A CONNECTED PLANE CONTINUA

BY

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ABSTRACT. A continuum $M$ is said to be $\lambda$ connected if any two distinct points of $M$ can be joined by a hereditarily decomposable continuum in $M$.

Recently this generalization of arcwise connectivity has been related to fixed point problems in the plane. In particular, it is known that every $\lambda$ connected nonseparating plane continuum has the fixed point property. The importance of arcwise connectivity is, to a considerable extent, due to the fact that it is a continuous invariant. To show that $\lambda$ connectivity has a similar feature is the primary purpose of this paper. Here it is proved that if $M$ is a $\lambda$ connected continuum and $f$ is a continuous function of $M$ into the plane, then $f(M)$ is $\lambda$ connected. It is also proved that every semiaposyndetic plane continuum is $\lambda$ connected.

Introduction. A nondegenerate metric space that is both compact and connected is called a continuum. It is known that every plane continuum that has a hereditarily decomposable boundary and does not separate the plane has the fixed point property [1]. Recently the author [4] proved that every arcwise connected nonseparating plane continuum has a hereditarily decomposable boundary. Hence all arcwise connected nonseparating plane continua have the fixed point property. In [7] it is pointed out that the author's theorem remains true if the word "arcwise" is replaced by "$\lambda". In fact, in [7] it is proved that a plane continuum that does not have infinitely many complementary domains is $\lambda$ connected if and only if its boundary does not contain an indecomposable continuum.

This paper is primarily concerned with the following questions:

(1) What other theorems about arcwise connected continua also hold for $\lambda$ connected continua?

(2) Are there general properties, other than arcwise connectivity for plane continua, that imply $\lambda$ connectivity?

Received by the editors September 13, 1971 and, in revised form, September 6, 1973.

AMS (MOS) subject classifications (1970). Primary 54C05, 54F25, 54F60, 57A05; Secondary 54F15.

Key words and phrases. Arcwise connectivity, $\lambda$ connected continua, aposyndesis, semiaposyndesis, planar continuous images of hereditarily decomposable continua.

(1) The author gratefully acknowledges conversations about these results with D. P. Bellamy, F. B. Fuller, E. E. Grace, F. B. Jones, and J. S. Rosasco.

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Here, in response to the first question, we prove that every planar continuous image of a \( \lambda \) connected continuum is \( \lambda \) connected.

To deal with the second question, we must consider general properties that do not imply arcwise connectivity. Examples have been given [13, Example 4], [10, Example 6], and [3, Example 1], which indicate that, for plane continua, arcwise connectivity is not a consequence of the aposyndetic property defined by F. Burton Jones. Jones' property can be generalized as follows.

Definition. A continuum \( M \) is said to be semiaposyndetic if for each pair of distinct points \( x \) and \( y \) in \( M \) there exists a subcontinuum \( F \) of \( M \) such that the sets \( M - F \) and the interior of \( F \) relative to \( M \) each contain a point of \( \{x, y\} \).

Although not all semiaposyndetic plane continua are arcwise connected, these properties are related. If \( X \) is a semiaposyndetic plane continuum and for any positive real number \( \epsilon \) there are at most a finite number of complementary domains of \( X \) of diameter greater than \( \epsilon \), then \( X \) is arcwise connected [6]. For another arc theorem involving semiaposyndesis see [5]. In the last section of this paper it is proved that all semiaposyndetic plane continua are \( \lambda \) connected.

Throughout this paper \( S^2 \) is a 2-sphere. The closure and the boundary of a given set \( Z \) are denoted by \( \text{Cl} \ Z \) and \( \text{Bd} \ Z \) respectively. The union of the elements of \( Z \) is denoted by \( \text{St} \ Z \).

Definitions. Let \( X \) be a continuum in \( S^2 \). A continuum \( L \) in \( X \) is said to be a link in \( X \) if \( L \) is either the boundary of a complementary domain of \( X \) or the limit of a convergent sequence of complementary domains of \( X \). A continuum \( T \) in \( X \) is said to be a 2-link in \( X \) if \( T \) is the union of two (not necessarily distinct) links in \( X \). An indecomposable subcontinuum \( I \) of \( X \) is said to be terminal in \( X \) if there exists a composant \( C \) of \( I \) such that each subcontinuum of \( X \) that meets both \( C \) and \( X - I \) contains \( I \).

Preliminary results.

Theorem 1. If \( X \) is a continuum in \( S^2 \) and \( I \) is an indecomposable subcontinuum of \( X \) that is contained in the union of finitely many links in \( X \), then every subcontinuum of \( X \) that contains a nonempty open subset of \( I \) contains \( I \).

