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EXTENSIONS OF THE v-INTEGRAL

BY

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ABSTRACT. In *Representations for transformations continuous in the BV norm* [J. R. Edwards and S. G. Wayment, Trans. Amer. Math. Soc. 154 (1971), 251–265] the v-integral is defined over intervals in E^1 and is used to give a representation for transformations continuous in the BV norm. The functions f considered therein are real valued or have values in a linear normed space X, and the transformation $T(f)$ is real or has values in a linear normed space Y. In this paper the v-integral is extended in several directions: (1) The domain space to (a) E^n, (b) an arbitrary space S, a field Σ of subsets of S and a bounded positive finitely additive set function μ on Σ (in this setting the function space is replaced by the space of finitely additive set functions which are absolutely continuous with respect to μ); (2) the function space to (a) bounded continuous, (b) C_c, (c) C_0, (d) C with uniform convergence on compact sets; (3) range space X for the functions and Y for the transformation to topological vector spaces (not necessarily convex); (4) when X and Y are locally convex spaces, then a representation for transformations on a C_1-type space of continuously differentiable functions with values in X is given.

1. **Introduction.** In [9] the v-integral is defined over finite intervals in E^1, Euclidean one-space, and is used to give a representation for transformations continuous in the BV norm. The functions f considered therein are real valued or have values in a linear normed space X and the transformation $T(f)$ is real valued or has values in a linear normed space Y. The purpose of this paper is to extend the v-integral in several directions. In §2 the v-integral is defined on finite intervals in E^n and representation theorems are given which are analogous to those in [9]. For simplicity in writing, we give the development in E^2, from which the extension to E^n is obvious. Then the definition of the v-integral is extended to the setting where E^n is replaced by an arbitrary space S with a field Σ of subsets and a bounded positive finitely additive set function μ defined on Σ. The analog of the absolutely continuous functions is the space of finitely additive set functions $\mathfrak{A}_F(\mu)$ which are absolutely continuous with respect to μ. Using a result due to C. Fefferman [10] and this generalized v-integral,

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a characterization of the continuous linear functionals on $\mathfrak{A}_F(\mu)$ is given. In §3 we return to the setting of functions defined on E^n and the v-integral is extended beyond the finite interval, and also an extended integral [8] is given. Representation theorems are given for transformations T into Y of spaces F of X valued functions where: (a) F is C_R, the space of continuous functions on an interval with the topology of uniform convergence, and X and Y are TVS's; (b) F is C_c, the space of continuous functions of compact support with the topology of uniform convergence, X and Y are TVS's; (c) F is C, the space of continuous functions with the topology of uniform convergence on compacta, X and Y are TVS's; (d) F is L_k, a Lebesgue-type space [8], X and Y are convex spaces. §4 parallels the development in [7] to give a general representation theorem for function spaces with topologies no stronger than a C_1-type topology (which is stronger than the BV topology). In this section X and Y are convex spaces.

2. **Extending the domain.** As mentioned in the introduction, the extension will be made to E^2 from which the extension to E^n is obvious. We shall assume further that the interval of interest is the square $\{(s, t) | 0 \leq s \leq 1;\ 0 \leq t \leq 1\}$. If g is a function of two variables, we generate the additive set function G defined on the collection $\mathfrak{J}$ of regular intervals I of the form $I = \{(s, t) | 0 \leq a < s \leq b \leq 1;\ 0 \leq c < t \leq d \leq 1\} = (a, b;\ c, d)$ as follows: $G(I) = g|_a^b|_c^d = g(b, d) - g(a, d) - g(b, c) + g(a, c)$. We shall use the notions of "bounded variation" and "absolutely continuous" in the sense of Hardy as defined in [12], that is, g is absolutely continuous or of bounded variation if G is. If we let $g^*(s, t) = g(s, t) - g(s, 0) - g(0, t) + g(0, 0)$ be the associated anchored [12] function, then $G^*(I) = G(I)$ for each $I \in \mathfrak{J}$. Except for Theorem 2.2 of this section, we shall henceforth assume the point functions g to be anchored on the axes.

It is known [4] that if g is absolutely continuous in the sense of Hardy then $Dg = g_{st} = g_{ts}$ almost everywhere and $g(s, t) = G(I) = \int_0^s \int_0^t Dg$ where $I = (0, s;\ 0, t)$ and the double integral is the Lebesgue integral in the plane, or the iterated one-dimensional Lebesgue integrals (from the Fubini and Tonelli theorems). If we let $V_R g = V_R G$ represent the variation in the sense of Hardy over $R = (0, 1;\ 0, 1)$, then it can be shown that for absolutely continuous functions $V_R g = \iint_R |Dg|$. We consider the normed space of anchored absolutely continuous functions with $\|g\| = V_R g$.

If $I = (a, b;\ c, d)$, denote the characteristic function of I by χ_I and let $|I| = (b - a)(d - c)$. Let the fundamental function Ψ_I be defined by the equation $\Psi_I(s, t) = |I|^{-1} \int_0^s \int_0^t \chi_I$. Note that $\Psi_I(s, t) = 1$ if $s \geq b$ and $t \geq d$. We remark that the linear span of the fundamental functions contains the doubly polygonal functions where by the statement that P_σ is doubly polygonal we mean the following: Let the regular mesh σ denote the intersections of a finite number $n + 1$ of

vertical lines determined by $\{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ and a finite number of horizontal lines $m+1$ determined by $\{0 = t < t_0 < t_1 < t_2 < \cdots < t_m = 1\}$, such that each rectangle formed is of order M for some fixed M [11]. If $s_{i-1} < s \leq s_i$ and $t_{j-1} < t \leq t_j$ then $u = P_\sigma(s, t)$ lies on the line determined by the plane parallel to the $s - u$ plane passing through t and the two piercing points of the two line segments joining $P_\sigma(s_{i-1}, t_{j-1})$ to $P_\sigma(s_{i-1}, t_j)$ and $P_\sigma(s_i, t_{j-1})$ to $P_\sigma(s_i, t_j)$ respectively and also lies on the line determined by the plane parallel to the $t - u$ plane passing through s and the two piercing points of the two line segments joining $P_\sigma(s_{i-1}, t_{j-1})$ to $P_\sigma(s_i, t_{j-1})$ and $P_\sigma(s_{i-1}, t_j)$ to $P_\sigma(s_i, t_j)$ respectively.

Let pg_σ denote the doubly polygonal function formed such that $pg_\sigma(s_i, t_j) = g(s_i, t_j)$. It follows as in [9] that if $\langle J_n \rangle$ is a sequence of regular meshes whose norm (maximum diameter of rectangles) tends to zero, then pg_σ converges to g in the BV norm since

$$pg_\sigma = \sum_{i,j=1}^{n,m} \left(g\Big|_{s_{i-1}}^{s_i}\Big|_{t_{j-1}}^{t_j}\right)\Psi_{I_{ij}}(s, t).$$

In analogy with [9] we shall say that the set function K defined on I is *convex with respect to area* provided that, if I is the disjoint union of rectangles $\langle I_i \rangle_{i=1}^k$, then $K(I) = \sum_{i=1}^k \lambda_i K(I_i)$ where $\lambda_i = \text{area}(I_i)/\text{area}(I)$. Let the *fundamental bound* WK *for* K be defined by $WK = \sup_{I \in \mathfrak{I}} K(I)$. Define *the v-integral over* R as follows:

$$v \iint_R K\,dg = v \iint_R K\,dG = \lim_{|\sigma| \to 0} \sum_{i,j=1}^{n,m} K(I_{ij})\Delta_{ij}G \quad \text{where } \Delta_{ij}G = g\Big|_{s_{i-1}}^{s_i}\Big|_{t_{j-1}}^{t_j}$$

and the meshes are restricted to regular meshes.

