COMPLEX APPROXIMATION FOR VECTOR-VALUED FUNCTIONS
WITH AN APPLICATION TO BOUNDARY BEHAVIOUR(1)

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ABSTRACT. This paper deals with the qualitative theory of uniform approximation by holomorphic functions. The first theorem is an extension to vector-valued mappings of N. U. Arakelian's theorem on uniform holomorphic approximation on closed sets. Our second theorem is on asymptotic approximation and yields, as in the scalar case, applications to cluster sets.

Let $D$ be a proper domain of the Riemann sphere, $D^*$ its one-point compactification, $F$ a (relatively) closed subset of $D$, $X$ a locally convex (complex) topological vector space, $H(D, X)$ the holomorphic mappings from $D$ to $X$, $H_F(D, X)$ the uniform closure of restrictions to $F$ of mappings in $H(D, X)$, and $A(F, X)$ the continuous mappings from $F$ to $X$ holomorphic on $F^0$.

Mergelian's celebrated theorem [11] gives necessary and sufficient conditions in order that $H_F(D, X) = A(F, X)$, for the case where $D = X = \mathbb{C}$, and $F$ is compact. Recently, E. Briem, K. B. Laursen, and N. W. Pedersen [5] have extended Mergelian's theorem to the case where $D = \mathbb{C}$, $F$ is compact, and $X$ is a locally convex space.

In a different direction Arakelian ([1], [2], and [3]) has generalized Mergelian's theorem to the case where $D$ is an arbitrary proper domain of the Riemann sphere, $F$ a (relatively) closed subset of $D$, and $X = \mathbb{C}$. Theorem 1 of the present paper extends Arakelian's theorem to the case where $X$ is a Fréchet space. We remark that the necessity of Arakelian's conditions holds for an arbitrary locally convex space.

The closed set $F \subset D$ is said to be a set of tangential, asymptotic, or Carleman approximation provided that for each $f \in A(F, \mathbb{C})$ and each positive continuous function $\epsilon$ on $F$, there is a function $g \in H(D, \mathbb{C})$ such that $|f(z) - g(z)| < \epsilon(z)$, $z \in F$. Arakelian [3] has given necessary and sufficient conditions for
tangential approximation in case $F^0 = \emptyset$. For general closed $F$, the problem has been completely solved by Nersesian [12]. Our Theorem 2 gives a vector-valued theorem on asymptotic approximation for the case where $F^0 = \emptyset$. As in the scalar case (see [8]), Theorem 2 yields powerful corollaries on boundary behaviour of holomorphic functions. In the scalar case, asymptotic approximation has also found deep applications to value distribution theory [3].

The proof for Banach spaces is a modification of the proof in the scalar case and requires the use of the Hahn-Banach theorem at several points. We include it in the Appendix as Theorem 3, partially for expository reasons, as the proof for the scalar case is not easily accessible and many details are left out. Our proof of Theorem 3 is modelled on Arakelian's proof for the case where $D = X = \mathbb{C}$ as presented by W. Fuchs [7] and Arakelian's proof where $D$ is a general domain and $X = \mathbb{C}$ [2].

We use the following notations. $S^2$ is the Riemann sphere; $\rho$ the chordal metric on $S^2$. For $E \subset S^2$, $\delta > 0$, $N_\rho(E, \delta) = \{z \in S^2; \rho(E, z) < \delta\}$, and $\bar{E}$ denotes the $S^2$-closure of $E$, $\partial E$ the $S^2$-boundary. For $z \in \mathbb{C}$, $\delta > 0$, $B(z, \delta) = \{\zeta \in \mathbb{C}; |z - \zeta| < \delta\}$. Let $D$ be a proper domain in $S^2$. A boundary curve $\alpha$ in $D$ is a continuous curve $\alpha: [0, 1) \to D$ such that $\alpha(t) \to \partial D$, as $t \to 1$. If $E$ and $X$ are topological spaces, $C(E, X)$ denotes the continuous mappings from $E$ to $X$; we write $C(E) = C(E, \mathbb{C})$ and $A(E) = A(E, \mathbb{C})$ etc. For $X$ a Fréchet space, $\{p_n\}$ will denote an increasing sequence of seminorms which generate the topology of $X$.

**Theorem 1.** Let $X$ be a Fréchet space, $D$ a proper domain of the Riemann sphere, and $F$ a (relatively) closed subset of $D$. Then

\[ H_F(D, X) = A(F, X) \]

if and only if $D^* \setminus F$ is connected and locally connected.

**Proof.** For the case where $X = \mathbb{C}$, the necessity was shown by Arakelian [2] (see also [9]). Now suppose $D^* \setminus F$ is not locally connected and connected. Then there is a function $f \in A(F, \mathbb{C})$ and an $\epsilon > 0$ such that

\[ |f(z) - g(z)| < \epsilon, \quad z \in F \]

is satisfied by no function $g \in H(D, \mathbb{C})$. Fix $x \in X$, $x \neq 0$, and choose a seminorm $p$ such that $p(x) = 1$. Then by the Hahn-Banach theorem, there is an $x^* \in X^*$ for which $x^*(x) = 1$, and $|x^*(y)| \leq p(y)$, for all $y \in X$. Suppose, to obtain a contradiction, that (1) holds. Then there is a holomorphic mapping $G \in H(D, X)$ for which

\[ p(f(z)x - G(z)) < \epsilon, \quad z \in F. \]

Thus

\[ |x^*(f(z)x - G(z))| < \epsilon; \]
that is, \(|f(z) - x^*(G(z))| < \epsilon\), which contradicts (2) and proves the necessity.

