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GLOBAL DIMENSION OF TILED ORDERS OVER COMMUTATIVE NOETHERIAN DOMAINS

BY

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ABSTRACT. Let R be a commutative noetherian domain and $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order. The main result of this paper is the following

Theorem. *Let* gl dim $R = d < \infty$ *and* Λ *a triangular tiled R-order (i.e.,* $\Lambda_{ij} = R$ *whenever* $i \leq j$*). Then the following three conditions are equivalent:*

(1) gl dim $\Lambda < \infty$;

(2) $\Lambda_{i,i-1} = R$ *or* gl dim $(R/\Lambda_{i,i-1}) < \infty$, *whenever* $2 \leq i \leq n$;

(3) gl dim $\Lambda \leq d(n-1)$.

If $d = 1$ or 2 then the upper bound in the above theorem is best possible. We give a sufficient condition for an arbitrary tiled R-order Λ to be of finite global dimension.

Introduction. Throughout this paper R denotes a commutative noetherian domain with quotient field K. As usual, an R-order Λ in the $n \times n$ matrix ring $M_n(K)$ is an R-subalgebra of $M_n(K)$ which is finitely generated as an R-module and spans $M_n(K)$ over K. The object of this paper is to establish an intimate connection between the finiteness of global dimension of Λ and the structure of Λ, in case Λ is a tiled R-order contained in $M_n(R)$.

An R-order Λ in $M_n(K)$ is *tiled* if it contains n orthogonal idempotents. Clearly Λ is left as well as right noetherian. A certain amount of normalization is possible. Thus, by conjugating if necessary, we may assume that Λ contains the idempotents e_{ii}, $1 \leq i \leq n$, where $\{e_{ij}: 1 \leq i, j \leq n\}$ is the system of usual matrix units in $M_n(K)$. Evidently, the (i, j)th entries of all elements of Λ form a nonzero fractional R-ideal of K, which we shall denote as Λ_{ij}. It is equally clear that $R \subseteq \Lambda_{ii}$, $\Lambda_{ij}\Lambda_{jk} \subseteq \Lambda_{ik}$ and $\Lambda = (\Lambda_{ij})$. Note that if $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$, then all Λ_{ij} are integral and $\Lambda_{ii} = R$.

The present author ([8], [9]) and independently R. B. Tarsey [18] have shown

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that if R is a discrete valuation ring (DVR) with maximal ideal $\mathfrak{m}$ and if $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ is a triangular tiled R-order (i.e., $\Lambda_{ij} = R$ whenever $i \leq j$), then gl dim $\Lambda < \infty$ if and only if $\mathfrak{m} \subseteq \Lambda_{i,i-1}$ for all i. The present author ([8], [9]) gave a sharp upper bound, viz., $n-1$, on their global dimension. The main result (Theorem 3.6) shows that these results can be retained in a suitably modified form even when R is an arbitrary noetherian domain of finite global dimension. The sufficient condition for finiteness of gl dim Λ, obtained in §2, is of independent interest (Theorem 2.7).

Several results from this paper will be needed in the sequel in which we investigate global dimension of arbitrary tiled orders over a DVR.

We now state some known results which will be needed in the following sections.

Lemma 0.1. *If S is a ring and S is not semisimple, then*

$$\mathrm{r\ gl\ dim}\ S = 1 + \sup_I \{\mathrm{hd}_S I\},$$

where the supremum is taken over all right ideals of S.

Proof. See [14, p. 178, Theorem 9.14].

Lemma 0.2. *Let S be any commutative ring, Λ a right noetherian S-algebra. Then*

$$\mathrm{r\ gl\ dim}\ \Lambda = \sup_{\mathfrak{m}} \mathrm{r\ gl\ dim}(\Lambda \otimes_S S_{\mathfrak{m}}),$$

where $\mathfrak{m}$ runs through all maximal ideals of S.

Proof. See [1, Corollary to Proposition 2.6].

Lemma 0.3. *Let S be any ring, I a two-sided ideal which is projective as a left S-module, and $I^n = I^{n+1}$ for some integer $n \geq 1$. If* l gl dim $S < \infty$, *then* l gl dim $(S/I) < \infty$.

Proof. See [4, Theorem 1].

Lemma 0.4. *If S is a right noetherian ring and I is any two-sided ideal contained in the Jacobson radical $J(S)$ of S, then* r gl dim $S \leq \mathrm{lwd}_S(S/I) +$ r gl dim (S/I), *where* lwd *denotes left weak dimension.*

Proof. [16, Theorem 1].

Lemma 0.5. *Let S be any ring and e an idempotent in S. If* $\mathrm{lwd}_{eSe} eS = 0$, *then for every right eSe-module N, we have* $\mathrm{hd}_S(N \otimes_{eSe} eS) = \mathrm{hd}_{eSe} N$; *furthermore,* r gl dim $eSe \leq$ r gl dim S.

Proof. See [5, Proposition 15, Theorem 7].

Lemma 0.6. *Let S be any ring. Let*

$$0 \to B \to A \to C \to 0$$

be a short exact sequence of right S-modules. Then:

(1) *If two of the dimensions* $\mathrm{hd}_S A$, $\mathrm{hd}_S B$, $\mathrm{hd}_S C$, *are finite, then so is the third.*

(2) *If* $\mathrm{hd}_S A \gneq \mathrm{hd}_S B$, *then* $\mathrm{hd}_S C = \mathrm{hd}_S A$.

(3) *If* $\mathrm{hd}_S A \lneq \mathrm{hd}_S B$, *then* $\mathrm{hd}_S C = 1 + \mathrm{hd}_S B$.

(4) *If* $\mathrm{hd}_S A = \mathrm{hd}_S B$, *then* $\mathrm{hd}_S C \leq 1 + \mathrm{hd}_S B$.

Proof. See [11, p. 169, Theorem 2].

Lemma 0.7. *Let S be a ring and x a central nonzero divisor in S. Write $S^* = S/(x)$. Let A be a nonzero S^*-module with* $\mathrm{hd}_{S^*} A < \infty$. *Then* $\mathrm{hd}_S A = \mathrm{hd}_{S^*} A + 1$.

Proof. See [11, p. 172, Theorem 3].

Lemma 0.8. *Let S be a right noetherian ring and x a central element in the Jacobson radical $J(S)$ of S. Let A be a finitely generated right S-module. If x is a nonzero divisor on both S and A, then* $\mathrm{hd}_{S^*}(A/Ax) = \mathrm{hd}_S A$, *where* $S^* = S/(x)$.

Proof. See [11, p. 178, Theorem 9].

Lemma 0.9. *Let R be a regular local ring with maximal ideal $\mathfrak{m}$. Let I be a nonzero proper ideal of R. Then,* $\mathrm{gl\,dim}(R/I) < \infty$ *if and only if* $I = (x_1, x_2, \ldots, x_s)$, *where the x_i form a part of an R-sequence generating $\mathfrak{m}$.*

Proof. See [11, p. 184, Theorem 13], [19, p. 303, Theorem 26].

Lemma 0.10. *Let R be a regular local ring of dimension d. Let I be a proper nonzero ideal of R. If* $\mathrm{gl\,dim}(R/I) < \infty$, *then* $\mathrm{gl\,dim}(R/I) \leq d - 1$.

Proof. Follows easily from Lemma 0.9.

1. Preliminaries. In this paper all rings are associative and have unit element, all modules are unitary. $J(S)$ will denote the Jacobson radical of the ring S. By a principal right (left) projective of S we will mean a right (left) ideal of S generated by an indecomposable idempotent of S. If $\mathfrak{m}$ is a maximal ideal of S, then $S_{\mathfrak{m}}$ will denote the localization of S at $\mathfrak{m}$. If M is an S-module, then $\mathrm{hd}_S M$ will denote the projective dimension of M. If S is noetherian, then the right and the left global dimensions of S are equal and this common value will be denoted by gl dim S. $\mathfrak{Mod}$-S (S-$\mathfrak{Mod}$) will denote the category of right (left) S-modules. Throughout this paper R will denote a commutative noetherian domain.

Lemma 1.1. *Let S be any ring. Let M_i, $u_1 \le i \le u_2$, and N_i, $u_1 + 1 \le i \le u_2$, be two families of right S-modules. Assume that*

(1) *N_i is projective for all i,*

(2) *$N_{i+1} + M_i = M_{i+1}$ for $u_1 \le i \le u_2 - 1$,*

(3) *$N_{i+1} \cap M_i \cong M_i$ for $u_1 \le i \le u_2 - 1$,*

Then $\mathrm{hd}_S M_i \le \mathrm{hd}_S M_{u_1} + i - u_1$ for $u_1 \le i \le u_2$.

Proof. Follows easily from Lemma 0.6 and by induction.

Lemma 1.2. *Let R be a commutative, noetherian local domain and let Λ be a tiled R-order in $M_n(K)$, where K is the quotient field of R. Then* $\mathrm{gl\ dim}\ \Lambda = 1 + \mathrm{hd}_\Lambda J(\Lambda)$.

