RATIONAL APPROXIMATION ON PRODUCT SETS\(^{(1)}\)

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ABSTRACT. Our object here is to study pointwise bounded limits, decomposition of orthogonal measures and distance estimates for \(R(K_1 \times K_2)\) where \(K_1 \) and \(K_2 \) are compact sets in the complex plane.

1. Introduction and main results. When \(X \) is a compact subset of \(\mathbb{C}^N\), then \(R(X)\) denotes the algebra of continuous functions which can be approximated uniformly on \(X\) by rational functions with singularities off \(X\). A measure \(\mu\) on \(X\) is called orthogonal to \(R(X)\), we write \(\mu \in R(X)^\perp\), if \(\int f \, d\mu = 0\) for all \(f \in R(X)\). A positive measure \(\lambda\) on \(X\) is a representing measure for \(x \in X\) if \(\int f \, d\lambda = f(x)\) for all \(f \in R(X)\). The restriction of a measure \(\mu\) to a subset \(E\) of \(X\) is denoted \(\mu_E\), and \(E\) is called a nullset for \(R(X)^\perp\) if \(\mu_E = 0\) whenever \(\mu \in R(X)^\perp\). We refer to [6] for further details on terminology.

First we will obtain the following decomposition theorem.

Theorem 1. Let \(K_i\) be compact subsets of \(\mathbb{C}\) \((i = 1, 2)\), and let \(Q_i\) be the set of non peak points for \(R(K_i)\). If \(\mu\) is a measure on \(K_1 \times K_2\) orthogonal to \(R(K_1 \times K_2)\), then \(\mu\) decomposes uniquely into \(\mu = \mu_0 + \mu_1 + \mu_2\) where the \(\mu_j\)'s \((j = 0, 1, 2)\) are pairwise mutually singular and orthogonal to \(R(K_1 \times K_2)\). The measure \(\mu_0\) belongs to the band of measures on \(K_1 \times K_2\) generated by representing measures for points in \(Q_1 \times Q_2\). \(\mu_1\) is supported on a set \(E_1 \times K_2\), and \(\mu_2\) is supported on a set \(K_1 \times E_2\) where \(E_i\) are nullsets for \(R(K_i)^\perp\).

Such a decomposition theorem has been obtained by B. Cole (unpublished) for the bidisc algebra. His proof carries over to the algebra \(A(U \times V)\), but not to \(R(K_1 \times K_2)\). Here we make the appropriate modifications, using Vitushkin's technique, to obtain this extension of Cole's decomposition.

When \(Q\) is a subset of \(X\), we introduce the algebra \(B(Q, R(X))\) of pointwise bounded limits as follows.

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Our second main result is now

**Theorem 2.** Let $K_i$ and $Q_i$, $(i = 1, 2)$ be as in Theorem 1. Let $f: Q_1 \times Q_2 \to C$ be bounded. The following are equivalent,

(a) $f \in B(Q_1 \times Q_2, R(K_1 \times K_2))$.
(b) $f(z, \cdot) \in B(Q_2, R(K_2))$ for all $z \in Q_1$, and $f(\cdot, w) \in B(Q_1, R(K_1))$ for all $w \in Q_2$.
(c) There is a sequence $\{f_n\}$ in $R(K_1 \times K_2)$ with $\|f_n\| \leq \|f\|$ and $f_n(z, w) \to f(z, w)$ for $(z, w) \in Q_1 \times Q_2$.

Here $\|\|$ means the uniform norm over the respective sets of definition.

The proof of Theorem 2 employs Vitushkin techniques, especially the characterization of $B(Q, R(K))$ for compact $K \subseteq C$, in terms of analytic capacity due to Gamelin and Garnett [8], and also Davie's theorem in [4] telling that $B(Q, R(K))$ is uniformly closed. We also here rely heavily on some unpublished ideas of B. Cole.

When $\sigma$ is a positive measure on $K_1 \times K_2$, we define $H^\infty(\sigma)$ as the weak-star closure of $R(K_1 \times K_2)$ in $L^\infty(\sigma)$. Our third main result is

**Theorem 3.** Let $K_i$ and $Q_i$, $(i = 1, 2)$ be as in Theorem 1. Let $\sigma$ be the measure $\sigma = dx \, dy_{Q_1} \times dx \, dy_{Q_2}$. Assume $Q_i$ is dense in $K_i$ for $i = 1, 2$, and let $u: K_1 \times K_2 \to C$ be continuous. Then

$$\text{dist}(u, H^\infty(\sigma)) = \text{dist}(u, R(K_1 \times K_2)).$$

Distance equalities like those in Theorem 3 have been obtained first by Sarason [10] for the disc algebra, and more recently by Gamelin and Garnett [8] for $A(U)$ and $R(K)$. Our proof of Theorem 3 employs a general criterion in [8].

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2. $A$-measures and the $B$-norm. $A$-measures were introduced by Henkin [9], and were employed to pointwise bounded approximation for $A(D)$ when $D \subseteq C^N$ is strictly pseudoconvex by Cole and Range [3].

Here let $A$ be a uniform algebra on a compact metric space $X$, and $Q$ be a
fixed Borel set in \( X \). A measure \( \nu \) on \( X \) is called an \( A \)-measure for \( A \) on \( Q \) if
\[
|f_n| \to 0 \quad \text{weak-star in} \quad L^\infty(|\nu|) \quad \text{whenever} \quad |f_n| \quad \text{is bounded in} \quad A \quad \text{and} \quad f_n \to 0 \quad \text{pointwise on} \quad Q.
\]
To verify that certain measures are \( A \)-measures, the following property (cf. [9, Theorem 1.4]) turns out to be useful.