We remark that in the preceding it is not necessary to restrict one's considerations to regular meshes determined by a finite number of vertical lines intersecting a finite number of horizontal lines. It would suffice to consider partitioning a given rectangle into any collection of regular rectangles and then defining the integral to be the limit with respect to the norm of the partition, or the net-type limit with respect to refinement of regular partitions. The only change would be that then one would simply define a polygonal function to be anything in the algebraic linear span of the collection of fundamental functions. We chose to partition only with collections of horizontal and vertical lines because any regular partition can be refined to be a partition by horizontal and vertical lines, and for computational purposes such refinements would probably be used. In §3 where a Lebesgue-type space is developed we shall use the more general polygonal functions and the more general type of partitions.

2.1. **Theorem.** *Let $\mathcal{G}^*$ be the set of all anchored, absolutely continuous functions over R with values in E^1 and normed with $\|g\| = V_R g$. Then T is a continuous linear functional on $\mathcal{G}^*$ if and only if there exists a fundamentally bounded, convex with respect to area set function K defined on $\mathcal{I}$ such that $T(g) = v\iint_R K\,dg$. Further, $\|T\| = WK$, the fundamental bound on K.*

The proof of Theorem 2.1 is precisely the same as 4.2 in [9]. Now suppose we consider the class $\mathcal{G}$ of absolutely continuous functions of two variables defined on R and having the property that $g(s, 0)$ and $g(0, t)$ are absolutely continuous as functions of one variable. Define

$$\|g\| = V_R G + V_0^1 g(s, 0) + V_0^1 g(0, t) + |g(0, 0)|.$$

2.2. **Theorem.** *The linear functional T on $\mathcal{G}$ is continuous if and only if there exist* (1) *a fundamentally bounded and convex with respect to area set function K_0 defined on $\mathcal{I}$,* (2) *two fundamentally bounded and convex set functions K_1 and K_2 defined on one-dimensional intervals, and* (3) *a scalar α such that*

$$T(g) = v\iint K_0\,dG + v\int_0^1 K_1\,dg(s, 0) + v\int_0^1 K_2\,dg(0, t) + \alpha g(0, 0).$$

Furthermore, $\|T\| = \max\{WK_0, WK_1, WK_2, \alpha\}$, where WK_i is the fundamental bound on K for $i = 0, 1, 2$.

Proof. Let $g^*(s, t) = g(s, t) - g(s, 0) - g(0, t) + g(0, 0)$, $g_1(s) = g(s, 0) - g(0, 0)$, $g_2(t) = g(0, t) - g(0, 0)$. Thus

$$g(s, t) = g^*(s, t) + g_1(s) + g_2(t) + g(0, 0).$$

The proof follows from 2.1 and [9] except for the statement regarding the norm of T, which we defer to the Appendix of this paper (§5) since it is a special case of a result which leads away from the mainstream of our development. Note that $v\int_0^1 K_1\,dg_1 = v\int_0^1 K_1\,dg(s, 0)$ since addition of a constant does not change the v-integral.

Suppose K is a fundamentally bounded and convex with respect to area set function defined on R. Let $f(s, t) = stK(I_{st})$ where $I_{st} = (0, s; 0, t)$, and define $F(I) = f|_a^b|_c^d$ where $I = (a, b; c, d)$. Then $F(I) = |I|K(I)$ and $|F(I)|WK \leq |I|$ where WK is the fundamental bound on K. If we define the function f to be Lipschitz provided the associated set function F satisfies $|F(I)| \leq L|I|$ for some constant L, then the entire §6 of [9] can be repeated in this setting and we note that one has a generalization of the Hellinger integral and the transformation of Theorem 2.1 of this paper has the representation

$$T(g) = \lim_{|\sigma| \to 0} \sum_{\sigma} \frac{\Delta_i F \Delta_i G}{I_i} = H \iint_R \frac{dF\,dG}{dA} = H \iint_R \frac{df\,dg}{dA}$$

for some Lipschitz function f. We also note that the dual of the absolutely continuous functions is isomorphic and isometric to the space of Lipschitz functions which in turn is isomorphic and isometric to the space of L^∞ functions as in [9].

A further extension is immediate from a result due to Fefferman [10]. Suppose Σ is a field of subsets of a space S, and suppose μ is a bounded positive finitely additive set function on Σ. Let P be a partition of S into elements in Σ. Fefferman [10] has shown that if γ is absolutely continuous (in the ϵ-δ sense) with respect to μ, and f_P^γ is the simple function defined by $\Sigma_{E \in P}(\gamma(E)/\mu(E))\chi_E$ then the net of measures $\{\gamma_P\}$ defined by $\gamma_P(F) = \int_F f_P^\gamma\, d\mu$ converges to γ in the variational norm. We note that γ_P is the analog of the polygonal function in [9] and the doubly polygonal function of this paper, and that the analog of the fundamental function Ψ_I is the fundamental measure Ψ_E defined by $\Psi_E(F) = (1/\mu(E)) \int_F \chi_E$ when $\mu(E) \neq 0$. If we let $\mathfrak{A}_F(\mu)$ denote the Banach space of finitely additive set functions on Σ which are absolutely continuous with respect to μ endowed with the variational norm, we have [10] the polygonal measures dense in $\mathfrak{A}_F(\mu)$. If T is a continuous linear functional on $\mathfrak{A}_F(\mu)$, then

$$T(\gamma) = \lim_P T(\gamma_P) = \lim_P T\Big(\sum \gamma(E_i)\Big) = \lim_P \sum \gamma(E_i) T(\Psi_{E_i}) = v \int K\,d\gamma$$

where $K(E) = T(\Psi_E)$ defines the bounded convex set function K. We thus have the following theorem.

2.3. **Theorem.** *Let μ be a positive bounded finitely additive set function. Then T is a linear functional on $\mathfrak{A}_F(\mu)$ if and only if there exists a bounded convex set function K such that $T(\gamma) = v\int K\,d\gamma$ for each $\gamma \in \mathfrak{A}_F(\mu)$. Furthermore, $\|T\| = WK$, the least bound on K.*

Although the subsequent generalizations given in this paper (§3) where the range spaces for the "functions" in the domain of T and for the range of T are topological vector spaces could be given in a straightforward manner in the setting of the previous paragraph (where the "functions" are measures), the authors choose to return to the setting where the domain is E^n (and in particular E^2) for the remainder of this paper. We do so for two reasons: First, the apparent "computability" of the v-integral is not lost in this setting, in the sense that the approximating sums for the integral converge to the integral as the norms of the partitions tend to zero (as opposed to net-type convergence). Secondly and more importantly, we wish to impose norms on the function spaces which are not stronger

than the BV norm. These norms are simpler, appear more natural, and perhaps are more useful in the setting where functions (as opposed to finitely additive measures) and E^n (as opposed to a finitely additive measure space) are involved.

3. **Other representation theorems with the v-integral.** In this section we consider several function spaces with differing topologies. Since the forms of the representation theorems appear essentially the same in each theorem, we remark that the conditions on the set function K vary as the function space is varied. To the authors' knowledge, these are the only known integral representation theorems in the setting of X and Y being TVS's (not necessarily convex) without separation properties being imposed on Y except for [5]. In most of this section we refer the reader to other literature for the techniques of proof to be used. We remark, however, that the theorems could also be proved by stating a general theorem for all function spaces with norm not stronger than BV norm much like Theorem 4.1 of this paper, except that X and Y would need only to be TVS's. The present structural development of the paper seemed to be more expedient than giving two such general theorems.