Proof. Assume \( I \) is contained in the union of \( \alpha \) (\( \alpha \) is a natural number) links in \( X \). Suppose \( Z \) is a collection of \( \alpha + 1 \) disjoint circular regions in \( S^2 \) such that each element of \( Z \) intersects \( I \). There exist points \( c \) and \( d \) belonging to distinct elements of \( Z \) such that \( \{c, d\} \) is contained in a complementary domain of \( X \). The theorem now follows directly from [4, Theorem 1 (proof)].

Theorem 2. Suppose \( M \) is a hereditarily decomposable continuum and \( f \) is a continuous function of \( M \) into \( S^2 \). Then no indecomposable subcontinuum of \( f(M) \) is terminal in \( f(M) \).
Proof. Assume there exists an indecomposable subcontinuum \( I \) of \( f(M) \) that is terminal in \( f(M) \). There exists a composant \( C \) of \( I \) such that each subcontinuum of \( f(M) \) that meets both \( C \) and \( f(M) - I \) contains \( I \). Note that since \( f(M) \) is decomposable, \( I \) is a proper subcontinuum of \( f(M) \). Let \( p \) be a point of \( f^{-1}(C) \) and let \( Z \) be the \( p \)-component of \( f^{-1}(I) \).

For each positive integer \( n \), let \( G_n \) be an open set in \( M \) such that (1) \( Z \subset G_n \), (2) \( \text{Bd } G_n \cap f^{-1}(I) = \emptyset \), (3) \( M - G_n \) is not the empty set, and (4) the distance from each point of \( G_n \) to \( Z \) is less than \( n^{-1} \). For each \( n \), let \( Y_n \) be the \( p \)-component of \( f^{-1}(C) \). Note that, for each \( n \), the continuum \( Y_n \) meets \( \text{Bd } G_n \) [12, Theorem 50, p. 18] and consequently \( f(Y_n) \) is a continuum in \( f(M) \) that meets both \( C \) and \( f(M) - I \). Hence for each \( n \), the continuum \( f(Y_n) \) contains \( I \). It follows from the continuity of \( f \) that \( f(Z) \) is \( I \). Since \( Z \) is hereditarily decomposable, this is a contradiction. Hence no indecomposable subcontinuum of \( f(M) \) is terminal in \( f(M) \).

Theorem 3. If \( X \) is a continuum in \( S^2 \) and no finite collection of links in \( X \) contains an indecomposable continuum in its union, then \( X \) is \( \lambda \) connected.

Proof. Let \( p_1 \) and \( q_1 \) be distinct points of \( X \). We shall construct a hereditarily decomposable continuum \( Q \) in \( X \) that contains \( p_1 \) and \( q_1 \).

Let \( A \) be an arc in \( S^2 \) from \( p_1 \) to \( q_1 \) ordered by \( < \).

Notation. For points \( x \) and \( y \) of \( A \) \((x < y)\) let \([x, y], [x, y), (x, y], \) and \((x, y)\) denote the sets \( \{z \in A \mid x < z < y\} \), \( \{z \in A \mid x < z < y\} \), \( \{z \in A \mid x < z < y\} \), and \( \{z \in A \mid x < z < y\} \) respectively.

Let \( \{G_n\} \) be the countable collection consisting of all complementary domains of \( X \). If no element of \( \{G_n\} \) intersects \( A \), then \( A \) is in \( X \) and the construction is trivial. Assume this is not the case. Suppose without loss of generality that \( G_1 \cap A \neq \emptyset \).

1. The construction of a continuum \( Z_1 \) in \((X \cup A) - G_1\) from \( p_1 \) to \( q_1 \). Let \( H_0 \) be a link in \( X \) whose intersection with \( A \) is \( p_1 \) (if no such link exists let \( H_0 \) be \( \emptyset \)). Let \( L_0 \) be a link in \( X \) that meets \( A \) only at \( q_1 \) (if no such link exists let \( L_0 \) be \( \emptyset \)). The links \( H_0 \) and \( L_0 \) will be referred to as selected 2-links.

With \( H_0 \), \( L_0 \), the arc \([p_1, q_1]\), and the complementary domain \( G_1 \) as sets of reference, we now construct either a continuum \( K_1 \) or a set \( T_1 \) as follows.

If \( H_0 \cap L_0 \neq \emptyset \), define \( K_1 = H_0 \cup L_0 \).