Proof. See Corollary 4.6 of [15].

The next lemma is a very handy tool in computation of the Jacobson radical $J(S)$ of the ring S containing a finite set of orthogonal idempotents with sum 1 [7].

Lemma 1.3. *Let S be any ring and let $\{e_i : 1 \le i \le n\}$ be a finite set of orthogonal idempotents in S with sum 1. Let $H_{ij} = \{e_i s e_j : e_i s e_j S e_i \subseteq J(e_i S e_i)\}$. Then $H_{ii} = J(e_i S e_i) = J(S) \cap e_i S e_i = e_i J(S) e_i$ and $J(S) = \bigoplus_{i,j} H_{ij}$.*

Proof. First we show that $\Sigma_{i,j} H_{ij} \subseteq J(S)$. Let $e_i s e_j$ be in H_{ij}. By Proposition 3 of [12, p. 57] it is enough to show that, for all r in S, $1 - e_i s e_j r$ is right invertible in S. Since $e_i s e_j S e_i \subseteq J(e_i S e_i)$, therefore by the same Proposition we have $e_i z e_i$ in $e_i S e_i$ such that

$$(e_i - e_i s e_j r e_i)(e_i - e_i z e_i) = e_i.$$

Hence $e_i s e_j r e_i z e_i = e_i s e_j r e_i + e_i z e_i$. Consequently, $(1 - e_i s e_j r e_i)(1 - e_i z e_i) = 1$. Now,

$$\begin{aligned}(1 - e_i s e_j r)(1 - e_i z e_i) &= 1 - e_i z e_i - e_i s e_j r \left(\sum_{k=1}^{n} e_k\right)(1 - e_i z e_i)\\ &= (1 - e_i z e_i) - e_i s e_j r e_i (1 - e_i z e_i) - \sum_{k \ne i} e_i s e_j r e_k (1 - e_i z e_i)\\ &= 1 - \sum_{k \ne i} e_i s e_j r e_k.\end{aligned}$$

However, since the e_i's are orthogonal, it is immediate that

$$\left(1 - \sum_{k \ne i} e_i s e_j r e_k\right)\left(1 + \sum_{k \ne i} e_i s e_j r e_k\right) = 1.$$

Hence $1 - e_i s e_j r$ is right invertible in S. Thus $\Sigma_{i,j} H_{ij} \subseteq J(S)$. Next we prove the assertion about H_{ii}. It is easy to see that $J(e_i S e_i) = H_{ii} \subseteq J(S) \cap e_i S e_i = e_i J(S) e_i$. Let $e_i s e_i$ be in $e_i J(S) e_i$ which is contained in $J(S)$. Hence there is an

r in S such that $(1 - e_i s e_i)(1 - r) = 1$; multiplying on both sides by e_i we get $(e_i - e_i s e_i)(e_i - e_i r e_i) = e_i$. Therefore $e_i J(S) e_i \subseteq J(e_i S e_i)$, by Theorem 1 of [6, p. 9].

We now finish the proof of the lemma by showing that $J(S) \subseteq \Sigma_{i,j} H_{ij}$. Let $s \in J(S)$. Then $e_i s e_j \in J(S)$ for all i, j. Hence $e_i s e_j S e_i \subseteq J(S) \cap e_i S e_i = J(e_i S e_i)$. Thus $e_i s e_j$ is in H_{ij} for all i, j. Since $\Sigma_{k=1}^n e_k = 1$, therefore $s = \Sigma_{i,j} e_i s e_j \in \Sigma_{i,j} H_{ij}$. That the sum is direct is obvious.

Corollary 1.4. *Let e be an idempotent in a ring S. Then $J(eSe) = eJ(S)e = J(S) \cap eSe$.*

Definition 1.5. Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order. Then Λ is called reduced if $\Lambda_{ij}\Lambda_{ji} \subsetneqq R$ whenever $i \neq j$; equivalently, $\Lambda_{ij} \subsetneqq R$ or $\Lambda_{ji} \subsetneqq R$ whenever $i \neq j$.

Lemma 1.6. *Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$, $\Gamma = (\Gamma_{ij}) \subseteq M_n(R)$ be two tiled R-orders. Then*

(1) *Λ is reduced if and only if $e_{ii}\Lambda \not\cong e_{jj}\Lambda$ as right Λ-modules, whenever $i \neq j$.*

(2) *If R is a local domain, Λ is reduced, and if Λ and Γ are isomorphic as rings, then Γ is also reduced.*

Proof. (1) If Λ is not reduced then $\Lambda_{ij} = R = \Lambda_{ji}$ for some $i \neq j$. Since $\Lambda_{il} \supseteq \Lambda_{ij}\Lambda_{jl} = \Lambda_{jl} \supseteq \Lambda_{ji}\Lambda_{il} = \Lambda_{il}$, therefore $\Lambda_{il} = \Lambda_{jl}$ for all $1 \leq l \leq n$. Hence $e_{ii}\Lambda \cong e_{jj}\Lambda$.

Conversely, assume that $e_{ii}\Lambda \cong e_{jj}\Lambda$ as right Λ-modules for some $i \neq j$. Let $\theta: e_{ii}\Lambda \to e_{jj}\Lambda$ be an isomorphism. Since $e_{ii}^2 = e_{ii}$, $\theta(e_{ii}) = \theta(e_{ii}^2) = \theta(e_{ii})e_{ii} \in e_{jj}\Lambda e_{ii}$, so that $\theta(e_{ii}) = ae_{ji}$ for some unit a in R. Hence we have $e_{jj}\Lambda = \theta(e_{ii})\Lambda = ae_{ji}\Lambda$. This yields $\Lambda_{jl} = \Lambda_{il}a = \Lambda_{il}$ for $1 \leq l \leq n$. In particular $\Lambda_{ji} = \Lambda_{ii} = R = \Lambda_{jj} = \Lambda_{ij}$. Thus Λ is not reduced.

(2) Since R is local, the e_{ii}'s are local idempotents. Hence, if $\theta: \Lambda \to \Gamma$ is a ring isomorphism, then by Proposition 3 of [12, p. 77] we may assume that $\theta(e_{ii}) = e_{\pi(i),\pi(i)}$, where π is a permutation of the numbers from 1 to n. If Γ is not reduced, then, by (1), we have $e_{ii}\Gamma \cong e_{jj}\Gamma$ as right Γ-modules for some $i \neq j$. From this, one at once gets that $e_{ll}\Lambda \cong e_{kk}\Lambda$ as right Λ-modules, where $l = \pi^{-1}(i)$ and $k = \pi^{-1}(j)$. Thus Λ is not reduced.

Lemma 1.7. *Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a reduced tiled R-order. Then there exist natural numbers $k, l \leq n$ such that $\Lambda_{ik} \neq R$ for all $i \neq k$ and $\Lambda_{li} \neq R$ for all $i \neq l$.*

Proof. We shall prove the existence of k. The proof for the existence of l is similar. We use induction on n. For $n = 2$, the assertion is trivial, since

$\Lambda_{12} \subsetneqq R$ or $\Lambda_{21} \subsetneqq R$. Let $n \geq 3$. If $\Lambda_{in} \neq R$ whenever $i \neq n$, then we are done. Suppose that $\Lambda_{jn} = R$ for some $j \leq n-1$. Observe that $e\Lambda e$ is also a reduced tiled R-order contained in $M_{n-1}(R)$, where $e = \Sigma_{i=1}^{n-1} e_{ii}$. Hence by the induction hypothesis there is $k \lneq n$ such that $\Lambda_{ik} \neq R$ whenever $i \neq k, n$. To complete the induction we show that $\Lambda_{nk} \neq R$. If $j = k$, then $\Lambda_{nk} = \Lambda_{nj} \neq R$, since Λ is reduced, and we are done. If $j \neq k$, then $\Lambda_{jk} \neq R$ as $j \leq n-1$. If $\Lambda_{nk} = R$, then $\Lambda_{jk} = R$, as $\Lambda_{jk} \supseteq \Lambda_{jn}\Lambda_{nk} = R$. Thus $\Lambda_{nk} \neq R$. This completes the induction and completes the proof of the lemma.

Definition 1.8. Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order. Λ is called a (super) triangular tiled R-order if $\Lambda_{ij} = R$ whenever $i \leq j$.

The next proposition gives a necessary condition for a reduced tiled R-order $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$, R a commutative noetherian local domain with quotient field K, to be conjugate to a triangular tiled R-order in $M_n(K)$.

Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order over a commutative noetherian local domain R with maximal ideal $\mathfrak{m}$. Let $\overline{\Lambda} = (\Lambda_{ij}/\Lambda_{ij}\mathfrak{m})$, where the multiplication in $\overline{\Lambda}$ is induced from that in Λ, i.e., if (λ_{ij}), $(\lambda'_{ij}) \in (\Lambda_{ij})$ then $(\lambda_{ij} + \Lambda_{ij}\mathfrak{m})(\lambda'_{ij} + \Lambda_{ij}\mathfrak{m}) = (\Sigma_{k=1}^{n} \lambda_{ik}\lambda'_{kj} + \Lambda_{ij}\mathfrak{m})$. Then $\overline{\Lambda}$ is a finite dimensional $R/\mathfrak{m}$-algebra which is naturally isomorphic with $\Lambda/\Lambda\mathfrak{m}$ as an $R/\mathfrak{m}$-algebra. Also, if Λ is reduced then, by Lemma 1.3, $J(\overline{\Lambda})$ is obtained from $\overline{\Lambda}$ be replacing the diagonal entries $R/\mathfrak{m}$ by zero. If M is a finitely generated right $\overline{\Lambda}$-module then by Exercise 29 of [4, p. 80], we have $J(M) = MJ(\overline{\Lambda})$, where $J(M)$ is the intersection of all maximal submodules of M. Let $\overline{e}_{ii}$ denote the indecomposable idempotent in $\overline{\Lambda}$ with 1 at the (i, i)th place and zero elsewhere.

Proposition 1.9. *Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a reduced tiled R-order over a commutative noetherian local domain R with maximal ideal $\mathfrak{m}$ and quotient field K. Let $\overline{P}_i = \overline{e}_{ii}\overline{\Lambda}$, $1 \leq i \leq n$. Assume that Λ is conjugate to a triangular tiled R-order $\Gamma = (\Gamma_{ij}) \subseteq M_n(R)$, i.e., $\Lambda = u\Gamma u^{-1}$ for some unit u in $M_n(K)$. Then, for some $j \leq n$,*

$$(*) \qquad \overline{P}_j \supsetneqq \overline{P}_j J(\overline{\Lambda}) \supsetneqq \overline{P}_j J^2(\overline{\Lambda}) \supsetneqq \cdots \supsetneqq \overline{P}_j J^{n-1}(\overline{\Lambda}) \supsetneqq \overline{P}_j J^n(\overline{\Lambda}) = 0$$

is a composition series of $\overline{P}_j$ considered as a right $\overline{\Lambda}$-module.

Proof. Let $\overline{\Gamma} = (\Gamma_{ij}/\Gamma_{ij}\mathfrak{m})$, and let $\overline{f}_{ii}$ denote the indecomposable idempotent in $\overline{\Gamma}$ with 1 at the (i, i)th place and zero elsewhere. Let $\overline{Q}_i = \overline{f}_{ii}\overline{\Gamma}$, $1 \leq i \leq n$. Then $\overline{P}_i$ and $\overline{Q}_i$, $1 \leq i \leq n$, are, up to isomorphism, the only principal right projectives of $\overline{\Lambda}$ and $\overline{\Gamma}$ respectively. Since Λ and Γ are R-isomorphic, $\overline{\Lambda}$ and $\overline{\Gamma}$ are isomorphic as $R/\mathfrak{m}$-algebras. Hence to prove the proposition it is enough to show that $\overline{Q}_1$ satisfies the condition $(*)$.

By Lemma 1.6(2) we have that Γ is reduced, so that $\Gamma_{ij} = R$ for $i \leq j$,

and $\Gamma_{ij} \subseteq \mathfrak{m}$ for $i \gneq j$. Also, by Lemma 1.3 we have $J(\bar{\Gamma})$ is obtained from $\bar{\Gamma}$ by replacing the diagonal entries $R/\mathfrak{m}$ by zero. Since the multiplication in $\bar{\Gamma}$ is induced from that in Γ, it follows that for $i \gneq j$ we have

$$(\Gamma_{1i}/\Gamma_{1i}\mathfrak{m}) \cdot (\Gamma_{ij}/\Gamma_{ij}\mathfrak{m}) = 0 \quad \text{in } (\Gamma_{1j}/\Gamma_{1j}\mathfrak{m}) = R/\mathfrak{m},$$

and for $i \leq j$ we have

$$(\Gamma_{1i}/\Gamma_{1i}\mathfrak{m}) \cdot (\Gamma_{ij}/\Gamma_{ij}\mathfrak{m}) = (\Gamma_{1j}/\Gamma_{1j}\mathfrak{m}) = R/\mathfrak{m}.$$

Now a direct computation shows that

$$\bar{Q}_1 \supsetneqq \bar{Q}_1 J(\bar{\Gamma}) \supsetneqq \bar{Q}_1 J^2(\bar{\Gamma}) \supsetneqq \cdots \supsetneqq \bar{Q}_1 J^{n-1}(\bar{\Gamma}) \supsetneqq \bar{Q}_1 J^n(\bar{\Gamma}) = 0$$

is a composition series of $\bar{Q}_1$ as a right $\bar{\Gamma}$-module. This completes the proof.

Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order. Let $e = \Sigma_{i=1}^{n-1} e_{ii}$. Then $e\Lambda e$ is a tiled R-order in $M_{n-1}(R)$. We shall treat $e\Lambda e$ as the $(n-1) \times (n-1)$ tiled R-order in $M_{n-1}(R)$ or the top $(n-1) \times (n-1)$ corner interchangeably. Let $P_i = e_{ii}\Lambda$ and $J_i = e_{ii}J(\Lambda)$ for $1 \leq i \leq n$. Let $\mathfrak{F}$: $\mathfrak{Mod}\text{-}\Lambda \to \mathfrak{Mod}\text{-}e\Lambda e$ and $\mathfrak{G}$: $\mathfrak{Mod}\text{-}e\Lambda e \to \mathfrak{Mod}\text{-}\Lambda$ be the functors defined by $\mathfrak{F}M = Me$ and $\mathfrak{G}N = N \otimes_{e\Lambda e} e\Lambda$. Then by Lemma 1.3 and Corollary 1.4 we have $J(e\Lambda e)$ is canonically isomorphic to $\bigoplus_{i=1}^{n-1} \mathfrak{F}J_i$. Since Λe is a projective left Λ-module and since $M \otimes_\Lambda \Lambda e \cong Me$ under the isomorphism $m \otimes e \mapsto me$, the functor $\mathfrak{F}$ is naturally equivalent to the functor $-\otimes_\Lambda \Lambda e$, and therefore is exact and additive. Also, since $e\Lambda \otimes_\Lambda \Lambda e \cong e\Lambda e$ [2, p. 68], the functors $\mathfrak{F}\mathfrak{G}$ and $1_{\mathfrak{Mod}\text{-}e\Lambda e}$ are naturally equivalent. Furthermore, if $e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module, then $e\Lambda = e\Lambda e \oplus e\Lambda e_{nn}$ is a progenerator in $e\Lambda e$-$\mathfrak{Mod}$, and therefore the functor $\mathfrak{G}$ is exact and additive; also by Lemma 0.5 we have, for every right $e\Lambda e$-module N, $\mathrm{hd}_\Lambda \mathfrak{G}N = \mathrm{hd}_{e\Lambda e} N$ and $\mathrm{gl\ dim}\ e\Lambda e \leq \mathrm{gl\ dim}\ \Lambda$. Thus we have the following proposition.

Proposition 1.10. *Let R be a commutative noetherian domain. Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order. Let $e = \Sigma_{i=j}^{n-1} e_{ii}$. If $e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module, then*

(1) *$\mathfrak{F}$ and $\mathfrak{G}$ are exact, additive functors; the functors $\mathfrak{F}\mathfrak{G}$ and $1_{\mathfrak{Mod}\text{-}e\Lambda e}$ are naturally equivalent; $e\Lambda$ is a progenerator in $e\Lambda e$-$\mathfrak{Mod}$; $J(e\Lambda e)$ is canonically isomorphic with $\bigoplus_{i=1}^{n-1} \mathfrak{F}J_i$.*

(2) *For every right $e\Lambda e$-module N, $\mathrm{hd}\ \mathfrak{G}N = \mathrm{hd}_{e\Lambda e} N$; and $\mathrm{gl\ dim}\ e\Lambda e \leq \mathrm{gl\ dim}\ \Lambda$.*

We conclude this section with a few remarks.

Remark 1. If R is a local domain then $\Lambda = (\Lambda_{ij})$ is a semiperfect ring, by Theorem 1 of [13]. Hence by using Proposition 3 of [12, p. 77] one can easily show that $e_{ii}\Lambda(\Lambda e_{ii})$, $1 \leq i \leq n$, are, up to isomorphism, the only principal right (left) projectives of Λ. Thus, by Theorem 2 of [13] or otherwise, every finitely generated indecomposable right (left) Λ-module is isomorphic to $e_{ii}\Lambda\ (\Lambda e_{ii})$ for some i.

Remark 2. If R is a local domain and if $\Lambda = (\Lambda_{ij})$ is reduced, then, by Lemma 1.3, $J(\Lambda)$ is obtained from Λ by replacing the diagonal entries R by $\mathfrak{m}$, the unique maximal ideal of R.