To prove sufficiency it is enough to show that \(A(F, X) \subset H_F(D, X)\) if \(D^* \setminus F\) is connected and locally connected. Let \(f \in A(F, X)\). We shall construct a Banach space \(Y \subset X\), with a stronger topology such that \(f\) is also in \(A(F, Y)\). Since the theorem is true for Banach spaces (Theorem 3), \(f \in H_F(D, Y)\). However, since the topology on \(Y\) is stronger than the topology on \(X\), \(H_F(D, Y) \subset H_F(D, X)\), which will establish (1).

Let \(\{\rho_n\}\) be a sequence of increasing seminorms which defines the topology for \(X\). If \(\{\omega_n\}\) is a sequence of positive numbers, then one easily shows that

\[\left\{ x \in X : \sup_n \frac{\omega_n \rho_n(x)}{} < \infty \right\} = Y \subset X\]

is a Banach space with a stronger topology than that of \(X\). Let \(\{D_n\}\) be an exhaustion of \(D\) by domains: \(\overline{D_n} \subset D_{n+1}\), \(n = 1, 2, \ldots\), and \(D = \bigcup_{n=1}^{\infty} D_n\). Set \(K_n = F \cap D_n\), and let \(K_n^i\) be an exhaustion of \(F^n\) by compact sets such that \(K_n^i \subset (K_n+1)^0\), \(n = 1, 2, \ldots\), and \(F^0 = \bigcup_{n=1}^{\infty} (K_n^i)\). We set

\[\omega_n^i = \frac{1}{1 + \sup_{K_n^i} \rho_n(f(z))}, \quad \omega_n^i = \frac{1}{1 + \sup_{K_n^i} \rho_n(f'(z))}\]

Lemma 1. For each \(j\),

\[\sup_{z \in K_j} \sup_n \omega_n^i \rho_n(f(z)) = M_j < \infty, \quad (3)\]

and

\[\sup_{z \in K_j^i} \sup_n \omega_n^i \rho_n(f'(z)) = M_j^i < \infty. \quad (4)\]

Proof. If \(z \in K_j\) and \(n \geq j\), then \(\omega_n^i \rho_n(f(z)) \leq 1\). While if \(n < j\), \(\rho_n \circ f\) is continuous and thus bounded on \(K_j\). This proves (3), and (4) is proved similarly.

Now set

\[\omega_n = \min\left(\omega_n^i, \omega_n^i\right)\]

and let \(Y\) be the Banach space associated with the sequence \(\{\omega_n\}\).

Lemma 2. With respect to the strong topology on \(Y\),

(i) \(f: F \to Y\) is continuous, and

(ii) \(f: F^0 \to Y\) is holomorphic.

Proof. Lemma 1 implies that \(f(F) \subset Y\) and \(f'(F) \subset Y\), where \(f'\) is the derivative with respect to the topology of \(X\).

To show (i) fix \(z \in F\) and choose \(j\) so that \(z \in K_j\). Then, for \(\zeta \in K_{j+1}\),
\[ \| f(z) - f(\zeta) \| = \sup_{n} \omega_{n} p_{n} (f(z) - f(\zeta)) \]
\[ \leq \sup_{n \geq N} \omega_{n} p_{n} (f(z) - f(\zeta)) + \sup_{n < N} \omega_{n} p_{n} (f(z) - f(\zeta)) \]
\[ \leq N^{-1} \sup_{n \geq N} \omega_{n} p_{n} (f(z) - f(\zeta)) + \sup_{n < N} \omega_{n} p_{n} (f(z) - f(\zeta)) \]
\[ \leq M_{j+1}/N + \sup_{n < N} \omega_{n} p_{n} (f(z) - f(\zeta)). \]

Given \( \epsilon > 0 \), choose \( N > 2M_{j+1}/\epsilon \), and choose \( \delta > 0 \) so that
\[ \zeta \in B(z, \delta) \cap F \Rightarrow \sup_{n < N} \omega_{n} p_{n} (f(z) - f(\zeta)) < \epsilon/2. \]

Hence for such \( \zeta \), \( \| f(z) - f(\zeta) \| < \epsilon \). This proves (i).

To show (ii), fix \( z \in F^0 \), and choose \( j \) and \( k \) such that \( z \in K_{j} \cap K_{k}^{'}. \) Now choose \( \delta > 0 \) so that \( B(z, \delta) \subset K_{j+1} \cap K_{k+1}^{'} \) and set \( \Gamma = \{ \zeta : |\zeta - z| = \delta \}. \) If \( |b| < \delta/2 \), then, for every \( n \),
\[ \omega_{n} p_{n} \left( \frac{f(z + b) - f(z)}{b} \right) = \omega_{n} p_{n} \left( \frac{1}{b} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} - f(\zeta) \frac{d\zeta}{\zeta - z} \right) \]
\[ \leq \frac{1}{2\pi} \int_{\Gamma} \frac{\omega_{n} p_{n} (f(\zeta)) |d\zeta|}{|\zeta - (z + b)||\zeta - z|} \]
\[ \leq M_{j+1} \frac{1}{2\pi} \int_{\Gamma} \frac{|d\zeta|}{|\zeta - (z + b)||\zeta - z|} \leq \frac{2}{\delta} M_{j+1}. \]

Consequently
\[ \sup_{|b| < \delta/2} \sup_{n} \omega_{n} p_{n} \left( \frac{f(z + b) - f(z)}{b} \right) \leq \frac{2}{\delta} M_{j+1}. \]