Assume that \( H_0 \) and \( L_0 \) are disjoint sets. Let \( x_1 \) and \( y_1 \) be the first and last points respectively of \([p_1, q_1] \cap \text{Bd } G_1 \) (with respect to the order of \( A \)). Let \( W_1 \) be the collection consisting of all links in \( X \) that meet \([p_1, x_1] \cup [y_1, q_1] \).

Let \( V_1 \) be the collection of all 2-links \( L \) in \( X \) such that \( L \) is the union of two (not necessarily distinct) elements of \( W_1 \).
If \( L \) is an element of \( V_1 \) that is the union of elements \( H \) and \( K \) (not necessarily distinct) of \( W_1 \), then we call any point set \( \{x, y\} \) such that \( x \) belongs to \( H \cap ([p_1, x] \cup [y_1, q_1]) \) and \( y \) belongs to \( K \cap ([p_1, x] \cup [y_1, q_1]) \) a basic set in \( L \).

Suppose no element of \( V_1 \) meets both \([p_1, x_1] \cup H_0 \) and \([y_1, q_1] \cup L_0 \) and no element of \( V_1 \) meets both \([p_1, x_1] \cup H_0 \) and \([y_1, q_1] \cup L_0 \). Define
\[
K_1 = [p_1, x_1] \cup \text{Bd} G_1 \cup [y_1, q_1].
\]

In the last part of this proof, \( \text{Bd} G_1 \) will be referred to as a selected 2-link.

Assume that \( V_1 \) does not have this property.

Suppose there exists an element \( E \) of \( V_1 \) in \( X - \{p_1, q_1\} \) that meets both \( H_0 \) and \( L_0 \). Define \( K_1 = H_0 \cup E \cup L_0 \). Later \( E \) will be referred to as a selected 2-link.

Assume no element of \( V_1 \) misses \( [p_1, q_1] \) and meets both \( H_0 \) and \( L_0 \). Let \( d \) be the least upper bound (with respect to \( < \)) of the set of points \([y_1, q_1]\) that can be joined to \( H_0 \) by an element of \( V_1 \). Since the limit of a convergent sequence of links is a link, there is an element of \( V_1 \) that joins \( d \) to \( H_0 \) [12, Theorem 59, p. 24]. If possible, select an element \( H \) of \( V_1 \) joining \( d \) to \( p_1 \) and define \( K_1 = H \cup [d, q_1] \). If no such element exists, select an element \( E \) of \( V_1 \) missing \( p_1 \) that joins \( d \) to \( H_0 \) and define \( K_1 = H_0 \cup E \cup [d, q_1] \).

Assume that no element of \( V_1 \) meets both \( H_0 \) and \([y_1, q_1]\).

Suppose an element of \( V_1 \) meets \( L_0 \) and \([p_1, x_1]\). Let \( a \) be the first point of \([p_1, x_1]\) that can be joined to \( L_0 \) by an element of \( V_1 \). If possible, select an element \( L \) of \( V_1 \) that joins \( a \) to \( q_1 \) and define \( K_1 = [p_1, a] \cup L \). If no such element exists, select an element \( E \) of \( V_1 \) missing \( p_1 \) that joins \( a \) to \( L_0 \) and define \( K_1 = [p_1, a] \cup E \cup L_0 \).

Assume that no element of \( V_1 \) meets both \( L_0 \) and \([p_1, x_1]\). Let \( a_1 \) be the first point of \([p_1, x_1]\) that can be joined to \([y_1, q_1]\) by an element of \( V_1 \). There exist a continuum \( L_1 \) and a point \( q_2 \) of \([y_1, q_1]\) such that (1) \( L_1 \) is a selected element of \( V_1 \) that joins \( q_2 \) to \( a_1 \), and (2) \( q_2 \) is the first point of \([y_1, q_1]\) that can be joined to \( a_1 \) by an element of \( V_1 \).

Let \( d_1 \) be the last point of \([y_1, q_1]\) that can be joined to \([p_1, x_1]\) by an element of \( V_1 \). There exist a continuum \( H_1 \) and a point \( p_2 \) such that (1) \( H_1 \) is a selected element of \( V_1 \) that joins \( p_2 \) to \( d_1 \), and (2) \( p_2 \) is the last point of \([p_1, x_1]\) that can be so joined to \( d_1 \).

Suppose an element \( E \) of \( V_1 \) contains \( \{a_1, d_1\} \). Define \( K_1 = [p_1, a_1] \cup E \cup [d_1, q_1] \).

Assume \( \{a_1, d_1\} \) is not contained in an element of \( V_1 \).

Suppose there exists an element \( E \) of \( V_1 \) that does not contain \( p_1 \) and meets
both $H_0$ and $H_1$. Define $K_1 = H_0 \cup E \cup H_1 \cup [d_1, q_1]$.