Remark 3. If $A = \Sigma_{i=1}^{n} Ru_i$ is a free left R-module on the basis $u_1, u_2, \cdots, u_n$ then A is a right $M_n(R)$-module naturally. Since $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ is a subring of $M_n(R)$, this $M_n(R)$-module structure induces an $(R - \Lambda)$-bimodule structure on A. Further, if B is a nonzero Λ-submodule of A, then $B = \Sigma_{i=1}^{n} B_i u_i$, where B_i's are nonzero ideals of R. Hence A is uniform as a right Λ-module, i.e., if B and C are nonzero Λ-submodules of A, then $B \cap C \neq 0$. Further, B and xB are isomorphic as Λ-modules whenever $0 \neq x \in R$. Also, a nonzero Λ-module $B = \Sigma_{i=1}^{n} B_i u_i$ of A is isomorphic to $e_{kk}\Lambda$ for some $k \leq n$ if and only if for some $0 \neq x$ in R we have $B_i = x\Lambda_{ki}$ for $1 \leq i \leq n$. Similar results are true for a free right R-module of rank n.

Remark 4. Let A be as in Remark 3. Then $P_i = e_{ii}\Lambda$, $1 \leq i \leq n$, can be identified with a Λ-submodule of A. This identification makes expressions like $P_i + P_j$, $P_i \cap P_j$ unambiguous. In later sections without mentioning this identification we will use expressions like $P_i + P_j$, $P_i \cap P_j$, etc.

2. A sufficient condition.

From now on R will always denote a commutative noetherian domain with the quotient field K. Furthermore, if R is local then $\mathfrak{m}$ will always denote the unique maximal ideal of R. We will always assume that a tiled R-order $\Lambda = (\Lambda_{ij})$ in $M_n(K)$ is contained in $M_n(R)$. We reserve e for the idempotent $\Sigma_{i=1}^{n-1} e_{ii}$, and $P_i = e_{ii}\Lambda$, $J_i = e_{ii}J(\Lambda)$, $1 \leq i \leq n$.

Lemma 2.1. *Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order, where R is a local domain. Then,*

(1) *$e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module if and only if $e\Lambda e_{nn} \cong e\Lambda e_{kk}$ in $e\Lambda e$-$\mathfrak{Mod}$ for some natural number $k \lneq n$.*

(2) *$e\Lambda e_{nn} \cong e\Lambda e_{kk}$ in $e\Lambda e$-$\mathfrak{Mod}$ for some $k \lneq n$ if and only if there exists $0 \neq a$ in R such that $\Lambda_{in} = \Lambda_{ik}a$ for $1 \leq i \leq n-1$.*

Proof. Follows easily from Remarks 1 and 3 at the end of §1.

Lemma 2.2. *Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a reduced tiled R-order over a local domain R. By Lemma 2.1(1) if $e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module, then there exists a natural number $k \lneq n$ such that $e\Lambda e_{nn} \cong e\Lambda e_{kk}$ as left $e\Lambda e$-modules. Further,*

(1) $\mathrm{hd}_\Lambda J_i = \mathrm{hd}_{e\Lambda e}\mathfrak{F}J_i$ *for* $1 \leq i \leq n-1$, $i \neq k$.

(2) $\mathrm{hd}_\Lambda(\mathfrak{F}J_i)e\Lambda = \mathrm{hd}_{e\Lambda e}\mathfrak{F}J_i$ *for* $i = n, k$.

(3) *$\mathfrak{F}J_n$ is isomorphic to a right ideal of $e\Lambda e$.*

Proof. By Lemma 2.1(2) we have $0 \neq a$ in R such that $\Lambda_{in} = \Lambda_{ik}a$ for $1 \leq i \leq n-1$. Therefore $\Lambda_{kn} = Ra$ and $a\Lambda_{ni} = \Lambda_{kn}\Lambda_{ni} \subseteq \Lambda_{ki}$ for $1 \leq i \leq n$. By Remark 2 at the end of §1, $J(\Lambda)$ is obtained from Λ by replacing the diagonal entries R by $\mathfrak{m}$. It follows that $\theta: \mathfrak{F}J_n \to \mathfrak{F}P_k$ defined by $\theta(x) = ae_{kn}x$ is a monomorphism in $\mathfrak{M}\text{od-}e\Lambda e$. This proves (3). Since $e\Lambda = e\Lambda e \oplus e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module and since $\mathfrak{F}J_i$, $1 \leq i \leq n$, is isomorphic to a right ideal of $e\Lambda e$, therefore the sequence $0 \to \mathfrak{F}J_i \otimes_{e\Lambda e} e\Lambda \to e\Lambda e \otimes_{e\Lambda e} e\Lambda$ is exact. This yields $\mathfrak{G}\mathfrak{F}J_i \cong (\mathfrak{F}J_i)e\Lambda$, so that $\mathrm{hd}_\Lambda(\mathfrak{F}J_i)e\Lambda = \mathrm{hd}_\Lambda \mathfrak{G}\mathfrak{F}J_i = \mathrm{hd}_{e\Lambda e}\mathfrak{F}J_i$, for $1 \leq i \leq n$, by Proposition 1.10(2). By using matrix multiplication one can easily check that $(\mathfrak{F}J_i)e\Lambda = J_i$ for $i \neq n, k$. This completes the proof of the lemma.

Proposition 2.3. *Let R be a local domain. Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a reduced tiled R-order. Assume that $e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module and that* gl dim $e\Lambda e < \infty$. *Set $I = \Sigma_{i \neq n} \Lambda_{ni}\Lambda_{in}$. If* gl dim $(R/I) < \infty$, *then*

(1) $\mathrm{hd}_\Lambda J_n \leq \mathrm{hd}_{e\Lambda e}\mathfrak{F}J_n + \mathrm{gl\ dim}\,(R/I)$.

(2) $\mathrm{gl\ dim}\, e\Lambda e \leq \mathrm{gl\ dim}\,\Lambda \leq \mathrm{gl\ dim}\, e\Lambda e + \mathrm{gl\ dim}\,(R/I) + 1$.

Proof. Since Λ is reduced and R is local, $I \subseteq \mathfrak{m}$. Also, $J(\Lambda)$ is obtained from Λ by replacing the diagonal entries R by $\mathfrak{m}$. Let gl dim $e\Lambda e = \alpha < \infty$ and gl dim $(R/I) = \beta < \infty$. By Lemma 2.1 we have a natural number $k \lneq n$ and $0 \neq a$ in R such that $\Lambda_{in} = \Lambda_{ik}a$ for $1 \leq i \leq n-1$. Hence by Proposition 1.10(2), Lemma 2.2 and Lemma 0.1 we have

$$\mathrm{gl\ dim}\, e\Lambda e \leq \mathrm{gl\ dim}\,\Lambda,$$

$$\mathrm{hd}_\Lambda J_i \leq \alpha - 1 \quad \text{for } i \neq n, k,$$

$$\mathrm{hd}_\Lambda(\mathfrak{F}J_i)e\Lambda = \mathrm{hd}_{e\Lambda e}\mathfrak{F}J_i \leq \alpha - 1 \quad \text{for } i = n, k.$$

Since $\Lambda_{ki}\Lambda_{in} = \Lambda_{ki}\Lambda_{ik}a \subseteq \mathfrak{m}a$ for $i \neq k, n$, therefore by using matrix multiplications one shows that $(\mathfrak{F}J_n)e\Lambda$ is obtained from J_n by replacing the (n, n)th entry $\mathfrak{m}$ by I and that $(\mathfrak{F}J_k)e\Lambda$ is obtained from J_k by replacing the (k, n)th entry Ra by $\mathfrak{m}a$. Since $a\Lambda_{ni} = Ra\Lambda_{ni} = \Lambda_{kn}\Lambda_{ni} \subseteq \Lambda_{ki}$ and $a\Lambda_{nk} = \Lambda_{kn}\Lambda_{nk} \subseteq \mathfrak{m}$, therefore we have

$$((\mathfrak{F}J_k)e\Lambda) + aP_n = J_k \quad \text{and} \quad ((\mathfrak{F}J_k)e\Lambda) \cap aP_n = aJ_n \cong J_n.$$

From this we get a short exact sequence

$$(*) \qquad 0 \to aJ_n \xrightarrow{\theta} ((\mathfrak{F}J_k)e\Lambda) \oplus aP_n \xrightarrow{\phi} J_k \to 0$$

where $\theta(b) = (b, b)$ and $\phi(c, c') = c - c'$. We now determine an upper bound on $\mathrm{hd}_\Lambda J_n$. As observed before $I \subseteq \mathfrak{m}$. If $I = \mathfrak{m}$, then $\beta = 0$ and $(\mathfrak{F}J_n)e\Lambda = J_n$, so that $\mathrm{hd}_\Lambda J_n = \mathrm{hd}_{e\Lambda e}\mathfrak{F}J_n \leq \alpha - 1$. Since $\mathrm{hd}_\Lambda(\mathfrak{F}J_k)e\Lambda \leq \alpha - 1$, therefore from the short exact sequence $(*)$ and Lemma 0.6 we have $\mathrm{hd}_\Lambda J_k \leq \alpha$. Hence by Lemma 1.2

we have gl dim $\Lambda \leq \alpha + 1$. Now assume that $I \subsetneqq \mathfrak{m}$. Since gl dim $(R/I) = \beta < \infty$ and R is local, therefore $\overline{R} = R/I$ is a regular local ring of dimension $\beta \neq 0$ and having the unique maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/I$. Hence $\overline{\mathfrak{m}} = \Sigma_{i=1}^{\beta} \overline{x}_i \overline{R}$ for some $\overline{R}$-sequence $\overline{x}_1, \cdots, \overline{x}_\beta$. This yields $\mathfrak{m} = I + \Sigma_{i=1}^{\beta} x_i R$, and

$$(\#) \qquad x_\nu R \cap \left(I + \sum_{i=1}^{\nu-1} x_i R\right) = x_\nu \left(I + \sum_{i=1}^{\nu-1} x_i R\right), \qquad 1 \leq \nu \leq \beta.$$