Let \( |b| < \delta/2 \) and \( f' \) be the derivative of \( f \) with respect to the topology of \( X \). Then
\[ \left\| \frac{f(z + b) - f(z)}{b} - f'(z) \right\| = \sup_{n} \omega_{n} p_{n} \left( \frac{f(z + b) - f(z)}{b} - f'(z) \right) \]
\[ \leq \sup_{n \geq N} \omega_{n} p_{n} (f(z + b) - f(z)) + \sup_{n < N} \omega_{n} p_{n} (f(z) - f(\zeta)). \]

As in the proof of (i), it is sufficient to show that, for given \( \epsilon \),
\[ \sup_{n \geq N} \omega_{n} p_{n} \left( \frac{f(z + b) - f(z)}{b} - f'(z) \right) < \frac{\epsilon}{2}, \]
provided \( N \) is sufficiently large. Now (7) is less than or equal to
The last inequality follows from (6) and (4). This establishes (7), which completes the proof of Lemma 2 and of the theorem.

Having considered the possibility of uniform approximation, we now turn to the possibility of asymptotic approximation.

Theorem 2. Let $X$ be a Fréchet space, $D$ a proper domain of the Riemann sphere, and $F$ a (relatively) closed subset of $D$ with empty interior. Then, for each mapping $f \in \mathcal{C}(F, X)$ and positive function $\epsilon \in \mathcal{C}(D, \mathbb{R}^+)$, there is a holomorphic mapping $g \in \mathcal{H}(D, X)$ such that for each continuous seminorm $p$,

$$(8) \quad p(f(z) - g(z)) = o(\epsilon(z)),$$

as $z \to \partial D$, $z \in F$, if and only if $D \setminus F$ is connected and locally connected.

Proof. In case $X = \mathbb{C}$, this theorem is essentially a corollary of Theorem 1. However in the general case, it seems that a detailed proof is required.

We remark that it is sufficient to show that for $n = 1, 2, \ldots$

$$(9) \quad p_n(f(z) - g(z)) = O(\epsilon(z)),$$

as $z \to \partial D$, $z \in F$. This is so, because for each $p$ there is an $n$ and a constant $A$ such that $p < Ap_n$, and by choosing an appropriate $\epsilon(z)$ we can replace $o$ by $O$.

We now prove (9). Let $D, F, f$, and $\epsilon$ be as in the hypothesis of the theorem. If $F$ is compact, then the theorem is vacuously true.

Suppose $F$ is not compact. Then we may assume $\lim \epsilon(z)$ is zero as $z$ approaches the boundary of $D$.

If $K$ is a compact set in $D$ we may choose a compact set $K^1 \subset D$ and containing $K$ such that $(D^* \setminus F) \setminus K^1$ is connected. Also set

$$(10) \quad K^2_n = \{z \in F: \epsilon(z) \geq \epsilon(\zeta), \text{ for some } \zeta \in K^1 \} \cup K^1.$$

Now let $\{K_n\}$ be a normal exhaustion of $D$ by compact sets inductively constructed so that $K_0 \cap F \neq \emptyset$ and $K^2_n \subset K^0_n$, $n = 1, 2, \ldots$. We may choose the $K_n$ in such a manner so that the $\epsilon_n$ strictly decrease to zero, where $\epsilon_n = \min \{\epsilon(z): z \in K^2_n\}$. Set $F_0 = F \cap K^2_0$ and, for $n = 1, 2, \ldots$, set

$$(11) \quad F_n = K^2_{n-1} \cup (F \cap K^2_n) = K^2_{n-1} \cup \{z \in F: \epsilon(z) \geq \epsilon_n\}.$$

It is not difficult to verify that $D^* \setminus F_n$ is connected. Thus $F_n$ is a "Mergelian set."
Set $f_0 = f$, for $z \in F_0$. Then by Mergelian's theorem [5], there is a mapping $g_0$ holomorphic on all of $D$ such that $p_1(g_0(z) - f_0(z)) < \epsilon_1/4$, $z \in F_0$. Now set $K_{-1} = \emptyset$, $\epsilon_{-1} = +\infty$, $g_{-1} = g_0$ and suppose mappings $g_j$, $j = 0, 1, \ldots, n - 1$, holomorphic on all of $D$, have been defined satisfying

\((10)\) \quad \rho_j(f(z) - g_j(z)) \leq \epsilon_j/2, \quad \text{if } \epsilon_j \leq \epsilon(z) \leq \epsilon_{j-1},
\]
\[(11)\) \quad \rho_{j+1}(f(z) - g_{j+1}(z)) \leq \epsilon_{j+1}/4, \quad \text{if } \epsilon(z) = \epsilon_j,
\]
and

\[(12)\) \quad \rho_j(g_j(z) - g_{j-1}(z)) < \epsilon_j/2^{j+1}, \quad z \in K_{j-1}^2.
\]

We wish to define $g_n$, but first we must define $f_n$. Set $f_n = g_{n-1}$ on $K_{n-1}^2$. The function $f_n$ is defined on the closed set $K_{n-1}^2$ and we shall define it on all of $F_n$. Set $H_n = f_n - f$. Then $H_n$ is defined on a closed set which does not meet the set

\[(13)\) \quad \{z \in F: \epsilon(z) = \epsilon_n\}.
\]

Set $H_n = 0$ on the set (13). $H_n$ is continuously defined on a closed subset of the normal space

\[(14)\) \quad \{z \in F: \epsilon_{n-1} \geq \epsilon(z) \geq \epsilon_n\}
\]
and is $p_n$-bounded by $\epsilon_n/4$. By Dugundji's extension of Tietze's theorem [6], we extend $H_n$ continuously to all of (14) so that the extension retains the same bound. Now $f_n = f + H_n$ is continuously defined on all of $F_n$ and is holomorphic on the interior. By Mergelian's theorem, there is a function $g_n$ holomorphic on all of $D$ such that $p_{n+1}(g_n(z) - f_n(z)) < \epsilon_{n+1}/2^{n+1}$, $z \in F_n$. From the way in which $f_n$ was defined, it follows that $g_n$ satisfies (10), (11), and (12).