Assume that each element of $V_1$ that meets both $H_0$ and $H_1$ contains $p_1$.

Suppose an element $E$ of $V_1$ in $X - \{q_1\}$ meets $L_0$ and $L_1$. Define $K_1 = [p_1, a_1] \cup L_1 \cup E \cup L_0$.

Assume that no element of $V_1$ in $X - \{q_1\}$ meets both $L_0$ and $L_1$.

Suppose an element of $V_1$ meets $[p_1, a_1]$ and $H_1$. Let $a$ be the first point of $[p_1, a_1]$ that can be joined to $H_1$ by an element of $V_1$. Select an element $E$ of $V_1$ that joins $a$ to $H_1$. Define $K_1 = [p_1, a] \cup E \cup H_1 \cup [d_1, q_1]$.

Assume that no element of $V_1$ meets both $[p_1, a_1]$ and $H_1$.

Suppose an element of $V_1$ meets $L_1$ and $[d_1, q_1]$. Let $d$ be the last point of $[d_1, q_1]$ that can be joined to $L_1$ by an element of $V_1$. Select an element $E$ of $V_1$ that joins $d$ to $L_1$. Define $K_1 = [p_1, a_1] \cup L_1 \cup E \cup [d, q_1]$.

Assume that no element of $V_1$ meets both $L_1$ and $[d_1, q_1]$.

Suppose an element of $V_1$ meets $H_0$ and $[p_2, x_1]$. Let $a$ be the first point of $[p_2, x_1]$ that can be joined to $H_0$ by an element of $V_1$. If possible, select an element $E$ of $V_1$ that joins $a$ to $p_1$ and define $K_1 = H_0 \cup [p_2, a] \cup H_1 \cup [d_1, q_1]$.

If no such element exists, select an element $E$ of $V_1$ missing $p_1$ that joins $a$ to $H_0$ and define $K_1 = H_0 \cup E \cup [p_2, a] \cup H_1 \cup [d_1, q_1]$.

Assume that no element of $V_1$ meets both $H_0$ and $[p_2, x_1]$.

Suppose an element of $V_1$ meets $[y_1, q_2]$ and $L_0$. Let $d$ be the last point of $[y_1, q_2]$ that can be joined to $L_0$ by an element of $V_1$. If possible, select an element $E$ of $V_1$ that joins $d$ to $q_1$ and define $K_1 = [p_1, a_1] \cup L_1 \cup [d, q_2] \cup L_0$.

If no such element exists, select an element $E$ of $V_1$ missing $q_1$ that joins $d$ to $L_0$ and define $K_1 = [p_1, a_1] \cup L_1 \cup [d, q_2] \cup E \cup L_0$.

Assume that no element of $V_1$ meets both $[y_1, q_2]$ and $L_0$.

Suppose an element of $V_1$ meets $[p_1, a_1]$ and $[p_2, x_1]$. Let $a$ be the first point of $[p_1, a_1]$ that can be joined to $[p_2, x_1]$ by an element of $V_1$. There exist a continuum $E$ and a point $b$ such that (1) $E$ is a selected element of $V_1$ that joins $b$ to $a$, and (2) $b$ is the first point of $[p_2, x_1]$ that can be so joined to $a$.

Define $K_1 = [p_1, a] \cup E \cup [p_2, b] \cup H_1 \cup [d_1, q_1]$.

Assume no element of $V_1$ meets both $[p_1, a_1]$ and $[p_2, x_1]$.

Suppose an element of $V_1$ meets $[y_1, q_2]$ and $[d_1, q_1]$. Let $d$ be the last point of $[d_1, q_1]$ that can be joined to $[y_1, q_2]$ by an element of $V_1$. There exist a continuum $E$ and a point $c$ such that (1) $E$ is a selected element of $V_1$ that joins $c$ to $d$, and (2) $c$ is the last point of $[y_1, q_2]$ that can be so joined to $d$.

Define $K_1 = [p_1, a_1] \cup L_1 \cup [c, q_2] \cup E \cup [d, q_1]$.

Assume no element of $V_1$ meets both $[y_1, q_2]$ and $[d_1, q_1]$.

If an element $E$ of $V_1$ can be selected that meets $L_1$ and $H_1$, define $K_1 = [p_1, a_1] \cup L_1 \cup E \cup H_1 \cup [d_1, q_1]$.