Let X and Y be TVS's and define a *polygonal function* from $R = (0, 1; 0, 1)$ into X to be a function of the form $\Sigma_{i=1}^n \Psi_{I_i} x_i$ where each $x_i \in X$ and each Ψ_I is a scalar valued fundamental function as defined in §2. Let the set of all such polygonal functions be denoted by P and note that P is a vector space over the real numbers where multiplication by a scalar is the usual pointwise multiplication and addition of functions is pointwise addition of function values. Let τ be a topology on P and let $L[X, Y]$ denote the continuous linear operators from X into Y. The convex with respect to area set function K from $\mathcal{I}$ into $L[X, Y]$ is said to be *τ-quasi-Gowurin* provided that given a neighborhood V of θ_Y (the origin in Y) there is a neighborhood U of θ_P (the origin in P, i.e., the function from R into X whose value is θ_X, the origin in X, for each point in R) such that if $\{I_i\}$ is a disjoint collection in $\mathcal{I}$ and $\{x_i\} \subset X$ satisfies $\Sigma_{i=1}^n \Psi_i x_i \in U$, then $\Sigma_{i=1}^n [K(I_i)](x_i) \in V$.

3.1. **Definition.** Suppose τ is a vector topology on P and F is contained in the completion of P under τ. If K is a convex set function which is τ-quasi-Gowurin, then for $f \in F$ define $v\int\int_R K\,df = \lim_\alpha v\int\int_R K\,dp_\alpha$, provided this limit exists, where p_α is a net of polygonal functions converging to f in τ.

Due to the fact that K is convex with respect to length it follows as in [9] for each polygonal function $p = \Sigma \Psi I_i x \in P$ that the v-integral of p with K is $\Sigma_{i=1}^n [K(I)](x_i) \in Y$.

Finally, since K is τ-quasi-Gowurin it follows that the integral is well defined in $\overline{Y}$, the closure of Y. For suppose $f \in F$ and suppose p_α and p_β are nets converging to f in τ. Then for an open set V about the origin in Y there

exists an open set U of θ_P such that if $p = \Sigma \Psi_{E_i} x_i \in U$, then $\Sigma[K(E_i)]x_i \in V$. But there exists an open set U' of θp such that $U' + U' \subset U$ and since p_α and p_β converge to f, there exists α_0 and β_0 respectively such that $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$ implies $p_\alpha - f$ and $p_\beta - f$ are in U'. Then $p_\alpha - p_\beta \in U' + U' \subset U$ and since $p_\alpha - p_\beta$ is polygonal, say $p_\alpha - p_\beta = \Sigma_{i=1}^n \Psi_{E_i} x_i - \Sigma_{j=1}^m \Psi_{F_j} z_j$, we can conclude

$$\sum_{i=1}^{n} [K(E_i)]x_i - \sum_{j=1}^{m} [K(F_j)]z_j \in V.$$

Hence we see by letting $p_\alpha = p_\beta$ that the approximations to the integral form a Cauchy net, and if we let p_α and p_β be distinct nets converging to f, then the limit, i.e. integral, is independent of the net.

Also, since the integral is defined on F in such a way that its value for g is the limiting value of the integrals of any net of polygonals converging to f, it follows that the integral generates a continuous linear operator on F for any r-quasi-Gowurin set function K. We make this remark since in general it is possible for a functional (for example) to be continuous on a dense set and discontinuous on the whole space (e.g., any discontinuous linear functional and its kernel).

In the following theorems, μ denotes the topology of uniform convergence. We shall assume throughout the remainder of this section that Y is complete. We mention also that if one wishes to talk of *the* limit as opposed to *a* limit for the approximating sums, then Y should be assumed to be Hausdorff. If Y is not complete, then in each of the theorems in this section the transformations are given by the desired integrals, but given the set functions K_0, K_1, K_2, we can only conclude that the integrals lie in $\bar{Y}$.

3.2. **Theorem.** *The linear transformation T from C_R (the continuous functions on R) into Y is continuous if and only if* (1) *there is a uniquely determined $L[X, Y]$-valued function K_0 which is convex with respect to area and is μ-quasi-Gowurin*; (2) *there are uniquely determined $L[X, Y]$-valued set functions K_1 and K_2 which are convex with respect to length and are μ-quasi-Gowurin*; *and* (3) *there is a uniquely determined $\alpha \in L[X, Y]$ such that $T(g) = v\iint K_0\, dg + v\int K_1\, dg(s, 0) + v\int K_2\, dg(0, t) + \alpha(g(0, 0))$ for each $g \in C_R$.*

Proof. Recall that $g(s, t) = g^*(s, t) + g_1(s) + g_2(t) + g(0, 0)$. Since T is linear we can view T as follows: $T = T_0 + T_1 + T_2 + \alpha$, where T_0 operates on anchored continuous functions, T_1 and T_2 operate on the functions of a single variable $g(s) = g(s, 0) - g(0, 0)$ and $g(t) = g(0, t) - g(0, 0)$ and $\alpha \in L[X, Y]$. From Corollary 2.3 in [5] we have the desired representation for T_1 and T_2, and we are left to characterize T_0 on C_R^0, the anchored continuous functions.

However, that $T_0(g^*) = v \iint K_0\, dg^* = v \iint K_0\, dg$ can be shown in a fashion analogous to the proofs of Theorems 2.1 and 2.2 in [5] by observing that $\lim_\sigma pg_\sigma^* = g^*$ with convergence being in μ.

Let $\mathfrak{I}_E$ denote the set of all regular half-open intervals $(a, b; c, d)$ in E^2, and let P_c denote the space of continuous, doubly polygonal functions with compact support. By observations similar to those made in 3.2, it is possible to establish the following theorem whose proof we omit.

3.3. **Theorem.** *The linear transformation T from C_c into Y (complete) is continuous if and only if* (1) *there exists a unique $L[X, Y]$-valued set function K_0 which is convex with respect to area and which is μ-quasi-Gowurin,* (2) *there are unique $L[X, Y]$-valued set functions K_1 and K_2 which are convex with respect to length and which are μ-quasi-Gowurin, and* (3) *there is a unique $\alpha \in L[X, Y]$ such that $T(g) = v \iint K\, dg + v \int K\, dg(s, 0) + v \int K\, dg(0, t) + \alpha(g(0, 0))$ for each $g \in C_c$.*

In the next theorem, by C_0 we mean the continuous functions which vanish at infinity with the topology μ.

3.4. **Theorem.** *The linear transformation T from C_0 into Y is continuous if and only if* (1), (2), *and* (3) *of Theorem* 3.3 *hold such that $T(g) = v \iint K_0\, dg + v \int K_1\, dg(s, 0) + v \int K_2\, dg(0, t) + \alpha(g(0, 0))$ where integration is in the extended sense of Definition* 3.1.

Proof. If we restrict T to C_c then Theorem 3.3 implies that T has the desired form. Since C_c is dense in C_0, it follows that each $g \in C_0$ is v-integrable in the extended sense. Since T is continuous, T has the desired form on C_0.

Conversely, if T has the given form, it follows from Theorem 3.3 that T is continuous on C_c and since C_c is dense in C_0, we conclude T is continuous on C_0.

Let C denote the space of continuous functions and let μ_c denote the topology of uniform convergence on compact sets.