Thus we define a sequence $\{g_n\}$ satisfying (10), (11), and (12). From (12) it is clear that $\{g_n\}$ converges to a mapping $g$ holomorphic on all of $D$. Now fix $p_n$ and let $z$ be any point of $F$ so close to the boundary of $D$ that $\epsilon(z) \leq \epsilon_{n-1}$. Choose $N$ so that $\epsilon_n(z) \leq \epsilon(z) \leq \epsilon_{N-1}$. Then for any $m > N$,

\[p_n(f(z) - g(z)) \leq p_n(f(z) - g_N(z)) + \sum_{j=N+1}^m p_j(g_j(z) - g_{j-1}(z)) + p_m(g_m(z) - g(z)).\]

By choosing $m$ sufficiently large, the last term can be made as small as desired, and so from (10) and (12) it follows that $p_n(f(z) - g(z)) < \epsilon_N \leq \epsilon(z)$. Since this holds for any $z$ for which $\epsilon(z) \leq \epsilon_{n-1}$, the proof is complete.

As a consequence of the preceding proof we have

**Corollary 1.** Under the hypotheses of Theorem 2, for each seminorm $p$, there is a holomorphic mapping $g \in H(D, X)$ for which $p(f(z) - g(z)) < \epsilon(z)$, $z \in F$. 
As a further application of Theorem 2 we have the following result, first proved by Kaplan [10] for the case that $X = \mathbb{C}$.

**Corollary 2.** Let $X$ be a Fréchet space, $D$ a proper domain of the Riemann sphere, $\{\alpha_n\}_{n=1}^{\infty}$ a family of disjoint boundary curves, and let $f \in C(\alpha_n, X)$, $n = 1, 2, 3, \ldots$, and $\epsilon \in C(\Omega, \mathbb{R}^+)$, then there exists a holomorphic mapping $g \in H(D, X)$ such that for each continuous seminorm $p$, and each $n = 1, 2, \ldots, p(f(z) - g(z)) = o(\epsilon(z))$, as $z \to \partial D$, along $\alpha_n$.

**Proof.** The proof is the same as for the case $X = \mathbb{C}$ (see [8]).

A holomorphic mapping $g \in H(D, X)$ is said to have asymptotic value $x \in X$ along a boundary path $\alpha$ in $D$ provided $f(z) \to x$ as $z \to \partial D$ along $\alpha$.

**Corollary 3.** Let $X$ be a Fréchet space, $D$ a proper domain of the Riemann sphere, $\alpha_n$, $n = 1, 2, \ldots$, a family of disjoint boundary curves, and $x_n$, $n = 1, 2, \ldots$, a sequence of elements of $X$. Then there is a holomorphic mapping $g \in H(D, X)$ which has asymptotic value $x_n$ along $\alpha_n$ for $n = 1, 2, \ldots$.

Note that by an appropriate choice of $\epsilon(z)$, one also obtains a mapping which approaches prescribed values with a prescribed speed along prescribed boundary curves. For the case where $X = \mathbb{C}$, Corollary 3 was proved by Bagemihl and Seidel [4] and by Rudin [13].

We conclude this section with some open problems.

**Conjecture 1.** Theorem 1 fails if $X$ is allowed to be a general locally convex space. A counterexample is required.

**Conjecture 2.** Theorem 1 holds for $X$ locally convex, $K$ compact, and $D$ an open Riemann surface.

**Conjecture 3.** Theorem 2 holds for $X$ Fréchet and $D$ an open Riemann surface.

**APPENDIX**

Theorem 3. If $X$ is a Banach space then $H_p(D, X) = A(F, X)$ if and only if $D^* \setminus F$ is connected and locally connected.

**Proof.** The following proof is based upon Arakelian's proof for the case where $D$ and $X$ are the plane as presented by W. Fuchs [7] and Arakelian's proof where $D$ is a general domain and $X$ the plane [2].

Four lemmas are required. We begin with a slightly modified lemma of Rudin.

**Lemma 1** (Rudin [14, p. 383]). Suppose $\sigma$ is a compact and connected subset contained in a disc of diameter $2r$, $\Omega = S^2 \setminus \sigma$ is connected, and the diameter of $\sigma$ is at least $0.99r$. Then there is a function $g \in H(\Omega)$ and a constant $b$, with the following property: If

$$Q(\zeta, z) = g(z) + (\zeta - b)g^2(z),$$

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the inequalities

\[(16) \quad |Q(\zeta, z)| < 100/r,\]
\[(17) \quad |Q(\zeta, z) - 1/(z - \zeta)| < 1000 r^2/|z - \zeta|^3,\]

hold for all \(z \in \Omega\) and for all \(\zeta\) in the disc.