Assume no element of $V_1$ meets both $L_1$ and $H_1$. 
Suppose an element of $V_1$ meets $L_1$ and $[p_2, x_1]$. Let $a$ be the first point of $[p_2, x_1]$ that can be joined to $L_1$ by an element of $V_1$. Let $E$ be a selected element of $V_1$ that joins $a$ to $L_1$. Define $K_1 = [p_1, a] \cup L_1 \cup E \cup [p_2, a] \cup H_1 \cup [d_1, q_1]$.

Assume no element of $V_1$ meets both $L_1$ and $[p_2, x_1]$.

Suppose an element of $V_1$ meets $[y_1, q_2]$ and $H_1$. Let $d$ be the last point of $[y_1, q_2]$ that can be joined to $H_1$ by an element of $V_1$. Let $E$ be a selected element of $V_1$ that joins $d$ to $E$. Define $K_1 = [p_1, a] \cup L_1 \cup [d, q_2] \cup E \cup H_1 \cup [d_1, q_1]$.

Assume no element of $V_1$ meets both $[y_1, q_2]$ and $H_1$. Define $T_1 = [p_1, a] \cup L_1 \cup H_1 \cup [d_1, q_1]$.

If at this stage of the construction $K_1$ is defined, then let $Z_1 = K_1$. If $K_1$ is not defined, then repeat the process, using the continua $H_1$, $L_1$, the arc $[p_1, a]$, and the complementary domain $G_1$ as reference sets to define $K_2$ or $T_2$. Note that no 2-link in $X$ meets two distinct elements of $\{(p_1, a_1), (p_2, q_2), [d_1, q_1]\}$. Hence the resulting set, $K_2$ or $T_2$, meets $T_1$ only at the points $p_2$ and $q_2$. If $K_2$ is defined, let $Z_1 = K_2 \cup T_1$. If $K_2$ is not defined, repeat the process using $H_2$, $L_2$, $[p_3, q_3]$, and $G_1$ as reference sets.

If after this process is repeated $n$ (finitely many) times $K_n$ is defined, then let $Z_1 = K_n \cup \bigcup_{i=1}^{n-1} T_i$. If, for each positive integer $n$, the set $T_n$ is defined, let $Z_1 = \text{Cl} \bigcup_{i=1}^{\infty} T_i$. Note that $Z_1$ is a continuum in $S^2 - G_1$ that contains $\{p_1, q_1\}$.

2. The properties of $Z_1$. Let $M_1$ be the collection consisting of all selected 2-links that appear in the definition of $Z_1$. The continua that appear in the definition of $Z_1$ have the linear separating property; that is, for any two distinct continua $L$ and $E$ in $S^2 - \{p_1\}$ appearing in the definition of $Z_1$ (each of $L$ and $E$ is either a selected 2-link or an arc in $A \cap Z_1$ that is maximal with respect to not having an interior point in common with $\text{Cl}(\text{St} M_1)$), either $L$ separates $p_1$ from $E - L$ or $E$ separates $p_1$ from $L - E$ in $Z_1$ but not both. Note that only one continuum appearing in the definition of $Z_1$ contains $p_1$.

Note that $Z_1$ has the linear linking condition with respect to $M_1$; that is, no 2-link in $X$ meets more than one component of $Z_1 - \text{Cl}(\text{St} M_1)$ and if $J$ is a component of $Z_1 - \text{Cl}(\text{St} M_1)$ and $r$ is an endpoint of $J$, there is at most one point $v$ that is an endpoint of a component of $Z_1 - (J \cup \text{Cl}(\text{St} M_1))$ that can be joined to $r$ by a 2-link in $X$.

3. The second phase of the construction of $Q$. If no element of $\{G_n\}$ intersects $Z_1$, define $Q = Z_1$.

Assume that some element of $\{G_n\}$ meets $Z_1$. Define $Q_1 = Z_1$. Suppose without loss of generality that $G_2 \cap Q_1 \neq \emptyset$. Since no 2-link in $X$ meets more than
one component of \( Q_1 - \text{Cl(St } M_1) \), the domain \( G_2 \) intersects only one component \( (p_1^2, q_1^2) \) of \( Q_1 - \text{Cl(St } M_1) \cup \{p_1, q_1\} \). Using the construction methods of §1, define a continuum \( Z_2 \) such that \( Q_2 = (Q_1 - (p_1^2, q_1^2)) \cup Z_2 \) is a continuum in \( S^2 - (G_1 \cup G_2) \) that contains \( \{p_1, q_1\} \). Here we start with two selected 2-links in \( Z_1 \), the arc \( [p_1^2, q_1^2] \), and the domain \( G_2 \) as sets of reference for the construction of \( Z_2 \). Note that each basic set in any 2-link that is selected in the construction of \( Z_2 \) is by definition in \( [p_1^2, q_1^2] - G_2 \).