Definition 2.1. A measure \( \nu \) on \( X \) has property (H) if there is a subfamily \( S \) of \( C(X) \) with the algebra generated by \( A \) and \( S \) dense in \( C(X) \), such that for each \( g \in S \) and each bounded sequence \( f_n \) in \( A \) with \( f_n \to 0 \) pointwise on \( Q \), there exists a bounded sequence \( \{F_n\} \) in \( A \) satisfying

(a) \( F_n \to 0 \) pointwise on \( Q \).

(b) \( F_n - g f_n \to 0 \) weak-star in \( L^\infty(|\nu|) \).

It is easy to prove the following.

Lemma 2.2. If \( \nu \) is orthogonal to \( A \) or is a representing measure for a point in \( Q \), then \( \nu \) is an \( A \)-measure if and only if \( \nu \) has property (H).

Let now \( \mathfrak{M}_0 \) denote the band of measures on \( X \) generated by representing measures for points in \( Q \). We want to show that \( A \)-measures are in \( \mathfrak{M}_0 \). First however we prove the following Forelli-type lemma (cf. [6, II. 7.3]).

Lemma 2.3. Let \( A \) be a uniform algebra on a compact \( X \). Let \( \phi_k \) be multiplicative linear functionals on \( A \) with \( M_{\phi_k} \) as the set of representing measures for \( \phi_k \), \( k = 1, 2, \ldots \). Let \( F \) be an \( F_\sigma \)-set which is a nullset for \( M_k \) for \( k = 1, 2, \ldots \). Then there is a sequence \( \{f_n\} \) in \( A \) with \( \|f_n\| \leq 1 \) such that \( |f_n| \to 1 \) on \( F \), and \( f_n \to 0 \) weak-star in \( L^\infty(\lambda) \) for each \( \lambda \in M_k \) for \( k = 1, 2, \ldots \).

Proof. Let first \( E \subset X \) be closed, \( \phi \in M_A \) with \( M_{\phi}(E) = 0 \). As in [6, II. 7.3], we get \( b_n \in A \) with \( \Re b_n > 0 \), \( \Re b_n > n^2 \) on \( E \) and with \( \Re \phi(b_n) < 1/n \) and \( \Im \phi(b_n) = 0 \). Now \( a_n = b_n^{-1} \in A \) and satisfies \( \Re a_n > 0 \), \( \Re a_n < n^{-2} \) on \( E \) and \( \Re \phi(a_n) > n \). Next we return to our situation, and let \( F = \bigcup_n F_n \) where each \( F_n \) is closed and \( F_n \subset F_{n+1} \). By the arguments above there are \( a_n^{(k)} \in A \) with \( \Re a_n^{(k)} > 0 \), \( \Re a_n^{(k)} < n^{-2} \) on \( F_n \) and \( \Re \phi_k(a_n^{(k)}) > n \). For each \( n \) we put \( a_n = a_n^{(1)} + \cdots + a_n^{(n)} \), and \( f_n = \exp(-a_n) \).

This now applies to

Proposition 2.4. Assume \( Q \) is the union of countably many parts for \( A \). If \( \nu \) is an \( A \)-measure, then \( \nu \in \mathfrak{M}_0 \).

Proof. Decomposing \( \nu \) relative to the band \( \mathfrak{M}_0 \), we may assume \( \nu \in \mathfrak{M}_0 \). Choose one \( \phi_k \) in each of the countably many parts whose union is \( Q \). Let \( M_k \) be as in Lemma 2.3. By [6, II. 7.4], there is an \( F_\sigma \)-set \( F \) with \( |\nu|(X \setminus F) = 0 \) and \( \lambda(F) = 0 \) for \( \lambda \in M_k \), \( k = 1, 2, \ldots \). Let now \( \{f_n\} \) be a sequence as in Lemma
2.3. If \( x \in Q \) with representing measure \( \lambda_x \), then there is \( \lambda \) in some \( M_k \) with 
\( \lambda_x \ll \lambda \) [6, pp. 143–144], so \( \lambda_x = g \lambda \) with \( g \in L^1(\lambda) \). Now \( f_n(x) = \int g f_n \, d\lambda \to 0 \), 
so \( f_n \to 0 \) pointwise on \( Q \). Then \( f_n \to 0 \) weak-star in \( L^\infty(|\nu|) \). Since \( |f_n| \to 1 \) 
on \( F \), we have \( |\nu|(F) = 0 \), so \( \nu = 0 \).

When \( \sigma \) is a positive measure on \( Q \), we define

\[
B(\sigma, A) = \{ f \in L^\infty(\sigma); \text{there is bounded } |f_n| \text{ in } A \\
\quad \text{with } f_n \to f \text{ a.e. } \sigma \}.
\]

On \( B(\sigma, A) \) we introduce a norm, called the \( B \)-norm \( \| \cdot \|_B \), by

\[
\|f\|_B = \inf \{ \sup \|f_n\|; f_n \in A, f_n \to f \text{ a.e. } \sigma \}.
\]

Out of B. Cole's more general scheme of double duals and reducing bands (unpublished) one obtains the following answer to when the \( B \)-norm is a "sup-norm". A proof is included in [1].