3.5. **Theorem.** *The linear transformation T from C into Y is continuous if and only if* (1), (2), *and* (3) *of Theorem* 3.3 *hold with μ_c-quasi-Gowurin and such that T is given by the extended integral representation of Theorem* 3.4.

Proof. The space C is the completion of P_c in μ_c. Thus, if T has the above integral representation it is continuous on P_c and hence T is continuous on C.

If T is continuous on C, then T has the above integral representation on P_c. From the definition of the extended integral we conclude that the above extended integral representation for T on C holds.

Next we develop a Lebesgue space in a fashion analogous to the developments in [2] and [8]. For the remainder of the paper X and Y are assumed to be convex spaces and K is μ-quasi-Gowurin. For each continuous seminorm q on Y define the seminorm ρ_q on P_c by

$$\rho_q\left(\sum_{i=1}^{n} \Psi_{I_i} \cdot x_i\right) = \sup\left\{q\left(\sum [K(I_i \cap J_j)](\alpha_j x_i)\right)\right\}$$

where the supremum is over partitions of E^2 over $\mathfrak{J}_E$ and corresponding collections of scalars $\{\alpha_j\}$ which satisfy $|\alpha| \leq 1$ (since K is μ-quasi-Gowurin, it follows that such a supremum does exist). Identify equivalence classes in P_c as follows: p is equivalent to p' if and only if $\rho_q(p - p') = 0$ for each continuous seminorm q. As is usual, no distinction is drawn between a function and its equivalence class in the material which follows. Let κ denote the topology on P_c defined by the above mentioned seminorms and let L_K denote the completion of P_c under κ. It is a straightforward observation that κ is not stronger than μ, and hence, it follows that C_0 is a subspace of L_K.

3.6. **Definition.** Suppose Z is a convex topological space with seminorms $\{\nu\}$. Then, a set function G from $\mathfrak{J}_E$ with values in $L[X, Z]$ which is convex with respect to area is said to be absolutely continuous with respect to K providing there is a pairing (q, ν) and constants $P_{q-\nu}$ (Goodrich's notation [11]) such that for the collection of intervals $\{I_i\}$ and vectors $\{x_i\}$

$$\nu\left(\sum [G(I_i)](x_i)\right) \leq P_{q-\nu} \sup\left\{q\left(\sum [K(I_i \cap J_j)](\alpha_j x_i)\right)\right\}$$

where the supremum is taken as before. We shall assume $P_{q-\nu}$ represents the least such constant.

3.7. **Theorem.** *The linear transformation T from L_K to Z is continuous if and only if there is a set function G which is convex with respect to area and which is absolutely continuous with respect to K such that $T(g) = v \iint G\,dg$ where integration is in the extended sense. Furthermore, if T is (ρ_q, ν) related, then $|T|_{P_{q-\nu}} = P_{q-\nu}$.*

Proof. Observe that a set function G is absolutely continuous with respect to K if and only if G is κ-quasi-Gowurin, and, hence, the proof of the theorem follows along the lines of the previous theorems.

Next we lift the results of §2 to the setting of vector-valued functions. In order to do this it is necessary to define the analogue of bounded variation in this setting, and we give three such definitions, each of which is equivalent to bounded variation in the scalar setting.

3.8. **Definition.** Suppose X is a convex space with continuous seminorms $\{p\}$. A function g: $E^2 \to X$ is said to be of bounded variation denoted by $g \in \mathrm{BV}(X)$ if for each continuous seminorm there is a constant $V_p(g)$ such that $\Sigma_\sigma p(\Delta g) \leq V_p(g)$ for all partitions σ over $\mathcal{J}_E$ of compact squares in E^2. We assume $V_p(g)$ is the least such constant.

The function space $\mathrm{BV}(X)$ is a convex space with the topology induced by the seminorms $\{\rho_p\}$ defined by $\rho_p(g) = V_p(g)$ for each continuous seminorm p.

3.9. **Definition.** Suppose X is a topological vector space. Then a function g: $E^2 \to X$ is said to be of semibounded variation denoted by $g \in \mathrm{SBV}(X)$ if there is a neighborhood U of Θ_X such that $\Sigma_\sigma \alpha_i \Delta G \in U$ for all partitions σ over $\mathcal{J}_E$ of compact squares in E^2, and for corresponding collection of scalars $\{\alpha_i\}$ such that $|\alpha_i| \leq 1$.

For each neighborhood U of Θ_X define V_U in $\mathrm{SBV}(X)$ by $V_U = \{g$: $\Sigma \alpha_i \Delta G \in U$ where σ and $\{\alpha_i\}$ are as above$\}$. Then if the analogous function identification is made that is made in §2, $\mathrm{SBV}(X)$ is a Hausdorff topological vector space under the topology generated by $\{V_U\}$ as neighborhoods of the identically zero function.

3.10. **Definition.** Again X is assumed to be a topological vector space. A function g: $E^2 \to X$ is said to be of weak bounded variation, denoted by $g \in \mathrm{WBV}(X)$, if there is a neighborhood U of Θ_X such that $\Sigma_\sigma \Delta G \in U$ for all partitions σ over $\mathcal{J}_E$ of compact squares in E^2.

For each neighborhood U of Θ_X define V_U to be the collection of all $g \in \mathrm{WBV}(X)$ such that there is a partition σ_g such that $\sigma > \sigma_g$ implies $\Sigma_\sigma \Delta G \in U$. As above, let the $\mathrm{WBV}(X)$ be generated by $\{V_U\}$ as neighborhoods of the identically zero function.

Recall that in §2 the absolutely continuous functions are characterized as the closure of the polygonal functions under the BV norm. We use this characterization to extend the notion of absolute continuity to our present setting, i.e., define $\mathrm{AC}(X)$ to be the closure of P_c in the $\mathrm{BV}(X)$ topology, $\mathrm{SAC}(X)$ to be the closure of P_c in the $\mathrm{SBV}(X)$ topology, and $\mathrm{WAC}(X)$ to be the closure of P_c in the $\mathrm{WBV}(X)$ topology. (Addendum: since absolutely continuous had not been previously defined in these settings when the authors wrote this paper prior to April 1970, these definitions seemed to be reasonable extensions. The second author has since shown in *On the* BV-*norm closure of vector-valued polygonal functions* [Rev. Roumaine Math. Pures Appl. 17 (1972), 1123–1126] that if one defines "absolutely continuous" for functions into a normed vector space by replacing absolute value signs by norm signs in the usual definition of absolute continuity [for example, H. L. Royden, *Real analysis*, Macmillan, New York, 1968], then the closure of the polygonal functions is a proper subset of the

"absolutely continuous" functions.) It is clear that it is possible to give an analytic representation of the continuous linear transformations into Y on each of these spaces in terms of extended integration. However, the characterization holds for ordinary integration. To show this it is sufficient to establish that, for $g \in \mathrm{AC}(X)$ ($\mathrm{SAC}(X)$) ($\mathrm{WAC}(X)$), $\lim_\sigma pg_\sigma = g$ in the $\mathrm{BV}(X)$ ($\mathrm{SBV}(X)$) ($\mathrm{WBV}(X)$) topology. The result follows essentially as in §2 for $\mathrm{AC}(X)$. The next lemma establishes the result for the other two cases. As in §2 we shall assume that our functions are anchored along the axes.

