

Then (W, T, π) *is equicontinuous if and only if* (Y, T, σ) *is equicontinuous.*

Proof. The implication

$$(W, T, \pi) \text{ equicontinuous} \Rightarrow (Y, T, \sigma) \text{ equicontinuous}$$

is known. However we shall incorporate a proof for the sake of completeness. This part of the argument will not use hypotheses (B) and (C).

First we observe that (i) (W, T, π) is equicontinuous if and only if (ii) for any sequence $\{t_n\}$ there is a subsequence $\{r_n\}$ such that the sequence $\{\pi_{r_n}\}$, where $\pi_r: W \to W$ is the mapping $\pi_r(w) = \pi(w, r)$, converges uniformly to some (necessarily) continuous function π^*. Also statement (ii) holds if and only if (iii) for every convergent sequence $\{w_n\}$ in W one has $\pi_{r_n}(w_n) \to \pi^*(w_0)$ where $w_n \to w_0$.

Now let $\{t_n\}$ be any sequence in T and choose a subsequence $\{r_n\}$ so that $\{\pi_{r_n}\}$ converges uniformly to π^*. Define σ^* by

$$\sigma^*(y) = p \circ \pi^* \circ p^{-1}(y).$$

We will now show that σ^* is well defined and that for any convergent sequence y_n in Y one has $\sigma_{r_n}(y_n) \to \sigma^*(y_0)$ where $y_n \to y_0$.

The fact that σ^* is well defined follows from the assumption that p is a homomorphism. Indeed, if $w, \hat{w} \in p^{-1}(y)$, then for all t, $p \circ \pi_t(w) = p \circ \pi_t(\hat{w})$. Now if we set $t = r_n$ and take limits we get $p \circ \pi^*(w) = p \circ \pi^*(\hat{w})$. (This also shows that $\sigma_t(y) = p \circ \pi_t \circ p^{-1}(y)$ for all t in T.)

Now if $\{y_n\}$ is a convergent sequence in Y, then we can choose a convergent subsequence $w_n \in p^{-1}(y_n)$ so that $w_n \to w_0$ and $y_n \to y_0$. Moreover $w_0 \in p^{-1}(y_0)$ and $\pi_{r_n}(w_n) \to \pi^*(w_0)$. Hence $p(\pi_{r_n}(w_n)) \to p(\pi^*(w_0))$, in other words, $\sigma_{r_n}(y_n) \to \sigma^*(y_0)$.

Now assume (Y, T, σ) is equicontinuous. Thus it is uniformly distal and we let $\eta(\xi)$ be its modulus of expansion. Let $\beta(\alpha)$ be the modulus of expansion of the mapping $p: W \to Y$ guaranteed by Lemma 2 and finally let $\delta(\epsilon)$ be the modulus of uniform continuity of p. Next let $K \subset T$ be a compact subset with the property that T is generated by any open neighborhood of K, and let $\nu(\mu, K)$ be the modulus of equicontinuity of K. Without any loss of generality we may assume that $\delta(\epsilon) \leq \rho$ for all $\epsilon > 0$. Let $\nu_0 = \nu(\rho, K)$ and for $0 < \nu \leq \nu_0$ define $\gamma(\nu) = \delta(\eta(\beta(\nu)))$. We will now show that (W, T, π) is uniformly distal by showing that $\gamma(\nu)$ is its modulus of expansion, i.e., that if $w_1, w_2 \in W$ with $d(w_1, w_2) \geq \nu$, then for all $t \in T$

$$d(w_1 \cdot t, w_2 \cdot t) \geq \gamma(\nu). \tag{1}$$

Case 1. Assume $w_2 \cdot t \in U_\nu^c(w_1 \cdot t)$ for some $t \in T$: In this case define $z_i = w_i \cdot t$ and note that Lemma 2 implies $d_Y(p(z_1), p(z_2)) > \beta(\nu)$. But since (Y, T, σ) is uniformly distal, we have, for all $t \in T$,

$$d_Y(p(z_1) \cdot t, p(z_2) \cdot t) \geq \eta(\beta(\nu)). \tag{2}$$

Now if (1) fails for some $t' \in T$ then $d(w_1 \cdot t', w_2 \cdot t') < \gamma(\nu)$, that is, for $\hat{t} = t^{-1}t'$, one has $d(z_1 \cdot \hat{t}, z_2 \cdot \hat{t}) < \gamma(\nu)$. From the uniform continuity of p we have

$$d_Y(p(z_1) \cdot \hat{t}, p(z_2) \cdot \hat{t}) = d_Y(p(z_1 \cdot \hat{t}), p(z_2 \cdot \hat{t})) < \eta(\beta(\nu))$$

which violates (2).