Theorem 2.5. Let \( \sigma \) be a positive measure on \( Q \) such that \( f_n \to 0 \) pointwise on \( Q \) whenever \( |f_n| \) is bounded in \( A \) and \( f_n \to 0 \) a.e. \( \sigma \). If every representing measure for points in \( Q \) are \( A \)-measures, then the \( B \)-norm on \( B(\sigma, A) \) satisfies

\[
\|f\|^2_B = \|f\|_B^2, \quad f \in B(\sigma, A).
\]

3. \( A \)-measures for \( R(K_1 \times K_2) \). When \( f \) is a bounded Borel function on \( C \), and \( \phi \) is a smooth function with compact support in \( C \), we define

\[
T_{\phi}f(z) = \phi(z)f(z) + \frac{1}{n} \int \frac{f(\xi)}{\xi - z} \, d\xi d\eta(\xi).
\]

For properties of this \( T_{\phi} \)-operator we refer to Chapter VIII of [6].

When \( K \subset C \) is compact, and \( Q \) is the set of non peak points for \( R(K) \), we also define

\[
H_{\phi}f(z) = \frac{1}{n} \int_{Q \setminus \xi} \phi(z) \, d\xi d\eta(\xi),
\]

and \( H_{\phi}f \in R(K) \) (cf. [8]). We finally define

\[
R_{\phi}f(z) = \frac{1}{n} \int_{Q} \phi(z) \, d\xi d\eta(\xi).
\]
and obtain \( \phi_f + R_{\phi_f} = T_{\phi_f} - H_{\phi_f} \in R(K) \) if \( f \in R(K) \).

We apply this to Lemma 3.1. Let \( K \subset \mathbb{C} \) be compact and let \( Q \) be the set of non peak points for \( R(K) \). If \( \lambda \) is a representing measure for a point in \( Q \), then \( \lambda \) is an \( A \)-measure for \( R(K) \) on \( Q \).

Proof. We verify property (H). Let \( S \) be the family of restrictions to \( K \) of smooth functions with compact support in \( \mathbb{C} \). \( S \) is dense in \( C(K) \). If \( \phi \in S \) and \( \{ f_n \} \) is bounded in \( R(K) \) with \( f_n(z) \to 0 \) for \( z \in Q \), we define \( F_n = \phi f_n + R_{\phi f_n} \) and obtain a sequence as in 2.1.

In the rest of this section let \( K_i \) and \( Q_i \) be as in Theorem 1. Proceeding as in Cole's band decomposition for the bidisc algebra (cf. Theorem 1), we introduce three bands of measures on \( K_1 \times K_2 \) as follows.

\( \mathfrak{M}_0 = \) band generated by representing measures for points in \( Q_1 \times Q_2 \).

\( \mathfrak{M}_1 = \) measures supported on sets of the form \( E_1 \times K_2 \) where \( E_1 \) is a nullset for \( R(K_1) \).

\( \mathfrak{M}_2 = \) measures supported on sets of the form \( K_1 \times E_2 \) where \( E_2 \) is a nullset for \( R(K_2) \).

It is well known (cf. [7, p. 200]) that a continuous function \( f: K_1 \times K_2 \to \mathbb{C} \) belongs to \( R(K_1 \times K_2) \) if \( f(z, \cdot) \in R(K_2) \) for each \( z \in K_1 \) and \( f(\cdot, w) \in R(K_1) \) for each \( w \in K_2 \).

Proposition 3.2. Let \( \nu \) be a measure on \( K_1 \times K_2 \). If \( \nu \) belongs to \( \mathfrak{M}_0 \), or \( \nu \) is orthogonal to \( R(K_1 \times K_2) \) and singular to both \( \mathfrak{M}_1 \) and \( \mathfrak{M}_2 \), then \( \nu \) is an \( A \)-measure for \( R(K_1 \times K_2) \) on \( Q = Q_1 \times Q_2 \).

Proof. Again we verify property (H). Here let \( S \) be the family of restrictions to \( K_1 \times K_2 \) of functions \( g \) of the form \( g(z, w) = \phi(z) \) or \( g(z, w) = \phi(w) \), where \( \phi \) is smooth with compact support in \( \mathbb{C} \). The algebra generated by \( S \) is dense in \( C(K_1 \times K_2) \). Let \( g \in S \), say \( g(z, w) = \phi(z) \). Let \( \{ f_n \} \) be bounded in \( R(K_1 \times K_2) \) converging pointwise to zero on \( Q_1 \times Q_2 \). Define

\[
R_g f_n(z, w) = \frac{1}{n} \int_{Q_1} f_n(\xi, w) \frac{\partial \phi}{\partial \xi} d\xi \, dx \, dy(\xi)
\]

and \( F_n = g f_n + R_g f_n \). We will show that \( \{ F_n \} \) satisfies the conditions of 2.1.
The comment before Lemma 3.1 implies \( F_n(z, w) \in R(K_1) \) for each \( w \in K_2 \). When \( r \) is orthogonal to \( R(K_2) \), then

\[
\int_{R} f_n(z, w) \, dr(w) = \frac{1}{\pi} \iint \frac{f_n(\xi, w)}{\xi - z} \, \partial \phi \, dr(w) \, dx \, dy(\xi) = 0
\]

by the Fubini theorem. Thus \( F_n(z, ) \in R(K_2) \) for each \( z \in K_1 \). Then \( \{F_n\} \) is a bounded sequence in \( R(K_1 \times K_2) \). Easily \( F_n(z, w) \to 0 \) for \( (z, w) \in Q_1 \times Q_2 \), and it remains to show that

\[
F_n - g_n = R f_n \to 0 \quad \text{weak-star in } L^\infty(\nu).
\]

It is enough to prove \( \int_{Q} b R f_n \, dv \to 0 \) for bounded \( b \). By the Fubini theorem we have

\[
\int_{Q} b R f_n \, dv = \frac{1}{\pi} \int \partial \phi \int \frac{f_n(\xi, w)}{\xi - z} b(z, w) \, dv(z, w) \, dx \, dy(\xi)
\]

and since \( \int |v|(z, w)/(\xi - z) \in L^1(dx \, dy) \), it is enough to prove that

\[
\int \frac{f_n(\xi, w)}{\xi - z} b(z, w) \, dv(z, w) \to 0 \quad \text{for a.a. } \xi.
\]