3.11. **Lemma.** *Suppose* $g \in \mathrm{SAC}(X)$ ($\mathrm{WAC}(X)$). *Then* $\lim_\sigma pg_\sigma = g$ *where the convergence is in the* $\mathrm{SBV}(X)$ ($\mathrm{WBV}(X)$) *topology.*

Proof. Suppose U is a symmetric neighborhood of Θ_X. Choose U' to be a symmetric neighborhood of Θ_X such that $U' + U' \subset U$. Since g is in the closure of P_c, then there is a polygonal function p such that $g - p \in V_{U'}$. Let σ denote the partition induced by p. Suppose $\sigma' > \sigma$. Then for scalars $\{\alpha_i\}$ such that $|\alpha_i| \leq 1$, $\Sigma_{\sigma'} \alpha_i \Delta(PG_{\sigma'} - P) = \Sigma_{\sigma'} \alpha_i \Delta(G - P) \in U'$. Here, as usual, we have used $PG_{\sigma'}$, G, and P to be the set functions associated with $pg_{\sigma'}$, g, and p respectively. Therefore, $p - pg_\sigma \in V_{U'}$, from which we conclude $g - pg_\sigma = (g - p) + (p - pg_\sigma) \in V_{U'} + V_{U'} \subset V_{U'+U'} \subset V_U$ and the lemma is established for $\mathrm{SAC}(X)$. To establish the result for $\mathrm{WAC}(X)$ replace $\Sigma_{\sigma'} \alpha_i \Delta(PG_{\sigma'} - P) = \Sigma_{\sigma'} \alpha_i \Delta(G - P)$ in the above argument by $\Sigma_{\sigma'} \Delta(PG_{\sigma'} - P) = \Sigma_{\sigma'} \Delta(G - P)$. Next the representation theorem is stated.

3.12. **Theorem.** *The linear transformation* T *from* $\mathrm{AC}(X)$ ($\mathrm{SAC}(X)$) ($\mathrm{WAC}(X)$) *to* Y *is continuous if and only if there is an* $L[X, Y]$*-valued set function* K *which is convex with respect to area and which is* $\mathrm{BV}(X)$*-quasi-Gowurin* ($\mathrm{SBV}(X)$*-quasi-Gowurin*) ($\mathrm{WBV}(X)$*-quasi-Gowurin*) *such that* $T(g) = v \iint K\,dg$ *for each* $g \in \mathrm{AC}(X)$ ($\mathrm{SAC}(X)$) ($\mathrm{WAC}(X)$). *Furthermore, if* T *operates from* $\mathrm{AC}(X)$ *to* Y *and is* (p, q) *related, then* $|T|_{p-q} = WK_{p-q}$.

4. A general representation theorem and an application to a C_1-type space of functions. Let X and Y denote locally convex topological spaces with continuous seminorms $\{q\}$ and $\{r\}$ respectively. As in [4], let $x = Df(s, t)$ provided for each open set U containing x there is a number $\delta > 0$ such that if $R = (a, b; c, d)$ is a rectangle containing (s, t) and diam $R < \delta$, then

$$[f(b, d) - f(a, d) - f(b, c) + f(a, c)]/[(b - a)(c - c)]$$

is in U whenever $(1/\alpha) \leq (b - a)/(c - d) \leq \alpha$ for some fixed $\alpha > 0$. Let F denote the collection of bounded X-valued functions f such that Df exists a.e. and such that there is a set $E \subset R$ of measure zero such that for each continuous seminorm

q on X there is a constant B_q satisfying $q(Df(s, t)) \leq B_q$ for all $(s, t) \in R \backslash E$. Define the τ_1 topology on F by the family of seminorms $\{\rho_q\}$ where

$$\rho_q(f) = \sup_{(s,t)\in R} q(f(s, t)) + \operatorname{ess\,sup}_{(s,t)\in R} q(Df(s, t)).$$

Let F_p denote the subspace of F such that $f \in F_p$ if and only if $\lim_\sigma pf_\sigma = f$ in the τ_1 topology. We next state a general representation theorem analogous to Theorem 2.2 in [7]. In this theorem we suppose the functions to be anchored. Again, in order to talk about *the* limit, we must assume our spaces to be Hausdorff.

4.1. **Theorem.** *Suppose G is a subspace of F_p with a convex topology τ which is not stronger than τ_1, which extends to $P + G$. By $P + G$ we simply mean the set of all functions of the form $p + g$ where $p \in P$ and $g \in G$. Furthermore, suppose that there is a linear map Θ from $P + G$ into G^+ which is $\rho - \rho''$ related (where ρ is a continuous seminorm on $P + G$ under τ and ρ'' is its extension in G'' and where G^+ is the weak sequential extension of G [12]). If T is a continuous linear operator from G into Y then there is an $L[X, Y^+]$-valued set function K which is convex with respect to area and which is τ-quasi-Gowurin such that $T(f) = v \iint K\,dg$. Furthermore, if T is (ρ, τ) related, then*

$$WK_{\rho-\tau''} \geq |T|_{\rho-q} \geq W_{\rho-\tau''}/|\Theta|_p \quad \textit{where } |\Theta|_\rho = \inf\{K\colon \rho''\Theta(p) \leq K\rho(p);\ p \in P\}.$$

The proof follows by using an adaptation of Theorem 2.2 in [7] to the setting of the v-integral [9].

Let C_1 denote the subspace of F such that $f \in C_1$ implies f and Df are continuous, and let C_1^0 denote the subspace of anchored functions in C_1. As an application of Theorem 4.1 we obtain the following theorem which is the main result of this section.

4.2. **Theorem.** *Suppose T is a continuous linear transformation from C_1^0 into Y. Then there is a $L[X, Y^+]$-valued set function K which is convex with respect to area and which is τ_1-quasi-Gowurin such that $T(f) = v \iint_R K\,df$ for each f in C_1^0. Furthermore, if T is (ρ_q, τ) related, then $|T|_{\rho_q-\tau} = WK_{\rho_q-\tau''}$.*

In order to apply Theorem 4.1 it is necessary to establish that C_1^0 is a subspace of F_p and to construct a Θ map. This will be accomplished in the following series of lemmas.

4.3. **Lemma.** *If f is in C_1^0 and Df is identically Θ_X, then $f(s, t) \equiv \Theta_X$.*

Proof of 4.3. We lift the argument for 2.2 in [6] to the present setting using the elementary techniques from [3].

4.4. **Lemma.** *If f is in C_1^0, then $L \int_0^t \int_0^s Df = f(s, t) \in R$.*

Proof of 4.4. Let $g(s, t) = L \int_0^t \int_0^s Df$. Since Df is continuous at (s_0, t_0), there exists a $\delta > 0$ such that if $\max(|s - s_0|, |t - t_0|) < \delta$, then $Df(s, t) - Df(s_0, t_0) = \epsilon(s, t)$ where $|\epsilon(s, t)| < \epsilon$. Then

$$\Delta g/\Delta A = \left(\int_{s_0}^{s} \int_{t_0}^{t} Df(s_0, t_0) + \int_{s_0}^{s} \int_{t_0}^{t} \epsilon(s, t)\right)\Big/\Delta A$$

and hence $q(Df(s_0, t_0) - \Delta g/\Delta A) \leq \epsilon$ which implies $\lim_{\Delta A \to 0} \Delta g/\Delta A = Df$ for each $(s_0, t_0) \in R$. Now since f and g are both anchored, we conclude they are equal.

4.5. Lemma. *If f is in C_1^0, then the difference function $\Delta f/\Delta A$ converges uniformly over R to Df as ΔA tends to zero.*

Proof of 4.5. This follows from 4.4 as 2.4 in [6] follows from 2.3 in [6].