Case 2. Assume $w_2 \cdot t \in U_\nu(w_1 \cdot t)$ for all $t \in T$: Then in particular $w_2 \in U_\nu(w_1)$ and therefore there exists $\hat{w}_1 \in p^{-1} \circ p(w_1)$ such that $d(w_2, \hat{w}_1) < \nu$. Then from Lemma 4 *either* $d(w_2 \cdot t, \hat{w}_1 \cdot t) < \nu$ for all $t \in T$ *or* $w_2 \cdot t \in U_\nu^c(\hat{w}_1 \cdot t)$ for some $t \in T$. But $U_\nu^c(\hat{w}_1 \cdot t) = U_\nu^c(w_1 \cdot t)$ and therefore the second possibility is ruled out since it violates the premise of Case 2. Setting $t = e$ we see that $d(w_2, \hat{w}_1) < \nu$ and since $d(w_1, w_2) \geq \nu$ we see that $w_1 \neq \hat{w}_1$. Since w_1 and $\hat{w}_1$ are in the same fiber, $d(w_1 \cdot t, \hat{w}_1 \cdot t) \geq 3\rho$ for all $t \in T$. Thus

$$\begin{aligned} d(w_1 \cdot t, w_2 \cdot t) &\geq d(w_1 \cdot t, \hat{w}_1 \cdot t) - d(\hat{w}_1 \cdot t, w_2 \cdot t) \\ &\geq 3\rho - \nu \geq 2\rho > \delta(\eta(\beta(\nu))) = \gamma(\nu) \end{aligned}$$

which establishes (1) and the theorem is proved.

We next show that the restriction on T may be dropped if W is locally connected.

Theorem 3. *Theorem* 2 *still holds if hypothesis* (C) *is replaced by* "*W is locally connected.*"

Proof. In proving that (W, T, π) is equicontinuous it suffices, since W is compact, to prove it pointwise, that is, given $w \in W$ and $\epsilon > 0$ there is a

$\delta = \delta(w, \epsilon) > 0$ such that $d(w', w) < \delta$ implies $d(w' \cdot t, w \cdot t) < \epsilon$ for all $t \in T$. Also by compactness of W there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$ and all $z \in W$, $p \mid B_\epsilon(z) = p_0$ is a homeomorphism and $p^{-1} \circ p_0(B_\epsilon(z))$ consists of N disjoint open sets $U_1 = B_\epsilon(z), U_2, \cdots, U_N$ where the restriction mapping $p \mid U_j$ is a homeomorphism.

Now let $w \in W$, $t \in T$ and ϵ, $0 < \epsilon \leq \epsilon_0$, be given. By Lemma 2 there is an $\epsilon' > 0$ such that $p(B_\epsilon(w \cdot t)) \supset B_{\epsilon'}(y \cdot t)$ where $y = p(w)$. By equicontinuity of (Y, T, σ) there is a $\delta' = \delta'(\epsilon') > 0$ such that $B_{\delta'}(y) \cdot t \subset B_{\epsilon'}(y \cdot t)$. Now let $V \subset W$ be an open connected subset such that $w \in V$ and $p(V) \subset B_{\delta'}(y)$. Then

$$V \cdot t \subset p^{-1}(p(V) \cdot t) \subset p^{-1}(B_{\epsilon'}(y \cdot t)) \subset U_1 \cup \cdots \cup U_N$$

while $w \cdot t \in U_1$. Since $V \cdot t$ is connected we have $V \cdot t \subset U_1 = B_\epsilon(w \cdot t)$. Finally let $\delta > 0$ be chosen so that $B_\delta(w) \subset V$. It is clear then that δ depends only on w and ϵ and not on t. Furthermore, if $d(w, w') < \delta$, then $w' \in V$ and $w' \cdot t \in V \cdot t \subset B_\epsilon(w \cdot t)$, or $d(w \cdot t, w' \cdot t) < \epsilon$, which completes the proof.

Remark 4. The results in this section are valid when W and Y are compact Hausdorff spaces provided the obvious changes in the argument are made.

Remark 5. If Y is a compact minimal transformation group and p is a finite extension, then hypothesis (B) of Theorem 2 is automatically satisfied as we have shown in Theorem 1. Therefore in the case that the spaces W and Y are metrizable and the acting group T belongs to the class $\mathfrak{T}$, we have presented a solution of a problem posed by R. Ellis [1, p. 56].

Remark 6. The equicontinuity of the family of mappings $\{\sigma_t : t \in T\}$, where $\sigma_t(y) = \sigma(y, t)$, does not depend on the topology on T. Therefore in order to apply Theorem 2 one need only find a topology on T so that (1) σ is continuous and that (2) $T \in \mathfrak{T}$.

BIBLIOGRAPHY

1. R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969. MR 42 #2463.

2. R. J. Sacker and G. R. Sell, *Skew-product flows and almost periodic ordinary differential equations*, Bull. Amer. Math. Soc. 79 (1973), 802–805.

3. N. G. Markley, *Lifting dynamical properties*, Math. Systems Theory 5 (1971), 299–305.

4. F. Rhodes, *On lifting transformation groups*, Proc. Amer. Math. Soc. 19 (1968), 905–908. MR 38 #5196.

5. R. Ellis, *The construction of minimal discrete flows*, Amer. J. Math. 87 (1965), 564–574. MR 32 #3053.

6. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR 33 #1824.

7. R. Ellis and W. H. Gottschalk, *Homomorphisms of transformation groups*, Trans. Amer. Math. Soc. 94 (1960), 258–271. MR 23 #A960.

8. J. Auslander, *Homomorphisms of minimal transformation groups*, Topology 9 (1970), 195–203. MR 42 #2455.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90007 (Current address of Robert J. Sacker)

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455