Since furthermore \( b(z, w)/(\xi - z) \in L^1(\nu) \) for a.a. \( \xi \), it is enough to show that for each \( \xi \in Q_1 / f_n(\xi, ) \to 0 \) weak-star in \( L^\infty(\nu) \). In fact we show that \( f_n(\xi, ) \to 0 \) a.e. \( \nu \). Put \( L = \{(z, w); f_n(\xi, w) \to 0\} \). Define \( b_n \in R(K_2) \) by \( b_n(w) = f_n(\xi, w) \). Then \( \{b_n\} \) is a bounded sequence in \( R(K_2) \) and \( b_n \to 0 \) pointwise on \( Q_2 \). Put \( E_2 = \{w; b_n(w) \not\to 0\} \). Lemma 3.1 tells that \( E_2 \) is a nullset for \( R(K_2) \), so if \( \nu \) is singular to \( \mathbb{M}_2 \), then \( |\nu|(L) = |\nu|(K_1 \times E_2) = 0 \). If \( \nu \) is a representing measure for \( (z_0, w_0) \in Q_1 \times Q_2 \), then the projection \( \pi: K_1 \times K_2 \to K_2 \) induces a representing measure \( \pi*\nu \) on \( K_2 \) for \( w_0 \) w.r.t. \( R(K_2) \). Again Lemma 3.1 gives \( \nu(L) = \pi*\nu(E_2) = 0 \), and this now completes the proof.

Since we know that each \( Q_i \) is the union of countably many parts for \( R(K_i) \) (cf. [6, p. 146]), we can combine the Propositions 2.4 and 3.2 to get

**Corollary 3.3.** A measure \( \nu \) on \( K_1 \times K_2 \) is an \( A \)-measure for \( R(K_1 \times K_2) \) on \( Q_1 \times Q_2 \) if and only if \( \nu \) belongs to the band \( \mathbb{M}_0 \).

**4. Proof of Theorem 1.** Let the bands \( \mathbb{M}_0, \mathbb{M}_1 \) and \( \mathbb{M}_2 \) be as in §3. The
Lemma 4.1. $\mathcal{M}_1$ and $\mathcal{M}_2$ are reducing bands (i.e. if $\mu \in R(K_1 \times K_2)^1$ decomposes $\mu = \mu_a + \mu_s$ relative to $\mathcal{M}_1$, then $\mu_a \in R(K_1 \times K_2)^1$. $\mathcal{M}_1$ and $\mathcal{M}_2$ are both singular to $\mathcal{M}_0$.

Proof. Let $\mu \in R(K_1 \times K_2)^1$ decompose $\mu = \mu_a + \mu_s$ relative to the band $\mathcal{M}_1$. There is an $F_\sigma$-set $E_1$ in $K_1$, $E_1$ being a nullset for $R(K_1)^1$ with $\mu_a$ supported on $E_1 \times K_2$ and $|\mu_s|(E_1 \times K_2) = 0$. When $E_1 = \bigcup_k F_k$ with each $F_k$ closed, then $F_k \times K_2$ is a peak set for $R(K_1 \times K_2)$ (cf. [6, p. 56]), so $\mu_a|_{F_k \times K_2} = \mu|_{F_k \times K_2} \in R(K_1 \times K_2)^1$. Then $\mu_a \in R(K_1 \times K_2)^1$. Next let $\nu$ be representing measure for some $(z, w) \in Q_1 \times Q_2$. Each $F_k$ consists of peak points for $R(K_1)$ so $z \not\in F_k$. Let $f \in R(K_1 \times K_2)$ peak at $F_k \times K_2$. Then $\nu(F_k \times K_2) = \lim_n \int E_n \nu = \lim_n \nu((z, w))^n = 0$. Thus $\nu(E_1 \times K_2) = 0$, and each measure in $\mathcal{M}_1$ is singular to $\mathcal{M}_0$. The proofs are similar for $\mathcal{M}_2$.

Lemma 4.2. $\mathcal{M}_1 \cap R(K_1 \times K_2)^1$ is singular to $\mathcal{M}_2 \cap R(K_1 \times K_2)^1$.

Proof. Let $\mu \in \mathcal{M}_1 \cap R(K_1 \times K_2)^1$ decompose $\mu = \mu_a + \mu_s$ relative to $\mathcal{M}_2$. Then $\mu_a$ is supported on $E_1 \times E_2$ where $E_i$ is a nullset for $R(K_1)^1$. Let $F_i \subset E_i$ be closed. Each $F_i$ is a peak interpolation set for $R(K_i)$, and $F_1 \times F_2$ is a peak set for $R(K_1 \times K_2)$. Then $R(K_1 \times K_2)|_{F_1 \times F_2}$ is closed, and $C(F_1 \times F_2) = C(F_1) \otimes C(F_2)$. Since $\mathcal{M}_2$ is reducing, $\mu_a \in R(K_1 \times K_2)^1$, and now $\mu_a|_{F_1 \times F_2} = 0$. Thus $\mu_a = 0$ and $\mu = \mu_s$.