4.6. Lemma. *Suppose f is an element of C_1^0. Then $\{pf_\sigma\}$ converges with σ to f in the τ_1 topology.*

Proof of 4.6. We lift the proof of 2.1 in [6] to this setting with the previous lemmas.

This establishes that $C_1^0 \subset F_p$ and we turn to the task of building the Θ map. Let $C_1^0 R$ denote C_1^0 when X is real. The next theorem characterizes weak convergence in $C_1^0 R$. It is analogous to 13.36 in [1] and the proof is elementary.

4.7. Theorem. *A sequence $\{\psi_n\}$ in $C_1^0 R$ is weakly convergent if and only if the sequence $\{D\psi_n\}$ is uniformly bounded and pointwise convergent.*

Let $p_\sigma = \sum_{i,j=1}^{n,m} \alpha_{ij} \Psi_{I_{ij}}$ be a real-valued polygonal function. Then Dp_σ exists a.e. on the open rectangles $\{I_{ij}\}$ and $Dp_\sigma = \alpha_{ij}$ on $\operatorname{int}(I_{ij})$, that is $Dp_\sigma = \sum \alpha_{ij} \chi_{I_{ij}}$ except on the edges of the rectangles. Let us define $Dp_\sigma(s, t) = \alpha_{ij}$ whenever $s = s_i$ or $t = t_j$, that is, we make $Dp_\sigma(s, t)$ continuous from the left as a function of either s or t. Recall that $I_{ij} = \{(s, t) | s_{i-1} < s \leq s_i;\ t_{j-1} < t \leq t_j\}$. Since the $\{I_{ij}\}$ from an essential [14] partition of the rectangle R, we proceed to define continuous functions which approximate Dp_σ and form pseudo partitions of unity as in Lemma 4.3 of [7] and hence construct the desired Θ map as in §4 of [7].

4.8. Lemma. *The linear map Θ is continuous.*

It is sufficient to show that if a net of polygonals $\{p_\alpha\}$ converges to $f \in C_1$ in τ_1 then $\{\Theta p_\alpha\}$ converges to f in τ_1'', the topology of $C_1^{0''}$. To establish that this is in fact the case it is convenient to use the following result which is established in [7].

4.9. **Lemma.** *If S is a locally convex space, then a sufficient condition to guarantee that a net $S_\alpha \in S^+$ converges to $s \in S$ is that for each α there is a sequence $\{s_{\alpha,n}\}$ such that:*

(a) *For each α, $\{s_{\alpha,n}\}$ converges weakly to s_α.*

(b) $\lim_\alpha(s_{\alpha,n} - s) = \Theta_S$, *the origin in S.*

Proof of 4.8. Suppose $\{p_\alpha\}$ is a net of polygonal functions which converge to $f \in C_1^0$. It follows as in 7.2 (iii) [8] that for each α there is a sequence of continuous functions $\{\psi_{\alpha,n}\}$ which are uniformly bounded and which converge pointwise to Dp_α, and such that $\lim_\alpha(\psi_{\alpha,n} - Df) = \Theta$ uniformly. Hence, it follows that $\Phi_{\alpha n}(s, t) = \int_0^s \int_0^t \psi_{\alpha,n}(s, t)$ converges weakly to p_α and that $\lim_\alpha \Phi_{\alpha n} - f = \Theta_{C_1^0}$, which implies that $\Theta(p_\alpha)$ converges to f in C_1^{0+}.

4.10. **Lemma.** *For each continuous seminorm q, $|\Theta|_{\rho_q} = 1$.*

Proof. Suppose q is a continuous seminorm. Since C_1^0 is in the closure of the polygonals and since $\Theta(f) = f$ for $f \in C_1^0$, in order to establish the lemma it is sufficient to show that $\rho_q(p) \geq \rho_q''(\Theta p)$ for each polygonal function. Suppose $p = \Sigma_{i=1}^n \Psi_{I_i} \cdot x_i \in P$. Then it follows from 7.2 (i) [8] that there are sequences of real functions $\{\psi_{ij}\}_{i=1}^\infty$, $j = 1, \cdots, n$, such that $0 \leq \Sigma_{j=1}^n \psi_{ij}(s, t) \leq 1$ and $\{\psi_{ij}\}$ converges pointwise to $D\Psi_{I_j}$ for each j. Hence, $\Sigma \Phi_{ij} \cdot x_j$ converges weakly to $\Sigma \Psi_{I_i} \cdot x_i$ where $\Phi_{ij}(s, t) = \int_0^s \int_0^t \psi(s, t)$. Hence,

$$\rho_q''(\Theta(p)) = \sup\left\{\lim_i \left\langle c', \sum \Phi_{ij} \cdot x_j\right\rangle : c' \in B_q^0\right\}$$

where $B_q^0 = \{c' \in C_1^{0\prime} : |\langle c', f\rangle| \leq 1$ for all $f \in C_1^0$; $\rho_p(f) \leq 1\}$. Therefore,

$$\begin{aligned}
\rho_q''(\Theta(p)) &\leq \limsup \rho_q\left(\sum \Phi_{ij} \cdot x_j\right) \\
&\leq \limsup \left\{\sup_{(s,t)\in R} q\left(\sum \Phi_{ij}(s, t) \cdot x_j\right) + \sup_{(s,t)\in R} q\left(\sum \Phi_{ij}(s, t) \cdot x_j\right)\right\} \\
&= \sup_{(s,t)\in R} q\left(\sum \psi_{I_j}(s, t) \cdot x_j\right) + \operatorname{ess\,sup}_{(t,s)\in R}\left(\sum (D\psi_{I_j}) \cdot x_j\right) \\
&= \rho_q\left(\sum \psi_{I_j}(s, t) \cdot x_j\right).
\end{aligned}$$

The lemma is established.

The proof of 4.2 now follows from 4.1, 4.6, 4.8, and 4.10. We have also the following theorem which combines Theorem 4.1 of [6] with Theorem 4.2 of this paper. We let C_1^* denote the subspace of C_1 such that $g \in C_1^*$ implies $g_s(s, 0)$ and $g_t(0, t)$ are continuous, with the stronger topology whose seminorms are

$$\rho_q(g) = \sup_{(s,t)} q(g(s, t)) + \sup_{(s,t)} q(Dg(s,t)) + \sup_{s} q(g_s(s, 0)) + \sup_{t} q(g_t(0, t)).$$

4.11. **Theorem.** *Suppose T is a continuous linear transformation from C_1^* into Y. Then there is a $L[X, Y^+]$-valued set function K_0 which is convex with respect to area and which is τ_1-quasi-Gowurin and there are two $L[X, Y^+]$-valued set functions K_1 and K_2 which are convex with respect to length and which are τ_1-quasi-Gowurin, and there exists an $\alpha \in L[X, Y]$ such that*

$$T(g) = v \iint_R K_0\, dg + v \int K_1\, dg(s, 0) + v \int K_2\, dg(0, t) + \alpha(g(0, 0)).$$

4.12. *Concluding remarks.* In the event that X is the reals, it follows that if f is in the closure of the polygonals, then $\lim_\sigma pf_\sigma = f$ in the τ_1 topology, and hence in any topology which is not stronger than τ_1. This gives a slightly more general version of Theorem 4.1. We also recall for the reader that the theorems of §3 follow from Theorem 4.1 where the Θ map is the identity.

5. **Appendix.** In this section we shall address three disjoint topics each supplemental to the mainstream of this paper. Appendix A is a general result concerning norm relationships on functionals on vector spaces which are the direct sum of certain subspaces. In Appendix B we examine an overview of integral representation theory, and in Appendix C we mention some of the literature which has appeared since this paper was written.