Set $E_0 = (\mathfrak{F} J_n)e\Lambda$, and for $1 \leq \nu \leq \beta$ set

$$E_\nu = x_\nu P_n + x_{\nu-1} P_n + \cdots + x_1 P_n + E_0 = x_\nu P_n + E_{\nu-1}.$$

It follows that $E_\beta = J_n$. By using (#) one gets that $E_\nu P_n \cap E_{\nu-1} = x_\nu E_{\nu-1} \cong E_{\nu-1}$, by Remarks 3 and 4 of §1. Hence the two families E_ν, $0 \leq \nu \leq \beta$, and $x_\nu P_n$, $1 \leq \nu \leq \beta$, satisfy the hypothesis of Lemma 1.1. Therefore $\mathrm{hd}_\Lambda J_n = \mathrm{hd}_\Lambda E_\beta \leq \mathrm{hd}_\Lambda E_0 + \beta \leq \alpha - 1 + \beta$. Since $\mathrm{hd}_\Lambda (\mathfrak{F} J_k)e\Lambda \leq \alpha - 1$, therefore from the short exact sequence $(*)$ and Lemma 0.6 we get $\mathrm{hd}_\Lambda J_k \leq \alpha + \beta$. Thus gl dim $\Lambda \leq \alpha + \beta + 1$, by Lemma 1.2. That gl dim $e\Lambda e \leq$ gl dim Λ follows from Proposition 1.10(2). This completes the proof of the proposition.

Corollary 2.4. *Let R be a local domain. Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a reduced tiled R-order. Assume that $e\Lambda e_{nn}$ and $e_{nn}\Lambda e$ are projective left and right $e\Lambda e$-modules. If* gl dim $e\Lambda e = \alpha < \infty$ *and* gl dim $(R/I) = \beta < \infty$, *where $I = \Sigma_{i \neq n} \Lambda_{ni}\Lambda_{in}$, then*

(1) $\mathrm{hd}_\Lambda J_n \leq \beta$.

(2) $\alpha \leq$ gl dim $\Lambda \leq \sup(\beta + 1, \alpha - 1) + 1$.

Proof. As $\mathfrak{F} J_n = J_n e = e_{nn}\Lambda e$, (1) follows from Proposition 2.3(1). To prove (2) we use notations of the proof of Proposition 2.3(1). By Lemma 2.2 we have $\mathrm{hd}_\Lambda J_i \leq \alpha - 1$ for $i \neq n, k$, and $\mathrm{hd}_\Lambda (\mathfrak{F} J_k)e\Lambda \leq \alpha - 1$. If $\mathrm{hd}_\Lambda (\mathfrak{F} J_k)e\Lambda \gneq \mathrm{hd}_\Lambda J_n$ then, by using the short exact sequence $(*)$ of Proposition 2.3 and Lemma 0.6, one gets that $\mathrm{hd}_\Lambda J_k = \mathrm{hd}_\Lambda (\mathfrak{F} J_k)e\Lambda \leq \alpha - 1$; and if $\mathrm{hd}_\Lambda (\mathfrak{F} J_k)e\Lambda \leq \mathrm{hd}_\Lambda J_n$, then $\mathrm{hd}_\Lambda J_k \leq 1 + \mathrm{hd}_\Lambda J_n \leq 1 + \beta$. Thus $\mathrm{hd}_\Lambda J(\Lambda) \leq \sup(\beta + 1, \alpha - 1)$. Now Lemma 1.2 and Proposition 1.10(2) complete the proof of the corollary.

In the next theorem we give a sufficient condition for a tiled R-order $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$, R a local domain, to be of finite global dimension, and in Theorem 2.7 we remove the hypothesis that R is local.

Theorem 2.5. *Let R be a commutative noetherian local domain with maximal ideal $\mathfrak{m}$. Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order. Let $e = \Sigma_{i=1}^{n-1} e_{ii}$. If $e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module, then the following conditions are equivalent:*

(1) gl dim $\Lambda < \infty$,

(2) gl dim $e\Lambda e < \infty$ *and if $I = \Sigma_{i \neq n} \Lambda_{ni}\Lambda_{in}$ then $I = R$ or* gl dim $(R/I) < \infty$.

Furthermore, if these conditions hold, then gl dim $e\Lambda e \leq$ gl dim $\Lambda \leq$ gl dim $e\Lambda e$ + gl dim $(R/I) + 1$, *where if* $I = R$, *then for the purpose of this theorem we set* gl dim $(R/I) = -1$.

Proof. By Lemma 2.1 we have a natural number $k \lneq n$ and $0 \neq a$ in R such that $\Lambda_{in} = \Lambda_{ik} a$ for $1 \leq i \leq n-1$. Hence $I = \Sigma_{i \neq n} \Lambda_{ni}\Lambda_{in} = \Lambda_{nk} a$. Since R is a local ring, $I = R$ if and only if $\Lambda_{nj} = R = \Lambda_{jn}$ for some $j \neq n$. Hence, if Λ is reduced, then $I \subsetneqq R$. We now prove (1) $\Rightarrow$ (2). By Proposition 1.10(2) we have gl dim $e\Lambda e \leq$ gl dim $\Lambda < \infty$. If $I = R$, then we are done. So assume that $\Lambda_{nk} a = I \subsetneqq R$. Let $\mathfrak{a}$ be obtained from Λ by replacing the (n, n)th entry R by I. One can easily check that $\mathfrak{a}$ is a two-sided ideal of Λ and that $\mathfrak{a}^2 = \mathfrak{a}$. Since $\mathfrak{a} = \bigoplus_{i=1}^{n-1} \Lambda e_{ii} \oplus \Lambda e_{kk} a$, $\mathfrak{a}$ is a projectvie left Λ-module. Hence, by Lemma 0.3, gl dim $\Lambda/\mathfrak{a} < \infty$. Obviously, the map θ: $\Lambda/\mathfrak{a} \to R/I$ defined by $\theta[(\lambda_{ij}) + \mathfrak{a}] = \lambda_{nn} + I$ is a ring isomorphism. Hence gl dim $(R/I) < \infty$.

We now prove (2) $\Rightarrow$ (1) by using induction on n. Let $n = 2$. Then $e\Lambda e \cong R$, so that gl dim R = gl dim $e\Lambda e < \infty$. If Λ is not reduced, then $\Lambda = M_2(R)$, and Λ and R are Morita equivalent. This yields gl dim R = gl dim $\Lambda < \infty$. If Λ is reduced, then the theorem follows from Proposition 2.3. Let $n \geq 3$. Again if Λ is reduced, then the theorem follows from Proposition 2.3. So assume that Λ is not reduced. In that case we have natural numbers $s \lneq l \leq n$ such that $\Lambda_{sl} = R = \Lambda_{ls}$. But then $\Lambda_{si} \supseteq \Lambda_{sl}\Lambda_{li} = \Lambda_{li} \supseteq \Lambda_{ls}\Lambda_{si} = \Lambda_{si}$, so that $\Lambda_{si} = \Lambda_{li}$ for $1 \leq i \leq n$. Similarly $\Lambda_{is} = \Lambda_{il}$ for $1 \leq i \leq n$. If $l = n$, then Λ and $e\Lambda e$ are Morita equivalent and therefore gl dim Λ = gl dim $e\Lambda e < \infty$. If $l \lneq n$, then consider Γ, the tiled R-order obtained from Λ by deleting the lth row and the lth column. Clearly, $\Gamma \subseteq M_{n-1}(R)$, and Λ and Γ are Morita equivalent. Let $f = \Sigma_{i \neq n; i \neq l} e_{ii}$. Then $e\Lambda e$ and $f\Gamma f$ are Morita equivalent. Further $f\Gamma e_{nn}$ is a projective left $f\Gamma f$-module, by Lemma 2.1 applied to Γ. Also, $I = \Sigma_{i \neq n; i \neq l} \Lambda_{ni}\Lambda_{in}$. Hence by the induction hypothesis we have gl dim $\Gamma < \infty$, and gl dim $f\Gamma f \leq$ gl dim $\Gamma \leq$ gl dim $f\Gamma f$ + gl dim $(R/I) + 1$. Since gl dim $f\Gamma f$ = gl dim $e\Lambda e$ and gl dim Γ = gl dim Λ, the induction is complete. This completes the proof of the theorem.