Finally we can conclude with

Proof of Theorem 1. Decompose $\mu = \mu_1 + \mu_s$ relative to the band $\mathcal{M}_1$, and decompose $\mu_s = \mu_2 + \mu_0$ relative to $\mathcal{M}_2$. Then $\mu = \mu_0 + \mu_1 + \mu_2$, $\mu_1 \in \mathcal{M}_1$ and $\mu_2 \in \mathcal{M}_2$, $\mu_0$, $\mu_1$ and $\mu_2$ are pairwise mutually singular and orthogonal to $R(K_1 \times K_2)$. Since $\mu_0$ is singular to $\mathcal{M}_1$ and $\mathcal{M}_2$, Proposition 3.2 says $\mu_0$ is an $A$-measure. Then $\mu_0 \in \mathcal{M}_0$ by Proposition 2.4. The decomposition is unique because of the two lemmas above. This completes the proof.

A measure is called completely singular if it is singular to all representing measures for our algebra. It is well known (cf. [6, p. 47]) that $R(K)^1$ has no nonzero completely singular measures. This is no longer true for $R(K_1 \times K_2)^1$. However, looking for extreme points in the unit ball of $R(K_1 \times K_2)^1$, we have the following result, which also has been obtained for the bidisc algebra by B. Cole.

Corollary 4.3. $\text{ball } R(K_1 \times K_2)^1$ has no completely singular extreme points.

Proof. Let $\mu \in \text{ball } R(K_1 \times K_2)^1$ be a completely singular extreme point. Being an extreme point, $\mu$ must belong to one of the bands $\mathcal{M}_0$, $\mathcal{M}_1$ or $\mathcal{M}_2$, and
being completely singular, \( \mu \notin \mathcal{M}_0 \). Say \( \mu \in \mathcal{M}_1 \), so \( \mu \) is supported on a set \( E \times K_2 \) where \( E \) is a nullset for \( R(K_1) \). If \( E_1 \) and \( E_2 \) are disjoint with \( E = E_1 \cup E_2 \), we define \( \mu_1 \) and \( \mu_2 \) as the restrictions of \( \mu \) to \( E_1 \times K_2 \) and \( E_2 \times K_2 \) respectively. Then \( \mu = \mu_1 + \mu_2 \), \( \mu_1 \) and \( \mu_2 \) are mutually singular and belong to \( \text{ball} R(K_1 \times K_2) \). Since \( \mu \) is extreme, \( \mu_1 = 0 \) or \( \mu_2 = 0 \). Thus the support of \( \mu \) must be a set of the form \( \{z\} \times K_2 \). When \( \delta_z \) is the point mass at \( z \), we may view \( \mu \) as \( \mu = \delta_z \times \mu_0 \) where \( \mu_0 \in R(K_2) \). By the Glicksberg-Wermer decomposition \( \mu_0 \) is absolutely continuous to some representing measure for \( R(K_2) \), and thus \( \mu \) cannot be completely singular. This contradiction proves the result.

Yet another consequence of the decomposition theorem is

**Corollary 4.4.** If \( Q_i \) is dense in \( K_i \) \((i = 1, 2)\), then \( \text{ball} \mathbb{M}_0 \cap R(K_1 \times K_2) \) is weak-star dense in \( \text{ball} R(K_1 \times K_2) \).

**Proof.** By the Krein-Milman theorem it is enough to show that every extreme point in \( \text{ball} R(K_1 \times K_2) \) is in the weak-star closure of \( \text{ball} \mathbb{M}_0 \cap R(K_1 \times K_2) \), so let \( \mu \) be such an extreme point. By the Glicksberg-Wermer decomposition and Corollary 4.3, then \( \mu \) is absolutely continuous to some representing measure \( \lambda \) for a non peak point \((z, w)\). If \((z, w) \in Q_1 \times Q_2\), then \( \mu \in \mathbb{M}_0 \). Assume \( z \) is a peak point for \( R(K_1) \). Then \( \lambda \) and \( \mu \) are supported on \( \{z\} \times K_2 \), and we may write

\[
\lambda = \delta_z \times \lambda_0 \quad \text{and} \quad \mu = \delta_z \times \mu_0
\]

where \( \lambda_0 \) is representing measure for \( w \) w.r.t. \( R(K_2) \) and \( \mu_0 \in R(K_2) \). Let now \( z_n \in Q_1 \) with \( z_n \to z \), and let \( \lambda_n \) be representing measure for \( z_n \) w.r.t. \( R(K_1) \).

Since \( z \) is a peak point, and any weak-star cluster point of \( \lambda_n \) must be a representing measure for \( z \), we have \( \lambda_n \to \delta_z \) weak-star. Defining \( \mu_n = \frac{1}{z_n - z} \times \mu_0 \), we obtain a sequence \( \mu_n \) in \( \text{ball} \mathbb{M}_0 \cap R(K_1 \times K_2) \) converging weak-star to \( \mu \).

**5. Proof of Theorem 2.** Let \( \sigma_i \) \((i = 1, 2)\) denote the area measure on the set \( Q_i \) of non peak points for \( R(K_i) \), and put \( \sigma = \sigma_1 \times \sigma_2 = dx dy_{Q_1} \times dx dy_{Q_2} \). Then \( \sigma \) is a positive measure on \( Q = Q_1 \times Q_2 \), and to simplify notation we put

\[
B(Q_1 \times Q_2) = B(Q_1 \times Q_2, R(K_1 \times K_2)), \quad B(\sigma) = B(\sigma, R(K_1 \times K_2)).
\]

We easily obtain (cf. [4])