5.A. *Concerning direct sums.* Theorem 2.2 of this paper is obtained by decomposing the function space into the direct sum of four vector spaces. In this section we shall comment briefly on a general result that implies $\|T\| = \max\{WK_0, WK_1, WK_2, \sigma\}$ in Theorem 2.2.

Let X be a normed vector space and let U and V be subspaces of X such that $X = U + V$, the direct sum of U and V. Suppose we denote the norm of x by $|x|$ for each $x \in X$ and let

$$|x|_1 = |u| + |v|, \qquad |x|_p = (|u|^p + |v|^p)^{1/p}, \qquad |x|_\infty = \max\{|u|, |v|\},$$

where $x = u + v$ and where $u \in U$, $v \in V$. For a continuous linear functional T on X, let $T_1 = T|U$ and let $T_2 = T|V$, the restrictions to U and V respectively. If $|x| = |x|_1$ for each $x \in X$, it follows that $\|T\| = \max\{\|T_1\|, \|T_2\|\} = \|T\|_\infty$ and if $|x| = |x|_\infty$ for each $x \in X$, it follows that $\|T\| = \|T_1\| + \|T_2\| = \|T\|_1$. In general if $|x| = |x|_p$ for each $x \in X$ and $1 \le p \le \infty$, then $\|T\| = \|T\|_q = (\|T_1\|^q + \|T_2\|^q)^{1/q}$ where $1/p + 1/q = 1$. These results apparently extend to the setting where X is the direct sum of a countable number of subspaces provided $p \neq \infty$.

To complete Theorem 2.2 we need only show that $|X| = |X|_1$ implies $\|T\| = \|T\|_\infty$. We have that

$$\|T\| = \sup_{x \in X} \frac{|T(x)|}{|x|} \geq \sup_{u \in U} \frac{|T(u)|}{|u|} \quad \text{and similarly} \quad \|T\| \geq \|T_2\|.$$

Also, since $|x| = |u| + |v|$ we have that

$$\frac{|T(x)|}{|x|} = \left| \frac{T(u) + T(v)}{|u| + |v|} \right|.$$

But $a/b \leq (a+c)/(b+d) \leq c/d$ whenever $a/b \leq c/d$ and therefore, assuming without loss that $|T(u)|/|u| \leq |T(v)|/|v|$, we have

$$\frac{|T(u)|}{|u|} \leq \frac{|T(x)|}{|x|} \leq \frac{|T(v)|}{|v|}$$

and hence $\|T\| \leq \max\{\|T_1\|, \|T_2\|\}$.

This completes the proof in the special case needed for Theorem 2.2, as induction extends the result to the setting where X is the direct sum of any finite number of subspaces.

To show the special case $|x| = |x|_\infty$ implies $\|T\| = \|T\|_1$ we note that for $\epsilon > 0$ there exists u, v with $|u| = |v| = 1$ such that $T(u) > |T_1| - \epsilon$ and $T(v) > |T_2| - \epsilon$. We assume $\epsilon < \max\{\|T_1\|, \|T_2\|\}/2$. Then $|u + v| = \max\{|u|, |v|\} = 1$ and $T(u+v) = T(u) + T(v) > \|T_1\| + \|T_2\| - 2\epsilon$ and hence $\|T\| \geq \|T_1\| + \|T_2\|$. Conversely, for $x \in X$, if we assume without loss that $x = u + v$ and $|u| \geq |v|$, then

$$\left| \frac{T(x)}{|x|} \right| = \left| \frac{T(u) + T(v)}{|u|} \right| < \left| \frac{T(u)}{|u|} \right| + \left| \frac{T(v)}{|u|} \right| < \left| \frac{T(u)}{|u|} \right| + \left| \frac{T(v)}{|v|} \right|$$

and hence $\|T\| \leq \|T_1\| + \|T_2\|. = \|T\|_1$.

For the general case with $1 < p < \infty$, $X = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ and $T_i = T|U_i$ we choose $e_i \in U_i$ such that $T(e_i) = \alpha_i = \|T_i\| - \epsilon_i$ where each $\epsilon_i \leq \epsilon < \max \|T_i\|$. Let $x_n = \xi_1 e_1 + \cdots + \xi_n e_n$ where $\xi_i = |\alpha_i|^{q-1}$. Then $T(x_n) = \Sigma_{i=1}^n \xi_i T(e_i) = \epsilon |\alpha_i|^q$ and hence $|T(x_n)| = \Sigma_{i=1}^n (\|T_i\| - \epsilon_i)^q \leq \|T\| |x_n|$. But

$$|x_n| = \left(\sum_{i=1} |\alpha_n|^{(q-1)p} \right)^{1/p} = \left(\sum_{i=1} (\|T_n\| - \epsilon_n)^q \right)^{1/p}$$

and so upon dividing the last inequality of the preceding sentence by $|x_n|$ we obtain $(\Sigma_{i=1}^n (\|T_i\| - \epsilon_i)^q)^{1/q} \leq \|T\|$, which implies since ϵ is arbitrary that $\|T\|_q \leq \|T\|$. Conversely for each x,

$$|T(x)| = \left|\sum_{i=1}^{n} T(u_i)\right| \leq \sum_{i=1}^{n} \|T_i\||u_i|$$

$$\leq \left(\sum_{i=1}^{n} \|T_i\|^q\right)^{1/q}\left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p} = \|T\|_q|x|,$$

and hence $\|T\| \leq \|T\|_q$.

5.B. *An overview of integral representation theory.* This section is an outgrowth of correspondence between the second author and the referee, whom we thank for his contribution.

In surveying the mass of papers on integral representations and continuous linear operations one wonders if it would not be possible to obtain a general road map-type theorem which would be economical and useful in unifying the theory. Such was the motivation behind Theorem 4.1 of this paper as well as Theorems 2.2 of [7] and 2.2 of [8]. The following is in the spirit of these theorems and generalizes them somewhat in that the v-integral can also be fitted into this scheme.

Let E and F be TVS's over the real or complex field, and let U and V denote generic neighborhoods of zero in E and F respectively. Let $L(E, F)$ be all continuous linear maps from E to F. If $A \subset E$, then for each T in $L(E, F)$ the function $T|A$ is an element of $\mathfrak{A} = \{\tau|\tau\colon A \to F$; given V there is some U such that $\{\lambda_i\}$ scalars, $\{\alpha_i\} \in A$, and $\Sigma_{i=1}^n \lambda_i\alpha_i \in U$ imply $\Sigma_{i=1}^n \lambda_i\tau(\alpha_i) \in V\}$. Conversely, if A is linearly dense in E and if F is complete and Hausdorff, then each τ on $\mathfrak{A}$ is $T|A$ for some unique T in $L(E, F)$. Thus under these assumptions on A and F there is a linear bijection $\rho\colon \mathfrak{A} \to L(E, F)$ and each τ is continuous (and as linear A as A is linear).

Let G be any set and Φ any map on G to E. Let $A = \Phi(G)$. For each $\tau \in \mathfrak{A}$, $\tau \cdot \Phi$ is on G to F and is an element of $\mathfrak{K} \equiv \{K|K\colon G \to F$; given V there is some U such that $\{\lambda_i\}$ scalars, $\{g_i\} \in G$, $\Sigma_{i=1}^n \lambda_i\Phi(g_i) \in U$ imply $\Sigma_{i=1}^n \lambda_i K(g_i) \in V\}$. The map $\rho(\tau) = \tau \cdot \Phi$ is injective on $\mathfrak{A}$ to $\mathfrak{K}$ and if F is Hausdorff, it is surjective.