**Lemma 2.** Let \(\sigma\) be a compact, connected subset of \(D\) and \(\gamma\) be a boundary curve originating at a point of \(\sigma\), \(E_0\) a closed subset of \(D\) such that \((\gamma \cup \sigma) \cap E_0 = \emptyset\). If \(Q(z)\) is a function holomorphic in \(S^2 \setminus \sigma\), \(\eta > 0\), then there exists a \(\psi\) in \(H(D)\) such that

\[(18) \quad |\psi(z) - Q(z)| < \eta \quad \text{for } z \in E_0.\]

**Proof.** The method of proof involves the standard technique of pole pushing. We include the proof for completeness.

Let \(\gamma\) be parameterized by \(z: [0, 1) \rightarrow D\). By finite induction one chooses

\[0 < \tau_1 < \tau_2 < \cdots < \tau_n < \tau_{n+1} < \cdots < 1\]

such that if \(t \geq \tau_n\) then \(\rho(z(t), \partial D) < 1/n\). Again by induction we choose domains \(G_n\) such that

(i) \(\overline{G_n} \cap E_0 = \emptyset\) for all \(n\),

(ii) \(G_1 \supset \sigma \cup z([0, \tau_1])\),

(iii) \(G_n \supset z([\tau_{n-1}, \tau_n])\), \(n > 1\),

(iv) \(G_n \subset N_{\rho}(\partial D, 1/(n - 2))\), \(n > 2\).

Using Runge’s theorem we construct inductively a sequence of rational functions \(\psi_n\) with the following properties:

(i) \(\psi_n\) has a unique pole at \(z(\tau_n)\),

(ii) \(|Q(z) - \psi_1(z)| < \eta/2\) for \(z \in S^2 \setminus G_1\),

(iii) \(|\psi_{n+1}(z) - \psi_n(z)| < \eta/2^n\) for \(z \in S^2 \setminus G_n\) and \(n > 1\).

On compact subsets of \(D\), \(\psi_n\) converges uniformly to a function \(\psi\) in \(H(D)\). If \(z \in E_0\), then

\[|Q(z) - \psi(z)| \leq |Q(z) - \psi_1(z)| + \sum_{n=2}^{\infty} |\psi_{n+1}(z) - \psi_n(z)| < \eta.\]

This completes the proof of the lemma.

**Lemma 3.** Let \(D\) be a domain of the finite plane. Then there exists a function \(\rho \in C^\infty(D)\) such that \(0 < \rho(z) < \rho(z, \partial D), |\overline{\partial} \rho| < 1\) for \(z \in D\).

**Proof.** See Arakélian [2, Lemma 4].

If \(\phi\) is a function from \(D\) into \(X\) which has continuous partial derivatives we say that \(\phi \in C^1(\overline{D}, X)\).
Lemma 4. Let $D$ be a domain of the finite plane and $f$ a continuous function from $D$ to $X$. For $e > 0$, there exist $\delta \in C^1(\mathbb{C})$ and $\phi \in C^1(D, X)$ such that $\delta|_{D \setminus D} = 0$, and for $z \in D$,

(19) \quad 0 < \delta(z) \leq \frac{1}{2} p(z, \partial D), \quad |\delta| < \min \{e, d^{1/2}(z)\},

(20) \quad \|\phi(z) - f(z)\| < e, \quad \|\phi(z)\| < e/\delta(z).

Furthermore, if for some point $z \in D$, the function $f$ is holomorphic in the disc $|\zeta - z| < \delta(z)$, then $\phi(z) = f(z)$.

Proof. Taking $\epsilon < 1$, we shall construct the function $\delta(z)$. For the compact set $K_\lambda \subset \mathbb{C}$, $K = \{z \in D: p(z, \partial D) \geq \lambda\}$, the condition $z \in K_\lambda$ and $|\zeta - z| \leq \lambda/2$ implies that $\zeta \in K_{\lambda/2}$. Thus for $\delta$ a constant, $\delta \leq \lambda/2$, we may set

$$
\omega(\lambda, \delta) = \max_{x \in K_\lambda, \zeta - z \leq \delta} \|f(\zeta) - f(z)\|.
$$

Note that $\omega(\cdot, \delta)$ is nonincreasing, $\omega(\lambda, \cdot)$ is nondecreasing and $\omega(\lambda, 0) = 0$.

Thus we may define $\delta_1(\lambda)$ as the unique value of $\delta$ for which $\omega(\lambda, \delta) + \delta = \epsilon \lambda/2$.

Note that $\delta_1(\lambda)$ is strictly increasing. Now define $\delta_2(\lambda)$ by

$$
\delta_2^{1/2}(\lambda) = \frac{1}{2} \int_0^\lambda \delta_1^{1/2}(t) \, dt, \quad 0 < \lambda < 1.
$$

Evidently, we have

(21) \quad \delta_2^{1/2}(\lambda) < \delta_1(\lambda) \leq \epsilon \lambda/2 < \lambda/2.

Also, a direct calculation gives

$$
\delta_2'(\lambda) = \delta_2^{1/2}(\lambda) \delta_1^{1/2}(\lambda) < \min \{\epsilon/2, \delta_2^{1/2}(\lambda)\}.
$$

For $z \in D$, we set $\delta(z) = \delta_2(p(z))$, where $p$ is the function in Lemma 3. (19) follows since

$$
0 < \delta(z) = \delta_2(p(z)) < p(z)/2 < p(z, \partial D)/2
$$

and

$$
|\delta(z)| = |\delta_2'(p(z))| p(z).
$$

For $z \in \mathbb{C} \setminus D$, set $\delta(z) = 0$. We must verify that $\delta \in C^1(\mathbb{C})$. This follows from (21) and the fact that $\delta_2'$ is continuous and $p \in C^1(D)$.