**Lemma 5.1.** If \( f \in B(\sigma) \) and \( \{|f|\} \) is bounded in \( R(K_1 \times K_2) \) and \( f \to f \) a.e. \( \sigma \), then in fact \( \{|f|\} \) converges everywhere on \( Q_1 \times Q_2 \) to a unique \( f \in B(Q_1 \times Q_2) \). Moreover \( \|f\| \leq \|f\|_B \).
Proof. Let $\epsilon > 0$ and $(z, w) \in Q_1 \times Q_2$. Define $P_{\epsilon/2}(z) = \{z' \mid |f(z) - f(z')| < \epsilon, f \in \text{ball}(R(K_1))\}$ and similarly $P_{\epsilon/2}(w)$ and $P_{\epsilon}(z, w)$ in terms of $R(K_2)$ and $R(K_1 \times K_2)$. Easily we have $P_{\epsilon/2}(z) \times P_{\epsilon/2}(w) \subseteq P_{\epsilon}(z, w)$. By Browder's metric density theorem (cf. [2]) we have $\sigma(P_{\epsilon}(z, w)) > 0$. Then there is $(z', w') \in P_{\epsilon}(z, w)$ with $f_n(z', w') \to f(z', w')$. Then $|f_n(z, w) - f_m(z, w)| \leq (2M + 1)\epsilon$ for $n$ and $m$ big enough, when now $M = \sup \|f\|$. Thus $|f_n(z, w)|$ converges for each $(z, w) \in Q_1 \times Q_2$, and we define

$$\widetilde{f}(z, w) = \lim_{n} f_n(z, w).$$

That $\widetilde{f}$ only depends on $f$ and not on the particular sequence $\{f_n\}$, is proved similarly. Finally, the proof also gives $\|\widetilde{f}\| \leq M$, which implies $\|\widetilde{f}\| \leq \|f\|_B$.

Again to simplify notation we let $B(Q_1) \# B(Q_2)$ be the bounded functions on $Q_1 \times Q_2$ satisfying (b) of Theorem 2. The following lemma is a trivial consequence of Davie's theorem [4].

Lemma 5.2. $B(Q_1) \# B(Q_2)$ is uniformly closed and contains $B(Q_1 \times Q_2)$.

To show that the two spaces in this lemma in fact are equal, we use the Vitushkin approximation scheme (cf. [6, p. 210]). First however let us state the following version of [6, VIII. 6.3].

Lemma 5.3. Let $E \subset \mathbb{C}$ be bounded with analytic capacity $\gamma(E) > 0$, analytic center $w_0$ and analytic diameter $\beta(E)$. Then there are functions $g$ and $h$ both analytic off some compact subset of $E$ and satisfying

(i) $g(\infty) = h(\infty) = 0$,

(ii) $g'(\infty) = 1$, $h'(\infty) = 0$,

(iii) $\beta(g, w_0) = 0$, $\beta(h, w_0) = 1$,

(iv) $\|g\| \leq 14/\gamma(E)$,

(v) $\|h\| \leq 6/\gamma(E)\beta(E)$.

Proof. Let $f_1$ and $f_2$ be as in the proof of [6, VIII. 6.3], and put $g = f_1/\gamma(E)$ and $h = f_2/\gamma(E)\beta(E)$.

We recall the approximation scheme from [6, VIII.7]. For each $\delta > 0$ cover the complex plane with discs $\Delta_k = \Delta(z_k, \delta)$ and choose smooth functions $\phi_k$ such that

(i) $\phi_k$ has support in $\Delta_k$,

(ii) $\sum_k \phi_k = 1$,

(iii) $\left\| \frac{\partial \phi_k}{\partial z_k} \right\| \leq 4/\delta$,

(iv) no point $z \in \mathbb{C}$ is contained in more than $M$ of the discs $\Delta_k$, where $M$ is a universal constant.
If now \( f \) is a bounded measurable function with compact support, and we define

\[
f_k(z) = T_{\phi_k} f(z) = \frac{1}{\pi} \int \frac{f(z) - f(\xi)}{z - \xi} \frac{\partial \phi_k}{\partial \xi} \, dx \, dy(\xi),
\]

then \( f = \sum_k f_k \).

Employing this approximation scheme we prove the equivalence of (a) and (b) in Theorem 2, which we here formulate as follows.

**Proposition 5.4.** \( B(Q_1 \times Q_2) = B(Q_1) \# B(Q_2) \).

**Proof.** Let \( f \in B(Q_1) \# B(Q_2) \). Extend \( f \) to be zero off \( Q_1 \times Q_2 \). For each \( \delta > 0 \) let \( \Delta_k \) and \( \phi_k \) be as above. Define

\[
\beta(f_k, t_k)(w) = \frac{1}{\pi} \int f(\xi, w) \frac{\partial \phi_k}{\partial \xi} \, dx \, dy(\xi),
\]

where \( t_k \) is an analytic center of \( E_k = \Delta(z, 3\delta) \setminus K_1 \). As usual we get \( f = \sum_k f_k, \|f_k\| \leq 32 \|f\| \). Since \( f(\cdot, w) \in B(Q_1) \), the characterization of \( B(\sigma_1) \) in [8] gives us the estimate \( |f_k(\infty)(w)| \leq A_1 \|f\| \gamma(E_k) \). Arguing as in [6, p. 216] we obtain

\[
|\beta(f_k, t_k)(w)| \leq A_2 \|f\| \gamma(E_k) \beta(E_k)
\]

where \( A_1 \) and \( A_2 \) are universal constants. By Lemma 5.3 applied to \( E = E_k \), we have functions \( g_k \) and \( h_k \) with the first three coefficients in their Laurent expansions at \( \infty \) as in (i)–(iii) and with norm estimates as in (iv)–(v) of that lemma.