Assume without loss of generality that the continua used to define \( Q_2 \) (i.e., all continua except \( [p_1^2, q_1^2] \) appearing in the definitions of \( Q_1 \) and \( Z_2 \)) have the linear separating property. Note that it may be necessary to delete an element of \( M_1 \) in the definition of \( Q_2 \) to get this property.

Let \( M_2 \) be the collection of selected 2-links used to define \( Q_2 \). The continuum \( Q_2 \) has the linear linking condition with respect to \( M_2 \).

4. Continuing until no complementary domain meets the constructed set. If no element of \( G \) meets \( Q_2 \), define \( Q = Q_2 \). If this is not the case, we continue the process.

Assume the continuum \( Q_n \) is defined and \( M_n \) is the collection of all selected 2-links in \( Q_n \). If \( Q_n \) is contained in \( X \), define \( Q = Q_n \). Assume \( Q_n \) intersects some element of \( G \). There exists an integer \( i \) greater than \( n \) such that \( Q_n \cap G_i \neq \emptyset \) and \( Q_n \cap \bigcup_{i=1}^{n-1} G_i = \emptyset \). The domain \( G_i \) intersects only one component \( (p_i^{n+1}, q_i^{n+1}) \) of \( Q_n - (\text{Cl(St } M_n) \cup \{p_1, q_1\}) \). Using the construction methods of §1, define \( Z_{n+1} \) such that \( Q_{n+1} = (Q_n - (p_i^{n+1}, q_i^{n+1})) \cup Z_{n+1} \) is a continuum in \( S^2 - \bigcup_{j=1}^i G_j \) containing \( \{p_1, q_1\} \). We can assume without loss of generality that the continua used to define \( Q_{n+1} \) have the linear separating property. The continuum \( Q_{n+1} \) has the linear linking condition with respect to \( M_{n+1} \) the collection of all selected 2-links used to define \( Q_{n+1} \).

Suppose that for each positive integer \( n \), a continuum \( Q_n \) is defined. Define \( Q \) to be the limit of \( \{Q_n\} \).

Since in all cases \( Q \) is a continuum in \( X \) that contains \( \{p_1, q_1\} \), the construction is complete. Now we must show that \( Q \) is hereditarily decomposable.

5. Derived subcontinua of \( Q \). Suppose there exist an arc \( [r, v] \) in \( A \) and links \( F_1, F_2, \) and \( F_3 \) contained in \( X - [r, v] \) such that for each \( i = 1, 2, \) and \( 3, F_i \) is the limit of a convergent sequence \( \{E_n^i\} \) of links in \( X - \bigcup_{j=1}^3 F_j \) with the following properties:

1. For each \( n \), the link \( E_n^i \) is contained in a selected 2-link \( L \) and a point \( v_n^i \) of a basic set in \( L \) is in \( [r, v] \cap E_n^i \).
2. The sequence \( \{v_n^i\} \) converges to \( v \).

We now show that one of \( F_1, F_2, \) and \( F_3 \) is contained in the union of the other two.
Assume that for \( j = 1, 2, \text{ and } 3 \), there exists a point \( w_j \) of \( F_j - \bigcup_{i \neq j} F_i \). There exists a point \( s \) of \((r, v)\), a positive integer \( m \), an element \( E \) of \( \{ E_n^1 \} \), and circular regions \( U_1, U_2, \text{ and } U_3 \) whose closures are disjoint containing \( w_1, w_2, \text{ and } w_3 \) respectively such that (1) \( \text{Cl}(U_1 \cup U_2 \cup U_3) \) is in \( S^2 - [r, v] \), (2) \( E \) meets both \([r, s) \text{ and } U_3 \) and does not intersect \( \text{Cl}(U_1 \cup U_2) \), and (3) for each \( n \) greater than \( m \),

\[
(F_1 \cup E_n^3) \cap \text{Cl}(U_2 \cup U_3) = (F_2 \cup E_n^3) \cap \text{Cl}(U_1 \cup U_3) = (F_3 \cup E_n^3) \cap \text{Cl}(U_1 \cup U_2) = \emptyset.
\]

Let \( X_1 \) be a continuum in \( S^2 - E \) that is the union of an arc in \((s, v)\), an element of \( \{ E_n^1 \} \) in \( X - \text{Cl}(U_2 \cup U_3) \) that meets both \((s, v) \text{ and } U_1 \), and an element of \( \{ E_n^2 \} \) in \( X - \text{Cl}(U_1 \cup U_3) \) that meets both \((s, v) \text{ and } U_2 \).