The mapping $\phi$ is defined by the vector-valued integral

(22) \quad \phi(z) = \{\pi \delta_2^2(z)\}^{-1} \int_{|\zeta - z| < \delta(z)} f(\zeta) \, ds_\zeta,

where $ds_\zeta$ is Lebesgue planar measure. It is clear from (22) that if $f$ is holomorphic in the disc $|\zeta - z| < \delta(z)$, then $x^*(\phi(z)) = x^*(f(z))$, for each $x^* \in X^*$.
which implies by the Hahn-Banach theorem, that \( \phi(z) = f(z) \).

Furthermore
\[
\| \phi(z) - f(z) \| \leq \max_{|\zeta - z| \leq \delta(z)} \| f(\zeta) - f(z) \| \leq \omega(\rho(z), \delta(z)) = \omega(\rho(z), \delta(\rho(z))) \leq \omega(\rho(z), \delta(\rho(z))) < \epsilon \rho(z)/2 < \epsilon/2 < \epsilon.
\]

To complete the proof, there remains only to verify the second inequality of (20).

\[
\overline{\partial} \phi(z) = -\frac{2\overline{\partial}(z)}{\delta(z)} \phi(z) + \frac{1}{\pi \delta^2(z)} \int_{|\zeta - z| \leq \delta(z)} \frac{\partial}{\partial \overline{\theta}} f(\zeta) d\overline{s}_\zeta + \frac{1}{\pi \delta^2(z)} \int_{|\zeta - z| \leq \delta(z)} f(\zeta) d\overline{s}_\zeta.
\]

Let \( \psi(z, \delta) = \int_{|\zeta - z| < \delta} f(\zeta) d\overline{s}_\zeta \). Then
\[
\overline{\partial} \psi(z, \delta) = \left[ \frac{\partial \psi}{\partial \overline{\theta}} (z, \delta) \overline{\partial}(z) + \frac{\partial \psi}{\partial z} (z, \delta) dz \right] |_{\delta = \delta(z)}.
\]

Since \( \overline{\partial} z = 0 \), we have
\[
\overline{\partial} \phi(z) = -\frac{2\overline{\partial}(z)}{\delta(z)} \phi(z) + \frac{1}{\pi \delta^2(z)} \int_{|\zeta - z| \leq \delta(z)} \frac{\partial}{\partial \overline{\theta}} f(\zeta) d\overline{s}_\zeta + \frac{1}{\pi \delta^2(z)} \int_{0}^{2\pi} f(z + \delta(z) e^{i\theta}) d\theta du du \overline{\partial}(z) = \frac{2\overline{\partial}(z)}{\delta(z)} \left[ -\phi(z) + \frac{1}{2\pi} \int_{0}^{2\pi} f(z + \delta(z) e^{i\theta}) d\theta \right]
\]
\[
= \frac{2\overline{\partial}(z)}{\delta(z)} \left[ f(z) - \phi(z) + \frac{1}{2\pi} \int_{0}^{2\pi} f(z + \delta(z) e^{i\theta} - f(z)) d\theta \right].
\]

Thus,
\[
\| \overline{\partial} \phi(z) \| \leq \frac{2\epsilon}{\delta(z)} \left[ \epsilon + \frac{1}{2\pi} \int_{0}^{2\pi} \omega(\rho(z), \delta(\rho(z))) d\theta \right] \leq \frac{3\epsilon^2}{\delta(z)},
\]
and the proof of Lemma 4 is complete.

Having established the required lemmas, we now proceed with the proof of the theorem. Let \( D^* \setminus F \) be connected and locally connected. If \( f \in A(F, X) \), then we shall prove that \( f \in H_F(D, X) \). Since \( X \) is an absolute retract [6], we may assume that \( f \in C(D, X) \). We may also assume \( \phi \neq D \). Given \( \epsilon > 0 \), let \( \phi \) and \( \delta \) be the functions from Lemma 4, and \( T \) the support of \( \overline{\partial} \phi \). Also, let \( \{ D_n \} \) be a normal exhaustion of \( D \). That is, for each \( n \), \( D_n \) is a domain bounded by finitely many disjoint smooth Jordan curves, \( \overline{D}_n \subset D_{n+1} \), and \( D = \bigcup_{n=1}^{\infty} D_n \). Using the Hahn-Banach theorem, one easily obtains the following generalization of Pompeiu's formula [7, p. 14]

\[
\phi(z) = \frac{1}{2\pi i} \int_{\partial D_n} \frac{\phi(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \int_{\partial D_n} \overline{\partial} \phi(\zeta) \overline{\partial}(\zeta) - z d\overline{s}_\zeta,
\]
for \( z \in D_n \).
Let $\zeta_0$ be such that $d(\zeta_0, D \setminus F) \leq \delta(\zeta_0)$. Therefore there is a point $\zeta' \in (D \setminus F) \cap B(\zeta_0, 1.001 \delta(\zeta_0))$. Consequently, since $D \setminus F$ is connected and locally connected $\zeta'$ can be joined to the boundary of $D$ by a curve $\gamma_0 \subset D \setminus F$, such that for each $m$ there is an $n > m$ such that

$$(24) \quad \gamma_0 \cap D_m = \emptyset \quad \text{if} \quad \zeta_0 \in D \setminus D_n.$$  