Now we define

\[
H_k = f_k'(\infty)g_k + \beta(f_k, t_k)h_k \quad \text{and} \quad F_\delta = \sum_k H_k.
\]

Since \( g_k \) and \( h_k \) are in \( R(K_1) \) for each \( k \), we get \( F_\delta(\cdot, w) \in R(K_1) \) for each \( w \in K_2 \). We use Fubini’s theorem to verify the condition of [8] characterizing \( B(\sigma_2) \) thus showing \( f_k'(\infty) \in B(\sigma_2) \) and \( \beta(f_k, t_k) \in B(\sigma_2) \) for each \( k \). Then \( H_k(z, ) \)
and in turn $F_\delta(z, \cdot) \in B(\sigma_2)$ for each $z \in K_1$. We have $\|F_\delta\| \leq A_4 \|f\|$, and as in [8, §3], one shows that for each $w \in Q_2$, then $F_\delta(z, w) \rightarrow f(z, w)$ for $\sigma_1$-a.a. $z$.

As in [4] (cf. our Lemma 5.1), we get $F_\delta(z, w) \rightarrow f(z, w)$ for $(z, w) \in Q_1 \times Q_2$. Next we do this argument all over again for each $F_\delta$, but now in the second variable. For $\rho > 0$ we get $g_{\delta, \rho} \in R(K_1 \times K_2)$ with $\|g_{\delta, \rho}\| \leq A_5 \|f\|$ and $g_{\delta, \rho} \rightarrow F_\delta$ pointwise on $Q_1 \times Q_2$. The bounded net $\{g_{\delta, \rho}\}$ has a subsequence $\{g_{\delta, \rho}\}$ converging weak-star in $L^\infty(\sigma)$. We may assume $\{g_{n}\}$ converges a.e. $\sigma$, and then $g_{n} \rightarrow f$ pointwise on $Q_1 \times Q_2$ by Lemma 5.1. Thus $f \in B(Q_1 \times Q_2)$, and the proof is completed.

As a consequence of this result and Lemma 5.2 (cf. [4]) we note Corollary 5.5. $B(Q_1 \times Q_2)$ is uniformly closed.

Trivially (c) $\Rightarrow$ (a) in Theorem 2, and now we turn to (a) $\Rightarrow$ (c). Then let $T : B(Q_1 \times Q_2) \rightarrow B(\sigma)$ be the inclusion map. Lemma 5.1 tells us that $T$ is one-to-one and onto, and $T^{-1}$ is continuous from $B(\sigma)$ with the $B$-norm to $B(Q_1 \times Q_2)$ with uniform norm. $B(\sigma)$ with the $B$-norm is complete, and we just established completeness of $B(Q_1 \times Q_2)$. Then $T$ is an isomorphism by the closed-graph theorem.

Our Lemma 5.1 also tells that the measure $\sigma$ satisfies the condition in Theorem 2.5. We proved in §3 that each representing measure for a point in $Q_1 \times Q_2$ is an $A$-measure, so by Theorem 2.5 we can conclude $\|f\|_B = \|f\|_B$, $f \in B(\sigma)$.

Thus $T$ is an isometry, which exactly means that (c) in Theorem 2 is satisfied for all $f \in B(Q_1 \times Q_2)$.

This now completes the proof of Theorem 2.

The Kreǐn-Smulian theorem and Theorem 2 give that $B(\sigma)$ is weak-star closed, and then must coincide with the weak-star closure $H^\infty(\sigma)$ of $R(K_1 \times K_2)$. Thus we can note (cf. [8]).

Corollary 5.6. If $f \in H^\infty(\sigma)$, then there is a sequence $\{f_n\}$ in $R(K_1 \times K_2)$ such that $\|f_n\| \leq \|f\|$ and $f_n \rightarrow f$ a.e. $\sigma$.

The equivalence of (a) and (b) in the following corollary was noted already in 3.3, but we include it here for completeness.

Corollary 5.7. Let $\mu$ be a measure on $K_1 \times K_2$ orthogonal to $R(K_1 \times K_2)$. The following are equivalent:

(a) $\mu$ belongs to the band $M_0$.
(b) $\mu$ is an $A$-measure for $R(K_1 \times K_2)$ on $Q_1 \times Q_2$.
(c) The inclusion map $H^\infty(\sigma + |\mu|) \rightarrow H^\infty(\sigma)$ is an isometric isomorphism.

Proof. Let $f \in H^\infty(\sigma)$, and let $\{f_n\}$ be a sequence in $R(K_1 \times K_2)$ with
\[ \|f_n\| \leq \|f\| \text{ and } f_n \rightharpoonup f \text{ a.e. } \sigma. \] Let \( \hat{f} \) be a weak-star cluster-point of \( \{f_n\} \) in \( H^\infty(\sigma + |\mu|) \). Then \( \hat{f} = f \text{ a.e. } \sigma \) and \( \|\hat{f}\| = \|f\| \). Thus (c) holds if and only if \( \hat{f} \) is uniquely determined by \( f \). Let \( B(\sigma + |\mu|) \) be the bounded weak-star closure of \( R(K_1 \times K_2) \) in \( L^\infty(\sigma + |\mu|) \). We have \( \hat{f} \in B(\sigma + |\mu|) \), and \( \hat{f} \) is unique in \( B(\sigma + |\mu|) \) for all \( f \in H^\infty(\sigma) \) if and only if \( \mu \) is an \( A \)-measure. In this case \( B(\sigma + |\mu|) \) becomes weak-star closed, and so coincides with \( H^\infty(\sigma + |\mu|) \).

As a direct consequence of the Corollaries 4.4 and 5.7 we note

**Corollary 5.8.** If \( Q_i \) is dense in \( K_i \) \((i = 1, 2)\), then the inclusion map \( H^\infty(\sigma + |\mu|) \to H^\infty(\sigma) \) is an isometric isomorphism for a weak-star dense set of measures \( \mu \) in ball \( R(K_1 \times K_2) \).