Lemma 2.6. *Let R be a commutative noetherian domain. Let I be a nonzero proper ideal of R. Assume that* gl dim $(R_{\mathfrak{m}}/I_{\mathfrak{m}}) < \infty$ *for all maximal ideals* $\mathfrak{m}$ *of R containing I. Then we have*:

(1) *If* gl dim $R = d < \infty$, *then* gl dim $(R/I) \leq d - 1$.

(2) *If I is contained in only finitely many maximal ideals of R, then* gl dim $(R/I) < \infty$.

Proof. Let $\mathcal{S}$ be the set of all maximal ideals $\mathfrak{m}$ of R containing I. Let $\overline{R} = R/I$. By Lemma 0.2 we have gl dim $R_{\mathfrak{m}} \leq$ gl dim R for all $\mathfrak{m}$ in $\mathcal{S}$ and gl dim $\overline{R} = \sup_{\mathfrak{m} \in \mathcal{S}}$ gl dim $\overline{R}_{\overline{\mathfrak{m}}}$, where $\overline{\mathfrak{m}} = \mathfrak{m}/I$. It is easy to see that if $\mathfrak{m}$ is in $\mathcal{S}$,

then $I_{\mathfrak{m}}$ is a nonzero proper ideal of $R_{\mathfrak{m}}$ and $\overline{R}_{\overline{\mathfrak{m}}} \cong R_{\mathfrak{m}}/I_{\mathfrak{m}}$. Thus (2) is obvious and (1) follows from 0.10 applied to $R_{\mathfrak{m}}$.

Theorem 2.7. *Let R be a commutative noetherian domain. Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order. Assume that $e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module. Let $I = \Sigma_{i \neq n} \Lambda_{ni}\Lambda_{in}$. If* gl dim $R < \infty$ *or if I is contained in only finitely many maximal ideals of R, then the following conditions are equivalent:*

(1) gl dim $\Lambda < \infty$,

(2) gl dim $e\Lambda e < \infty$, $I = R$ *or* gl dim $(R/I) < \infty$.

If these conditions hold then we have

$$\text{gl dim } e\Lambda e \leq \text{gl dim } \Lambda \leq \text{gl dim } e\Lambda e + \text{gl dim}(R/I) + 1,$$

where, if $I = R$, then for the purpose of this theorem we set gl dim $(R/I) = -1$.

Proof. The proof follows easily from the familiar localization, Lemmas 0.2, 2.6 and Theorem 2.5.

We conclude this section with the following proposition.

Proposition 2.8. *Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a triangular tiled R-order over a commutative noetherian domain R. If* gl dim $\Lambda < \infty$, *then* gl dim $R < \infty$.

Proof. Since Λ is a triangular tiled R-order, $\Lambda_{ij} = R$ for $i \leq j$. Therefore $e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module and $e\Lambda e \subseteq M_{n-1}(R)$ is a triangular tiled R-order. Hence by Proposition 1.10(2) we have gl dim $e\Lambda e < \infty$. Applying this argument successively we conclude that gl dim $R < \infty$. This completes the proof.

In the next section we characterize triangular tiled R-orders of finite global dimension, where R is a commutative noetherian domain of finite global dimension.

3. Main theorem. In this section we investigate the global dimension of a tiled R-order $\Lambda \subseteq M_n(R)$ over a commutative noetherian domain R of finite global dimension. In view of Lemma 0.2, the main case we have to deal with is when R is a regular local ring. In the first half of this section we derive some of the properties of a tiled R-order over a regular local ring R. We recall that when R is a local ring, then the maximal ideal is denoted by $\mathfrak{m}$.

Lemma 3.1. *Let R be a regular local ring of dimension d and let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order. Then,* gl dim $\Lambda \geq d$.

Proof. Let $A = \Sigma_{i=1}^n Ru_i$ be a free left R-module on the basis $u_1, u_2, \cdots, u_n$. Treat A as a right Λ-module naturally. Let $\text{hd}_\Lambda A = \alpha$. If $\alpha = \infty$, then we are done. So assume that $\alpha < \infty$. Let $\mathfrak{m}$ be generated by an R-sequence $x_1, x_2, \cdots, x_d$, i.e., $\mathfrak{m} = \Sigma_{i=1}^d x_i R$. Let $X_i = \text{diag } x_i$, $1 \leq i \leq d$. Then the matrices X_i are nonzero divisors in Λ and are contained in $J(\Lambda)$, by Lemma 1.3. Furthermore,

X_i is a nonzero divisor on the right Λ-module $A/\Sigma_{j=1}^{i-1} AX_j$. Since $\mathrm{hd}_\Lambda A = \alpha < \infty$, $\mathrm{hd}_\Lambda(A/AX_1) = \alpha + 1$, by Lemmas 0.7 and 0.8. An easy induction shows that $\mathrm{hd}_\Lambda(A/\Sigma_{j=1}^{d} AX_j) = \alpha + d$. This completes the proof of the lemma.

Remark. The techniques of the proof of the above lemma are similar to those used in the proof of Theorem 14 of [17]. The above lemma also follows from Corollary 3.2 of [20], but our proof is elementary. The notations established in Lemma 3.1 will be needed in the proof of Theorem 3.4 of this paper.

Lemma 3.2. *Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a reduced triangular tiled R-order, where R is a regular local ring of dimension d. Let $\mathfrak{m}$ be generated by an R-sequence $x_1, x_2, \cdots, x_d$. Assume that $\Lambda_{ij} = x_1R$ whenever $i = n$ or $n-1$ and $i \gneq j$. Then $\mathrm{hd}_\Lambda J_i \leq d-1$ for $i = n-1, n$. Hence, if $n = 2$, then $\mathrm{gl\ dim}\ \Lambda = d$, and if $d = 2$, then $\mathrm{hd}_\Lambda J_i = 1$ for $i = n-1, n$.*

Proof. Since Λ is reduced, $J(\Lambda)$ is obtained from Λ by replacing the diagonal entries R by $\mathfrak{m}$. Set $E_1 = P_n$ and $E_\nu = x_\nu P_{n-1} + x_{\nu-1}P_{n-1} + \cdots + x_2P_{n-1} + P_n$ for $2 \leq \nu \leq d$. Then $E_d = J_{n-1}$ and $E_\nu = x_\nu P_{n-1} + E_{\nu-1}$. Since $x_1, \cdots, x_d$ is an R-sequence, therefore $x_\nu R \cap \Sigma_{i=1}^{\nu-1} x_iR = x_\nu \Sigma_{i=1}^{\nu-1} x_iR$ for $2 \leq \nu \leq d$. It follows that $x_\nu P_{n-1} \cap E_{\nu-1} = x_\nu E_{\nu-1} \cong E_{\nu-1}$ for $2 \leq \nu \leq d$. Since $\mathrm{hd}\ E_1 = 0$, therefore Lemma 1.1, applied to the two families E_ν, $1 \leq \nu \leq d$, and $x_\nu P_{n-1}$, $2 \leq \nu \leq d$, of right Λ-modules, yields $\mathrm{hd}_\Lambda J_{n-1} = \mathrm{hd}_\Lambda E_d \leq d-1$. Similarly to show that $\mathrm{hd}_\Lambda J_n \leq d-1$, set $F_1 = x_1P_1$ and $F_\nu = x_\nu P_n + \cdots + x_2P_n + x_1P_1$ for $2 \leq \nu \leq d$. Then $F_\nu = x_\nu P_n + F_{\nu-1}$ and $F_d = J_n$. An argument similar to the above shows that $\mathrm{hd}_\Lambda J_n \leq d-1$. It is easy to see that when $d = 2$, J_{n-1} and J_n are not projective right Λ-modules. Therefore we must have $\mathrm{hd}_\Lambda J_i = 1$ for $i = n-1$ and n. The remaining assertion follows from Lemmas 1.2 and 3.1. This completes the proof.

Proposition 3.3. *Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order, where R is a regular local ring of dimension d, with the unique maximal ideal $\mathfrak{m}$ generated by an R-sequence $x_1, x_2, \cdots, x_d$. Assume that $\Lambda_{1j} = R$ for all j and $\Lambda_{ij} = x_1R$ or R whenever $i \neq 1$ or $i \neq j$. Then:*

(1) $\mathrm{gl\ dim}\ \Lambda \leq d(n-1)$.

(2) *If $\Lambda_{ij} = x_1R$ whenever $i \neq j$ and $i \neq 1$, then* $\mathrm{gl\ dim}\ \Lambda \leq d+1$.

(3) *If Λ is a triangular tiled R-order then* $\mathrm{gl\ dim}\ \Lambda = d$.

Proof. First consider the case $n = 2$. If Λ is reduced then $\mathrm{gl\ dim}\ \Lambda = d$, by Lemma 3.2. If Λ is not reduced then $\Lambda = M_2(R)$, so that $\mathrm{gl\ dim}\ \Lambda = d = \mathrm{gl\ dim}\ R$. Thus, all the assertions are true for $n = 2$.