There exists a continuum \( X_2 \) in \( S^2 - (E \cup X_1) \) that is the union of an arc in \((s, v)\), an element of \( \{ E_n^1 \} \) in \( X - \text{Cl}(U_2 \cup U_3) \) that meets both \((s, v) \text{ and } U_1 \), and an element of \( \{ E_n^2 \} \) in \( X - \text{Cl}(U_1 \cup U_3) \) that meets both \((s, v) \text{ and } U_2 \).

There exists a point \( t \) of \((s, v)\) such that no element of \( \{ E_n^3 \} \) meets both \((t, v) \text{ and } X_1 \cup X_2 \). Let \( T \) be the continuum \( F_1 \cup F_2 \cup X_1 \cup X_2 \cup \text{Cl}(U_1 \cup U_2) \). There is an element \( H \) of \( \{ E_n^3 \} \) in \( S^2 - T \) that meets both \((t, v) \text{ and } U_3 \). The disjoint continua \( F_1 \cup F_2, X_1, \text{ and } X_2 \) can be closely approximated by arcs in such a way that the existence of the continuum \( E \cup H \cup \text{Cl}(U_3) \) contradicts \([12, \text{ Theorem 116, p. 247}]\). Hence one of \( F_1, F_2, \text{ and } F_3 \) is contained in the union of the other two.

It follows that the collection of all links having the properties that \( F_1, F_2, \text{ and } F_3 \) have with respect to \([r, v]\) when partially ordered by inclusion has at most two maximal elements. Hence the union of this collection is a 2-link \( F \) in \( X \). Since \( r \) precedes \( v \) with respect to the order of \( A \), we shall refer to \( F \) as a continuum in \( Q \) that is derived from the left at \( v \). Similarly continua that are derived from the right at a point of \( A \) are 2-links in \( X \). A continuum that is derived from the left or from the right at a point \( z \) of \( A \) will be referred to as a derived continuum in \( Q \) with base point \( z \).

6. Definition of the collection \( C \). If \( E \) is an element of \( M_i \), for some \( i \), that is not deleted at a later stage of the construction to get the linear separating property or if \( E \) is an arc in \( A \cap Q \) that is maximal with respect to not having an interior point in common with \( \bigcup \text{Cl}(\text{St} M_i) \), then \( E \) is said to be a continuum used to define \( Q \). Let \( B \) denote the collection of all continua used to define \( Q \). Let \( D \) denote the collection consisting of all derived continua in \( Q \).

Define \( C \) to be the collection consisting of all continua \( Y \) that satisfy one of the following conditions:
(1) $Y$ is the union of an element $E$ of $B$ with all elements of $D$ that have their base points in $E$.

(2) There is an element $F$ of $D$ that does not have a base point in an element of $B$. The union of $F$ with all elements of $D$ that meet $F$ is $Y$.

Note that $Q - St C$ is a totally disconnected set in $A$.

For each continuum $E$ belonging to $B$, there are at most two elements of $D$ that have base points in $E$. Furthermore, for each element $F$ of $D$ at most one element of $B$ contains a base point of $F$. It follows that for each element $Y$ of $C$, there exist points $r$ and $t$ (not necessarily distinct) of $A$ and a collection $N$, consisting of finitely many links in $X$ and possibly one arc in $A \cap Q$ that is maximal with respect to not having an interior point in $\bigcup \operatorname{Cl}(St M_i)$, such that each element of $N$ meets $[r, t]$ and the union of $N$ is $Y$. The set $[r, t]$ will be referred to as the core set of $Y$.

7. An order relation in $C$ in terms of separating. Define the binary relation $\ll$ in $C$ as follows. For distinct elements $Y$ and $Z$ of $C$, $Y \ll Z$ if $Y$ contains $p_1$ or $Z$ separates $p_1$ from the set $Z - Y$ in $Q$. Since the continua at all stages of the construction have the linear linking condition and the sets used to define these continua have the linear separating property, $\ll$ is an order relation in $C$ (i.e., $\ll$ is transitive and satisfies the law of trichotomy). When checking the properties of $\ll$, it is helpful to note that for any two elements $Y$ and $Z$ of $C$, if $Y \ll Z$, then the $p_1$-component of $Q - Z$ meets $Y - Z$.

Note that if $V$, $Y$, and $Z$ are elements of $C$ such that $V \ll Y$ and $Y \ll Z$, then $V$ and $Z$ are mutually exclusive.