Moreover, one can find an arc $\sigma_0$ of $\gamma_0$ contained in $B(\zeta_0, 2\delta(\zeta_0))$ and of diameter $0.99 \delta(\zeta_0)$. An application of Lemma 1 establishes the existence of a function $Q_0(\zeta, z)$ linear in $\zeta$, $Q_0(\zeta, \cdot) \in H(S^2 \setminus \sigma_0)$,

$$|Q_0(\zeta, z)| < \frac{100}{\delta(\zeta_0)}, \quad \left| Q_0(\zeta, z) - \frac{1}{\zeta - z} \right| < \frac{1000 \delta^2(\zeta_0)}{|\zeta - z|^3},$$

where $\zeta \in B(\zeta_0, 2\delta(\zeta_0))$ and $z \in S^2 \setminus \sigma_0$. By restricting $\zeta$ to a sufficiently small disc $B_{\zeta_0}$ with center at $\zeta_0$ we arrive at the following inequalities:

$$(25) \quad |Q_0(\zeta, z)| < 200/\delta(\zeta),$$

$$(26) \quad |Q_0(\zeta, z) - 1/(\zeta - z)| < 2000 \delta^2(\zeta)/|\zeta - z|^3,$$

where $\zeta \in B_{\zeta_0}$ and $z \in S^2 \setminus \sigma_0$.

The family $\{B_{\zeta} : \zeta \in T\} = \mathcal{B}$ is an open cover of $D_n \cap T$ for all $n$. By finite induction and the Heine-Borel theorem we can extract a sequence $B_n$ with associated functions $Q_n(\zeta, z)$, $n = 1, 2, \ldots$, such that $T \cap \bigcup_{j=1}^{n} B_j$ is strictly increasing and for a subsequence of indices $n_k$, $D_n \cap T \subset \bigcup_{j=1}^{n_k} B_j$.

For $\zeta \in T$, we define $Q_i(\zeta, z) = Q_j(\zeta, z)$, where $j$ is the smallest index such that $\zeta \in B_j$. Let $G_n = T \cap (B_n \setminus \bigcup_{j=1}^{n-1} B_j)$. Then we have

$$(27) \quad Q(\zeta, z) \text{ is linear on } G_n.$$  

If $\zeta \in G_n$, $Q(\zeta, \cdot)$ is holomorphic on $\Omega_\zeta = S^2 \setminus \sigma_n \supset F$,

$$(28) \quad |Q(\zeta, z)| \leq 200/\delta(\zeta),$$

for $\zeta \in T$ and $z \in \Omega_\zeta$.

The integral

$$l = \frac{1}{\pi} \iint_{T \cap D_n} \overline{\phi'(\zeta)} \left( Q(\zeta, z) - \frac{1}{\zeta - z} \right) ds_\zeta, \quad z \in F,$$
exists by (27), and the fact that $T \cap D_n$ is contained in a finite union of the $G_k$'s. We shall show that
\begin{equation}
\|f\| < A\epsilon,
\end{equation}
where $A$ is a constant independent of $n$ and of $z \in E$. For $V_z = \{\zeta \in D : |\zeta - z| < 4\delta(\zeta)\}$,
\[\|f\| \leq \frac{1}{\pi} \int_{T \cap D_n} \frac{1}{\delta(\zeta)} \left| Q(\zeta, z) - \frac{1}{\zeta - z} \right| ds_\zeta \]
\[= \int_{T \cap D_n} \frac{1}{\delta(\zeta)} ds_\zeta + \int_{T \cap D_n} \frac{1}{\delta(\zeta)} ds_\zeta \leq A\epsilon[1 + I_2],
\]
by (29) and (30), where
\begin{equation}
I_1 = \int_{V_z} \frac{ds_\zeta}{\delta(\zeta)|\zeta - z|}, \quad I_2 = \int_{C \setminus V_z} \frac{\delta(\zeta)}{|\zeta - z|^3} ds_\zeta.
\end{equation}
Because of (19) and the mean-value theorem,
\begin{equation}
|\delta(\zeta) - \delta(z)| \leq 2\epsilon|\zeta - z|, \quad \zeta, z \in D.
\end{equation}
Hence if $\epsilon < 1/16$, we obtain for $\zeta \in V_z$, $\delta(z)/2 < \delta(\zeta) < 2\delta(z)$, which implies that $V_z \subset B(z, 8\delta(z))$. Therefore
\[I_1 < \frac{1}{8\delta(z)} \int_{B(z, 8\delta(z))} \frac{ds_\zeta}{|\zeta - z|} = \frac{32\pi}{3}.
\]
Analogously $C \setminus V_z \subset C \setminus B(z, 2\delta(z))$ and so
\[I_2 < 2\pi + \int_{B(z, 1) \setminus B(z, 2\delta(z))} \frac{\delta(\zeta)}{|\zeta - z|^3} ds_\zeta = 2\pi + \int_0^{2\pi} I_3(\theta) d\theta.
\]
Furthermore, taking (19) into account, and the estimate $\delta(\zeta) \leq |\zeta - z|$, $\zeta \in C \setminus B(z, 2\delta(z))$, and integrating by parts,
\[I_3 = -\int_{2\delta(z)}^1 \delta(z + \rho e^{i\theta}) d(1/\rho)
\]
\[< \frac{\delta(z + 2\delta(z) e^{i\theta})}{2\delta(z)} + \int_{2\delta(z)}^1 \frac{\delta(1/2(z + \rho e^{i\theta})}{\rho} d\rho < 1 + \int_0^1 \frac{d\rho}{\sqrt{\rho}} = 3.
\]
This completes the proof of (31); which says that the function
\[b_n(z) = \frac{1}{2\pi} \int_{\partial D_n} \frac{\phi(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \int_{T \cap D_n} \overline{Q(\zeta, z)} ds_\zeta
\]
satisfies
\begin{equation}
|\phi(z) - b_n(z)| < A\epsilon, \quad z \in F \cap D_n,
\end{equation}
where $A$ is a constant independent of $n$.