We now prove (3) by using induction on n. Let $n \geq 3$. If Λ is not reduced then Λ is Morita equivalent to a triangular tiled R-order in $M_{n-1}(R)$ satisfying

the hypothesis of the proposition. Hence by the induction hypothesis we have gl dim $\Lambda = d$. Now assume that Λ is reduced. Since Λ is a triangular tiled R-order $e\Lambda e_{nn} \cong e\Lambda e_{n-1,n-1}$ as left $e\Lambda e$-modules. Also, $e\Lambda e \subseteq M_{n-1}(R)$ satisfies the hypothesis of the proposition. Hence by Lemmas 1.2, 2.2 and the induction hypothesis we have $\mathrm{hd}_\Lambda J_i \leq d-1$ for $i \leq n-2$. Also, by Lemma 3.2 we have $\mathrm{hd}_\Lambda J_i \leq d-1$ for $i = n-1, n$. Thus by Lemmas 1.2 and 3.1 we have gl dim $\Lambda = d$. This completes the induction.

We now prove (1) and (2) simultaneously by using induction on n. Let $n \geq 3$. First assume that Λ is reduced. Since $\Lambda_{1j} = R$ for all j, therefore by Lemma 1.7 we have a natural number $l \geq 1$ such that $\Lambda_{li} \neq R$ whenever $i \neq l$, so that $\Lambda_{li} = x_1 R$ for $i \neq l$. Since the global dimension of a ring is isomorphism invariant, we may assume that $l = 2$. Let $y = (y_{ij})$ be in $M_n(K)$, where $y_{12} = x_1^{-1}$, $y_{21} = 1 = y_{ii}$ for $i \geq 3$, $y_{ij} = 0$ otherwise. Let $\Gamma = (\Gamma_{ij}) = y\Lambda y^{-1}$. Computation shows that $\Gamma_{1j} = R$ for all j; $\Gamma_{2j} = R$ for $j \geq 2$; $\Gamma_{i1} = x_1\Lambda_{i2}$ for $i \geq 2$; $\Gamma_{i2} = x_1 R$ for $i \geq 3$; $\Gamma_{ij} = \Lambda_{ij}$ otherwise. Hence $\Gamma = (\Gamma_{ij}) \subseteq M_n(R)$ is a tiled R-order and $e_{11}\Gamma e'$ is a projective right $e'\Gamma e'$-module, where $e' = \Sigma_{i=2}^n e_{ii}$. Further, $I' = \Sigma_{i \neq 1} \Gamma_{1i}\Gamma_{i1} = x_1 R$, so that gl dim $(R/I') = d - 1 < \infty$. Clearly $e'\Gamma e' \subseteq M_{n-1}(R)$ satisfies the hypothesis of the proposition in cases (1) and (2). We now complete the proof of (1). By the induction hypothesis gl dim $e'\Gamma e' \leq d(n-2)$; hence by the analogue of Theorem 2.5 we have gl dim $\Gamma \leq d(n-2) + d = d(n-1)$. Since Λ and Γ are conjugate, gl dim $\Lambda \leq d(n-1)$. For (2) we observe that by the hypothesis we have $\Lambda_{i2} = x_1 R$ for all $i \geq 3$. Thus $\Gamma_{i1} = \Gamma_{i2}x_1$ for $i \geq 2$. Hence by Lemma 2.1 (rather, its analogue), we have that $e'\Gamma e_{11}$ is a projective left $e'\Gamma e'$-module. Therefore by the analogue of Corollary 2.4 and the induction hypothesis we have

$$\text{gl dim } \Gamma \leq \sup(d-1+1,\ d+1-1) + 1 = d+1.$$

Thus gl dim Λ = gl dim $\Gamma \leq d+1$.

For the case Λ is not reduced, the argument is similar to the case (3) and we leave it to the reader. This completes the induction and the proof of the proposition too.

The author is thankful to the referee for suggesting the following corollary.

Corollary 3.4. *Let R be a regular local ring of dimension d, Λ a triangular tiled R-order satisfying the hypothesis of Proposition 3.3. Then each finitely generated R-free Λ-module is projective.*

Proof. Follows from Proposition 3.3 of this paper and Proposition 3.5 of [20].

Theorem 3.5. *Let R be a commutative noetherian domain of global dimension $d < \infty$. Let I be a nonzero proper ideal of R such that* gl dim $(R/I) < \infty$. *Let*

$\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a tiled R-order. Assume that $\Lambda_{1j} = R$ for all j and $\Lambda_{ij} = I$ or R whenever $i \neq j$ and $i \neq 1$. Then:

(1) gl dim $\Lambda \leq d(n-1)$.

(2) *If Λ is a triangular tiled R-order, then* gl dim $\Lambda = d$.

(3) *If $\Lambda_{ij} = I$ whenever $i \neq j$ and $i \neq 1$, then* gl dim $\Lambda \leq d+1$. *Furthermore, if R is local, $I = J(R)$ and $n \geq 3$, then* gl dim $\Lambda = d+1$.

(4) *If $n \geq 4$, R is local and if $\Lambda_{ij} = I$ whenever $i \neq j$ and $i \neq 1$, then Λ is not conjugate to a triangular tiled R- order.*

Proof. Since gl dim $R = d$, gl dim $R_{\mathfrak{m}} \leq d$ for all maximal ideals $\mathfrak{m}$ of R with equality occurring for at least one $\mathfrak{m}$. By Lemma 0.2 we have gl dim $\Lambda = \sup_{\mathfrak{m}}$ gl dim $\Lambda_{\mathfrak{m}}$, where the supremum is taken over all maximal ideals $\mathfrak{m}$ of R. If $I \not\subseteq \mathfrak{m}$, then $I_{\mathfrak{m}} = R_{\mathfrak{m}}$, and therefore $\Lambda_{\mathfrak{m}} = M_n(R_{\mathfrak{m}})$. Thus if $I \not\subseteq \mathfrak{m}$, then gl dim $\Lambda_{\mathfrak{m}} =$ gl dim $R_{\mathfrak{m}}$. If $I \subseteq \mathfrak{m}$, then gl dim $(R_{\mathfrak{m}}/I_{\mathfrak{m}}) \leq$ gl dim $(R/I) < \infty$. Thus to prove the theorem we may assume that R is a regular local ring of dimension d with maximal ideal $\mathfrak{m}$. But then, by Lemma 0.9, we have an R-sequence $x_1, x_2, \cdots, x_d$ such that $I = \Sigma_{i=1}^{\mu} x_i R$, $\mu \geq 1$, and $\mathfrak{m} = \Sigma_{i=1}^{d} x_i R$. Also, we have that $R/(x_\mu)$ is a regular local ring of dimension $d-1$ and $x_i + (x_\mu)$, $1 \leq i \leq d$, $i \neq \mu$, is an $R/(x_\mu)$-sequence generating the maximal ideal of $R/(x_\mu)$. Set $\mathfrak{a} = (\mathfrak{a}_{ij}) \subseteq \Lambda$, where $\mathfrak{a}_{ij} = (x_\mu)$ for $1 \leq i, j \leq n$. Clearly $\mathfrak{a}$ is a two-sided ideal contained in $J(\Lambda)$ and is projective as a right Λ-module. Hence, by Lemma 0.4, we have

$$(\#\,\#) \qquad \text{gl dim } \Lambda \leq 1 + \text{gl dim } \Lambda/\mathfrak{a}.$$

It is easy to see that $\Lambda/\mathfrak{a}$ is isomorphic to a tiled $R/(x_\mu)$-order $(\Lambda_{ij}/(x_\mu)) \subseteq M_n(R/(x_\mu))$. We now prove (1), (2) and the first part of (3) simultaneously by using induction on μ. For $\mu = 1$ the assertions follow from Proposition 3.3. Let $\mu \geq 2$. By the induction hypothesis we have

$$\begin{aligned} \text{gl dim } (\Lambda/\mathfrak{a}) &\leq (d-1)(n-1) && \text{in case (1),} \\ &= d-1 && \text{in case (2),} \\ &\leq d && \text{in case (3).} \end{aligned}$$

The assertions now follow from (# #) and Lemma 3.1.

We now prove the remaining part of (3). Since Λ is reduced, $J(\Lambda)$ is obtained from Λ by replacing the diagonal entries R by $\mathfrak{m}$. Hence, for $i \geq 2$, $J_i \cong \Sigma_{i=1}^{d} P_1 X_i$ as right Λ-modules, where $X_i = \text{diag } x_i$. Since P_1 is projective and is isomorphic to A, where A is a defined in Lemma 3.1, we have $\text{hd}_\Lambda(P_1/\Sigma_{i=1}^{d} P_1 X_i) = d$. Hence $\text{hd}_\Lambda J_i = d-1$ whenever $i \geq 2$. To complete the proof we show that $\text{hd}_\Lambda J_1 = d$, since then by Lemma 1.2 we have gl dim $\Lambda = d+1$. Set $E_\nu = P_2 + P_3 + \cdots + P_\nu$ for $2 \leq \nu \leq n$. Then, since $n \geq 3$, $E_n = J_1$; also

for $3 \leq \nu \leq n$ we have $E_\nu = E_{\nu-1} + P_\nu$, $E_{\nu-1} \cap P_\nu \cong J_2$ and E_ν is not projective as a right Λ-module. Hence, using short exact sequences, induction and Lemma 0.6, one gets that $\mathrm{hd}_\Lambda E_\nu = d$ for $\nu \geq 3$. Thus $\mathrm{hd}\, J_1 = d$.