8. All continua in $Q$ are decomposable. Suppose there exists an indecomposable continuum $I$ in $Q$. Since $I$ is not contained in the union of finitely many links and arcs in $X$ and $Q - St C$ is totally disconnected, there exist mutually exclusive elements $V$, $Y$, and $Z$ of $C$ such that $V \ll Y$, $Y \ll Z$, and each component of $I$ meets both $V$ and $Z$. Note that since $I$ is not separated by any of its proper subcontinua, $Y$ cannot be an arc in $A$.

Suppose that the core set of $Y$ consists of two distinct points $r$ and $t$. The continuum $Y$ separates $V$ from $Z$ in $Q$. Hence there exist six disjoint continua $\{P_i\}_{i=1}^6$ in $I - [r, t]$, each intersecting $V$, $Z$, and a link in $X$ that meets $[r, t]$. Let $R$ and $U$ be disjoint circular regions in $S^2 - (V \cup Z \cup \bigcup_{i=1}^6 P_i)$ that are centered on $r$ and $t$ respectively. For each integer $i$ $(1 \leq i \leq 6)$, let $e_i$ be a point of $P_i$ that can be joined to either $r$ or $t$ by a link in $X$ and let $R_i$ be a circular region centered on $e_i$ in $S^2 - (V \cup Z \cup \bigcup_{j \neq i} P_j)$. For each $i$, since $e_i$
belongs to a link in $X$ that meets \{r, t\}, there exists an arc $J_i$ in a complementary domain of $X$ that meets both $R_i$ and a component of $R \cup U$.

It follows that some three of \{$J_i\}_{i=1}^6$ meet one component of $R \cup U$. Assume without loss of generality that $J_1$, $J_2$, and $J_3$ each meet $R$. The continua $V$, $Z$, $P_1$, $P_2$, and $P_3$ can be closely approximated by arcs in such a way that the existence of $J_1$, $J_2$, and $J_3$ contradicts [12, Theorem 116, p. 247].

Using an argument similar to the preceding, we can show that assuming the core set of $Y$ consists of one point also involves a contradiction. Hence $Q$ is hereditarily decomposable.

Continuity.

**Theorem 4.** Suppose $M$ is a hereditarily decomposable continuum, $f$ is a continuous function of $M$ into the plane, and $R$ is the union of finitely many links in $f(M)$. Then $R$ does not contain an indecomposable continuum.

**Proof.** Assume there exists an indecomposable continuum $I$ in $R$. According to Theorem 1, every subcontinuum of $f(M)$ that contains a nonempty open subset of $I$ contains $I$. It follows that $I$ is terminal in $f(M)$ [8, Theorem 2]. Since $M$ is hereditarily decomposable, this contradicts Theorem 2. Hence $R$ does not contain an indecomposable continuum.

**Theorem 5.** If $M$ is a \(\lambda\) connected continuum and $f$ is a continuous function of $M$ into the plane, then $f(M)$ is \(\lambda\) connected.

**Proof.** Suppose $p$ and $q$ are distinct points of $f(M)$. Since $M$ is \(\lambda\) connected, there exists a hereditarily decomposable continuum $H$ in $M$ that meets both $f^{-1}(p)$ and $f^{-1}(q)$. According to Theorems 3 and 4, $f(H)$ is \(\lambda\) connected. Hence there exists a hereditarily decomposable continuum in $f(H)$ that joins $p$ and $q$. It follows that $f(M)$ is \(\lambda\) connected.

**Comment.** Theorem 5 cannot be generalized to include all continuous images in Euclidean 3-space. There are continuous images of the topologist’s sine curve in $E^3$ that are not \(\lambda\) connected.

**Aposyndesis.**

**Theorem 6.** If $X$ is a semiaposyndetic plane continuum, then $X$ is \(\lambda\) connected.

**Proof.** Assume there exists an indecomposable continuum $I$ that is contained in the union of finitely many links in $X$. Let $x$ and $y$ be distinct points of $I$. According to Theorem 1, every subcontinuum of $X$ that contains a point of \{x, y\} in its interior relative to $X$ contains $I$. This contradicts the assumption that $X$ is semiaposyndetic. It follows from Theorem 3 that $X$ is \(\lambda\) connected.
Corollary. Every aposyndetic plane continuum is \(\lambda\) connected.

Example. An aposyndetic continuum in Euclidean 3-space need not be \(\lambda\) connected. To see this let \(M\) be the product of the pseudo arc [11] with itself. \(M\) is aposyndetic [9, Theorem 7] and not \(\lambda\) connected. In fact, \(M\) does not contain a hereditarily decomposable continuum. Since the pseudo arc is chainable, \(M\) is embeddable in Euclidean 3-space [2].

BIBLIOGRAPHY


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