Recalling that for $\zeta \in G_j$, $Q(\zeta, z) = g_j(z) + (\zeta - b_j)g_j^2(z)$, where $g_j$ is holomorphic in $S^2 \setminus \sigma_j$, we invoke Lemma 2 to obtain a function $\psi_j(z)$ holomorphic in $D$ such that
\[ |g_j(z) - \psi_j(z)| < \eta_j, \quad z \in F \cup \overline{D}_{k(j)} \]

where \( k(j) \) is the largest index for which \( \overline{D}_k \) does not meet the boundary curve \( \gamma_j \) of which \( \sigma_j \) is an arc, and \( \eta_j \) is chosen so small that setting

\[ G(\zeta, z) = \psi_j(z) + (\zeta - b_j)\psi'_j(z), \]

\( \zeta \in G_j, \ z \in D, \) we have

\[ |G(\zeta, z) - Q(\zeta, z)| < \delta(\zeta)/ (1 + |\zeta|^3), \]

for \( \zeta \in G_j \) and \( z \in F \cup \overline{D}_{k(j)} \). From (35), \( G(\cdot, z) \) is integrable over \( T \cap D_n \), and we have

\[ \left\| \frac{1}{\pi} \int_{T \cap D_n} \delta(\zeta)|Q(\zeta, z) - G(\zeta, z)| \, ds \zeta \right\| \leq \frac{1}{\pi} \int_{T \cap D_n} \frac{\delta(\zeta)}{1 + |\zeta|^3} \, ds \zeta \]

\[ \leq \frac{\varepsilon}{\pi} \int_{C} \frac{ds \zeta}{1 + |\zeta|^3} = A \varepsilon, \]

for \( z \in F \), where \( A \) is a constant independent of \( z \) and of \( n \).

Thus we have shown that the function

\[ H_n(z) = \frac{1}{2\pi i} \int_{\partial D_n} \frac{\phi(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{\pi} \int_{T \cap D_n} \overline{\delta(\zeta)}G(\zeta, z) \, ds \zeta, \]

holomorphic in \( D_n \), satisfies

\[ \|f(z) - H_n(z)\| < A \varepsilon, \quad z \in F \cap D_n, \]

where \( A \) is independent of \( n \) and \( z \).

To complete the proof we need only show that as \( n \to \infty \), \( H_n \) converges to a mapping \( g \), holomorphic on all of \( D \). Let \( K \) be a compact set in \( D \). Choose \( n_0 \) such that \( K \subset D_{n_0} \). By (24) there is an \( N_1 \) such that \( j \geq N_1 \implies k(j) \geq n_0 \). Now choose \( N_2 > N_1 \) such that \( T \setminus D_{N_2} \subset \bigcup_{j=N_1}^{\infty} G_j \). Suppose \( N_2 < m < n \). Then, for \( z \in K \),

\[ H_n(z) - H_m(z) = \frac{1}{2\pi i} \left( \int_{\partial D_n} - \int_{\partial D_m} \right) \frac{\phi(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{\pi} \int_{T \cap (D_n \setminus D_m)} \overline{\delta(\zeta)}G(\zeta, z) \, ds \zeta. \]

The formula of Pompeiu then gives:

\[ H_n(z) - H_m(z) = \frac{1}{\pi} \int_{T \cap (D_n \setminus D_m)} \overline{\delta(\zeta)} \left\{ \frac{1}{\zeta - z} - G(\zeta, z) \right\} \, ds \zeta \]

\[ = \frac{1}{\pi} \int_{T \cap (D_n \setminus D_m)} \overline{\delta(\zeta)} \left\{ \frac{1}{\zeta - z} - Q(\zeta, z) \right\} \, ds \zeta \]

\[ + \frac{1}{\pi} \int_{T \cap (D_n \setminus D_m)} \overline{\delta(\zeta)}|Q(\zeta, z) - G(\zeta, z)| \, ds \zeta \]

\[ = I_1 + I_2. \]
The choice of $N_2$ yields the following estimate:

$$\| f_1 \| \leq \frac{1}{\pi} \int_{D_n \setminus D_m} \frac{\epsilon}{\delta(\zeta)} \frac{d^2(\zeta)}{|\zeta - z|^3} ds_{\zeta} \leq \frac{2000}{\pi} \int_{D_n \setminus D_m} \frac{ds_{\zeta}}{|\zeta - z|^3};$$

$$\| f_2 \| \leq \frac{1}{\pi} \int_{D_n \setminus D_m} \frac{\epsilon}{\delta(\zeta)} \frac{\delta(\zeta)}{1 + |\zeta|^3} ds_{\zeta} \leq \frac{1}{\pi} \int_{D_n \setminus D_m} \frac{ds_{\zeta}}{1 + |\zeta|^3}.$$  

These estimates show that $f_1$ and $f_2$ converge to zero, uniformly on $K$, as $n, m \to \infty$. Thus $H_n$ converges, uniformly on compact subsets of $D$, and the proof of Theorem 3 is complete.

REFERENCES


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