Lastly we prove (4). Let $n \geq 4$. Let $\bar{\Lambda}$, $\bar{P}_i = \bar{e}_{ii}\bar{\Lambda}$, $1 \leq i \leq n$, be as in Proposition 1.9. Since Λ is reduced, by Lemma 1.3, $J(\bar{\Lambda})$ is obtained from $\bar{\Lambda}$ by replacing the diagonal entries $R/\mathfrak{m}$ by zero. Computation shows that $\bar{P}_i J^3(\bar{\Lambda}) = 0$ for $1 \leq i \leq n$. Since $n \geq 4$, therefore none of $\bar{P}_j$ satisfies the condition $(*)$ of Proposition 1.9. Thus, by the same proposition, Λ is not conjugate to a triangular tiled R-order.

Remark 1. The assertion (2) in the above theorem is a generalization of Theorem 14 of [17].

Remark 2. The assertion (4) in the above theorem is not true in general when $n \lneq 4$. For if $n = 2$, then Λ is itself a triangular tiled R-order; and if $n = 3$ and $I = xR$ is a principal ideal, then $y\Lambda y^{-1}$ is a triangular tiled R-order, where $y = (y_{ij})$ is in $M_3(K)$, with $y_{12} = x^{-1}$, $y_{21} = y_{33} = 1$ and $y_{ij} = 0$ otherwise.

The next theorem characterizes triangular tiled R-orders of finite global dimension over a commutative noetherian domain R of finite global dimension.

Theorem 3.6. *Let R be a commutative noetherian domain of global dimension $d < \infty$. Then the following three conditions on a triangular tiled R-order $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ are equivalent:*

(1) $\mathrm{gl\ dim}\ \Lambda < \infty$.

(2) $\Lambda_{i,i-1} = R$ *or* $\mathrm{gl\ dim}\ (R/\Lambda_{i,i-1}) < \infty$ *for* $2 \leq i \leq n$.

(3) $\mathrm{gl\ dim}\ \Lambda \leq d(n-1)$.

Proof. First we note that $e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module, and since $\Lambda_{ni} = \Lambda_{ni}\Lambda_{i,n-1} \subseteq \Lambda_{n,n-1}$ for $i \neq n$, we have $I = \Sigma_{i \neq n} \Lambda_{ni}\Lambda_{in} = \Lambda_{n,n-1}$. We now prove (1) $\Rightarrow$ (2) by using an induction on n. If $n = 2$, then the assertion follows from Theorem 2.7. Let $n \geq 3$. Again by the same theorem we have $\mathrm{gl\ dim}\ e\Lambda e < \infty$, and $\Lambda_{n,n-1} = R$ or $\mathrm{gl\ dim}\ (R/\Lambda_{n,n-1}) < \infty$. Now the induction hypothesis completes the proof as $e\Lambda e$ is a triangular tiled R-order contained in $M_{n-1}(R)$. We now prove (2) $\Rightarrow$ (3) again by using an induction on n. Let $n = 2$. Then, by Theorem 3.5(2), we have $\mathrm{gl\ dim}\ \Lambda = d$. Let $n \geq 3$. By the induction hypothesis we have $\mathrm{gl\ dim}\ e\Lambda e \leq d(n-2)$. Hence by Theorem 2.7 and Lemma 2.6(1) we have

$$\mathrm{gl\ dim}\ \Lambda \leq \mathrm{gl\ dim}\ e\Lambda e + \mathrm{gl\ dim}(R/\Lambda_{n,n-1}) + 1 \leq d(n-2) + d - 1 + 1 = d(n-1).$$

(3) $\Rightarrow$ (1) is trivial.

This completes the proof of the theorem.

Remark. The above theorem is a generalization of Theorem 1 of [8], [9].

Corollary 3.7. *Let R be a Dedekind domain. Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$ be a triangular tiled R-order. The following conditions are equivalent:*

(1) gl dim $\Lambda < \infty$.

(2) $\Lambda_{i,i-1} = R$ *or* $\Lambda_{i,i-1}$ *is a product of distinct maximal ideals of* R.

(3) gl dim $\Lambda \leq n - 1$.

It is known that the upper bound $d(n-1)$ in Theorem 3.6 is attained when $d = 1$ ([9, Theorem 2], [17, Theorem 11]). By constructing an example we show that this is also the case when $d = 2$.

Proposition 3.8. *Let R be a regular local ring of dimension 2. Let $\mathfrak{m}$ be the unique maximal ideal of R generated by an R-sequence x, y. Let $\Lambda = (\Lambda_{ij}) \subseteq M_n(R)$, $n \geq 2$, where $\Lambda_{ij} = R$ whenever $i \leq j$, $\Lambda_{i,i-1} = xR$ for $2 \leq i \leq n$, and $\Lambda_{ij} = x\mathfrak{m} = x^2R + xyR$ otherwise. Then Λ is a triangular tiled R-order in $M_n(R)$ and*

$$\mathrm{hd}_\Lambda J_i = 2i - 1 \quad \textit{for } 1 \leq i \leq n-1,$$

$$\mathrm{hd}_\Lambda J_n = \sup(2(n-2), 1).$$

Thus gl dim $\Lambda = 2(n-1)$.

Proof. Computation shows that Λ is a ring, hence is a triangular tiled R-order. Obviously Λ is reduced; hence $J(\Lambda)$ is obtained from Λ by replacing the diagonal entries R by $\mathfrak{m}$. To prove the remaining assertions we use an induction on n. If $n = 2$, then the assertions follow from Lemma 3.2. Let $n \geq 3$. Clearly $e\Lambda e$ satisfies the hypothesis of the proposition. Hence by the induction hypothesis we have gl dim $e\Lambda e = 2(n-2)$. Since $e\Lambda e_{nn} \cong e\Lambda e_{n-1,n-1}$, therefore by Proposition 1.10(1), Lemma 2.2 and the induction hypothesis we have $\mathrm{hd}_\Lambda J_i = \mathrm{hd}_{e\Lambda e} \mathfrak{F}J_i = 2i - 1$ for $1 \leq i \leq n-2$, $\mathrm{hd}_\Lambda(\mathfrak{F}J_{n-1})e\Lambda = \mathrm{hd}_{e\Lambda e} \mathfrak{F}J_{n-1} = \sup(2(n-3), 1)$. Let $L_1 = yP_{n-1} + P_n$ and $L_2 = xP_{n-2} + P_n$. Then L_1 and L_2 are isomorphic to right ideals of Λ. Also one can easily check that $yP_{n-1} \cap P_n = yL_2 \cong L_2$ and $xP_{n-2} \cap P_n = xJ_{n-2} \cong J_{n-2}$. Thus by using obvious short exact sequences and Lemma 0.6 one concludes that $\mathrm{hd}_\Lambda L_1 = \mathrm{hd}_\Lambda J_{n-2} + 2 = 2(n-2) + 1 = 2n - 3$. Hence, by Lemma 0.1, we have gl dim $\Lambda \geq 2n - 2 = 2(n-1)$. But then Theorem 3.6 yields gl dim $\Lambda = 2(n-1)$. Since $I = \sum_{i \neq n} \Lambda_{ni}\Lambda_{in} = xR$, gl dim $(R/I) = 1$. Therefore, by Lemmas 0.1, 2.2 and Proposition 2.3, we have $\mathrm{hd}_\Lambda J_n \leq \mathrm{hd}_{e\Lambda e} \mathfrak{F}J_n + 1 \leq 2(n-2) = 2(n-1)\,2$. Hence, by Lemma 1.2 we must have $\mathrm{hd}_\Lambda J_{n-1} = 2(n-1)-1$. To complete the induction we show that $\mathrm{hd}_\Lambda J_n = 2(n-1)-2$. Computation shows that $(\mathfrak{F}J_{n-1})e\Lambda + P_n = J_{n-1}$ and $(\mathfrak{F}J_{n-1})e\Lambda \cap P_n = J_n$. Hence the sequence

$$0 \to J_n \xrightarrow{\theta} ((\mathfrak{F}J_{n-1})e\Lambda) \oplus P_n \xrightarrow{\phi} J_{n-1} - 0$$

is exact, where $\theta(a) = (a, a)$ and $\phi(b, c) = b - c$. But $\mathrm{hd}_\Lambda J_{n-1} = 2(n-1)-1$

and $\mathrm{hd}_\Lambda(\mathfrak{F}J_{n-1})e\Lambda = \sup(2(n-3), 1)$; therefore we must have $\mathrm{hd}_\Lambda J_n = 2(n-1) - 2$, by Lemma 0.6. This completes the induction and proves the proposition.

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