Hence given E, F, G, and Φ and letting $A = \Phi(G)$, we conclude (if F is complete and Hausdorff and A is linearly dense in E) that $L(E, F)$ and $\mathfrak{K}$ are linearly isomorphic under $\rho\colon L(E, F) \to \mathfrak{K}$ where $\rho(\tau) = T \cdot \Phi$. Clearly each K in $\mathfrak{K}$ inherits whatever linearity and continuity properties Φ possesses.

Now suppose S is a set and X is a TVS over the same field as F, and E is a space of X-valued function on S. If $\mathfrak{I}$ is a class of subsets of S and for some map Φ in $\mathfrak{I} \times X$ to E the set $\Phi(\mathfrak{I} \times X)$ is linearly dense in E, then the above scheme can be used. The most commonly used Φ is as follows: To each $I \in \mathfrak{I}$ a scalar function Φ_I on S is assigned such that if Φ is defined by $\Phi(I, x) = \Phi_I(\cdot)x$

then the set $A = \Phi(\mathcal{I} \times X)$ is linearly dense in E. This in turn is most frequently done by choosing Φ_I so that $\{\Phi_I\colon I \in \mathcal{I}\}$ is linearly dense in the space E for which X is the scalar field, and then showing, if it can be shown, that A is actually linearly dense in E. Then there is a linear isomorphism between $L(E, F)$ and $\mathcal{K} = \{K \mid K\colon \mathcal{I} \times X \to F$; given V there is some U such that $\{\lambda_i\}$ scalars, $\{I_i\} \in \mathcal{I}$, $\{x_i\} \in X$, and $\Sigma_{i=1}^n \lambda_i \Phi_{I_i}(\cdot)x_i \in U$, imply $\Sigma_{i=1}^n \lambda_i K(I_i, x_i) \in V\}$. The correspondence is: $K(I, x) = L(\Phi_I(\cdot)x)$ where $L \in L(E, F)$ corresponds to K. It is easy to see that:

(i) Each $K \in \mathcal{K}$ is linear in x.

(ii) Each $K \in \mathcal{K}$ is continuous in x if the map $\Phi_I(\cdot)x$, for fixed I, is continuous on X to E.

(iii) $\mathcal{K} = \{K \mid K$ is scalar homogeneous in x and given V there is U such that $\Sigma_{i=1}^n \Phi_{I_i}(\cdot)x_i \in U$ implies $\Sigma_{i=1}^n K(I_i, x_i) \in V\} = \{K \mid K$ is linear in x and given V there is some U such that if $\{I_i\}$ are distinct and $\Sigma_{i=1}^n \Phi_{I_i}(\cdot)x_i \in U$, then $\Sigma_{i=1}^n K(I_i, x_i) \in V\}$.

(iv) If given any finite sequence $\{I_i\}$ in $\mathcal{I}$ and any $x \in X$ there is a disjoint finite collection $\{I_j'\}$ in $\mathcal{I}$ such that each $\Phi_{I_i}(\cdot)x$ can be written as a sum in E of finitely many $\Phi_{I_j}(\cdot)x$, then $\mathcal{K} = \{K \mid K$ is linear in x and additive in I, and given V there is some U such that if $\{I_j'\}$ are disjoint in $\mathcal{I}$ and $\Sigma_{j=1}^n \Phi_{I_j'}(\cdot)x_j \in U$, then $\Sigma_{j-1} K(I_j', x_j) \in V\}$.

(v) Under the hypotheses of (ii) and (iv), $K(I, \cdot)$ is an additive function on $\mathcal{I}$ to $L(X, F)$ for each K. Thus $\{K(I, \cdot)\}$ is the set of additive functions on $\mathcal{I}$ to $L(X, F)$ that are quasi-Gowurin in the sense of the present paper, and for each such K the corresponding element L of $L(E, F)$ is v-integration. Thus the construction of the v-integral could be viewed as a way of identifying the appropriate set functions, the appropriate vector-valued set functions and the mapping Φ so that $\Phi(G)$ is linearly dense in E under the given norm of bounded variation. Herein, the Φ_I to be used is $|I|\Psi_I$ where Ψ_I is the fundamental function of §2 of this paper and similarly where K of $\mathcal{K}$ above is related to K_0 of the present paper by $K(I, \cdot) = |I| K_0(I)$, i.e. $K(I, x) = |I| K_0(I)[x]$.

5.C. *A survey.* Due to the extensive amount of time involved in refereeing this paper, several results have appeared regarding the v-integral in the interim. For the interested reader we provide a survey. In *A representation theorem for* AC(R^n)* [preprint] A. de Korvin and R. J. Easton extend the v-integral to the setting of Theorem 2.1 in this paper. Then in *Functions of bounded variation on indempotent semigroups* [R. A. Alo, and A. J. de Korvin, Math. Anal. Appl. 194 (1971), 1–11] a theorem similar to Theorem 2.3 of this paper is obtained in a setting of semigroups by using an identification between certain functions on

idempotent semigroups and finitely additive set functions from *Measure algebras and functions of bounded variation on idempotent semigroups,* [S. Newman, Bull. Amer. Math. Soc. 75 (1969), 1396–1400]. In *Vector valued absolutely continuous functions on idempotent semigroups* [A. Alo, A. de Korvin and R. J. Easton, Trans. Amer. Math. Soc. 172 (1972), 491–500] these results are extended to the vector-valued function setting, as alluded to following Theorem 2.3 of this paper. We remark that 5.B extends both results to the TVS setting.

Along different lines, a calculus for the v-integral in the real-valued function setting has been developed in *A derivative to match the v-integral* [L. Hatta and S. G. Wayment, J. Reine Angew. Math. 257 (1972), 16–28], *A Radon-Nikodym theorem for the v-integral* [L. Hatta and S. G. Wayment, J. Reine Angew. Math. 259 (1973), 137–146] and *A v-integral characterization for the dual of the Lipschitz function* [J. R. Edwards, L. Hatta and S. G. Wayment, Rev. Roumaine Math. Pures Appl. 18 (1973), 885–891]. Also we remark that in *On the* BV *norm closure of vector-valued polygonal functions* [S. G. Wayment, Rev. Roumaine Math. Pures Appl. 17 (1972), 1123–1126] it is shown that defining the space of absolutely continuous "functions" to be the closure of the "polygonals" is in some sense not justified, and perhaps one should pursue integral representations for the larger class of absolutely continuous "functions" where the definition parallels the standard definition of absolutely continuous function but absolute value signs are replaced by norm signs.

Also, by some very clever techniques, Dan Mauldin [preprint] studied the space $\mathfrak{M}[0, 1]$ of real-valued countably additive regular set functions defined on the σ-algebra of Borel subsets of $[0, 1]$ (which is identifiable with the functions of bounded variation) and considered "maximal subsets" of mutually singular elements in $\mathfrak{M}[0, 1]$ to construct a bounded (but not convex) set function K for a given functional T on $\mathfrak{M}[0, 1]$ such that for each $\mu \in \mathfrak{M}[0, 1]$ it follows that $T(\mu)$ is a certain integral of μ with respect to the set function K. That integral is precisely the v-integral in the setting of this paper's Theorem 2.3. Thus the v-integral can be used to obtain a representation (but not a characterization) for the second dual of $C[0, 1]$ in terms of a v-integral with respect to bounded (but not convex) set functions, a problem outstanding since the celebrated Riesz representation theorem.

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