6. Proof of Theorem 3. The localization property. When \( A \) is a uniform algebra on a compact metric space \( X \), \( \sigma \) is a positive measure on \( X \), and \( H^\infty(\sigma) \) is the weak-star closure of \( A \) in \( L^\infty(\sigma) \), then the fiber over \( p \in X \) in the maximal ideal space \( M_{H^\infty(\sigma)} \) of \( H^\infty(\sigma) \) is the set

\[ \mathbb{M}_p = \{ \Psi \in M_{H^\infty(\sigma)} ; \Psi(f) = f(p), f \in A \}. \]

Theorem 3 will follow by verifying the assumptions in the following general criterion of [8] for such distance equalities.

**Theorem 6.1** (Gamelin, Garnett). Let \( A \) be a uniform algebra on a compact \( X \). Let \( \sigma \) be a positive measure on \( X \) whose closed support coincides with \( X \). Suppose

\[ \text{(L)} \quad \text{if } p \in \text{supp } \sigma, u \in C(X), f \in H^\infty(\sigma) \text{ and } |f| \leq u \text{ a.e. } \sigma, \text{ then} \]

\[ |f| \leq u(p) \text{ on } \mathbb{M}_p. \]

\[ \text{(D)} \quad \text{The inclusion map } H^\infty(\sigma + |\mu|) \to H^\infty(\sigma) \text{ is an isometric isomorphism for a weak-star dense set of measures } \mu \text{ in ball } A^+. \]

For any continuous \( u : X \to \mathbb{C} \) we then have \( \text{dist}(u, A) = \text{dist}(u, H^\infty(\sigma)) \).

In Theorem 3 we apply this to \( X = K_1 \times K_2 \), \( A = R(K_1 \times K_2) \) and \( \sigma = dx dy_{Q_1} \times dx dy_{Q_2} \), and assuming \( Q_i \) is dense in \( K_i \) \((i = 1, 2)\). Property (D) was established in the preceding section, so the only thing left to verify is now the localization property (L). We have identified \( H^\infty(\sigma) \) with \( B(Q_1 \times Q_2) \), so (L) now follows from
Proposition 6.2. Let \( f \in B(Q \times Q) \), and let \( \Psi \) be in the fiber over \((z_0, w_0) \in K \times K \) in the maximal ideal space of \( B(Q \times Q) \). For \( \delta > 0 \) define \( W_\delta = \Delta(z_0, \delta) \times \Delta(w_0, \delta) \). For any \( \delta > 0 \) we have

\[
|\Psi(f)| \leq \|f\|_{(Q \times Q)_{\Lambda} W_\delta}.
\]

Proof. Put \( f = 0 \) outside \( Q \times Q \). Choose a smooth function \( \phi \) with compact support in \( \Delta(z_0, \delta) \) satisfying \( \phi = 1 \) on \( \Delta(z_0, \delta/2) \), \( \|\partial \phi / \partial \xi\| \leq 2/\delta \) and \( \|\phi\| \leq 1 \).

Define

\[
R_f(z, w) = \int \frac{\partial \phi}{\partial \xi} d\sigma_1(\xi) \quad \text{and} \quad F = \phi f + R_f.
\]

When \( \{f_n\} \) is bounded in \( R(K \times K) \) and \( f_n \to f \) pointwise on \( Q_1 \times Q_2 \), then we put \( f_n = \phi f_n + R_f f_n \). We saw in the proof of Proposition 3.2 \( f_n \in R(K_1 \times K_2) \), and \( \{f_n\} \) is bounded and \( f_n \to f \) pointwise on \( Q_1 \times Q_2 \). Thus \( f \in B(Q_1 \times Q_2) \).

When \( |z - z_0| < \delta/2 \), then

\[
(F - f)(z, w) = R_f(z, w) = \int \frac{\partial \phi}{\partial \xi} d\sigma_1(\xi)
\]

which is analytic in \( z \) for each \( w \in Q_2 \).

Next we define \( G_1 : Q \times Q_2 \to C \) by

\[
G_1(z, w) = \frac{(F - f)(z, w) - (F - f)(z_0, w)}{z - z_0}
\]

and want to show that \( G_1 \in B(Q_1 \times Q_2) \). When \( \{f_n\} \) is as above and \( r \in R(K_2) \), then \( \int R_f f_n(z_0, w) d\sigma(w) = 0 \) by the Fubini theorem. Thus \( R_f f_n(z_0, w) \in R(K_2) \), and we get a bounded sequence satisfying \( R_f f_n(z_0, w) \to R_f f(z_0, w) = (F - f)(z_0, w) \) for \( w \in Q_2 \). Then \( (F - f)(z_0, w) \in B(Q_2) \). For each \( z \in Q_1 \), \( z \neq z_0 \), then \( G_1(z, w) \in B(Q_1) \). Furthermore

\[
G_1(z_0, w) = \int \frac{\partial \phi}{\partial \xi} d\sigma_1(\xi),
\]

and similar use of Fubini gives \( G_1(z_0, w) \in B(Q_2) \). Totally \( G_1(z, w) \in B(Q_1) \) for all \( z \in Q_1 \). Next we fix \( w \in Q_2 \). Now \( (F - f)(z_0, w) \in B(Q_1) \), is analytic and zero at \( z_0 \), and then \( G_1(z, w) \in B(Q_1) \) (see [8, 6.1]).
Define next $h: Q_1 \times Q_2 \rightarrow \mathbb{C}$ by $h(z, w) = (F - f)(z_0, w)$ and put $b = 0$ outside $Q_1 \times Q_2$. Choose a new smooth function $\phi$ with compact support in $\Delta(w_0, \delta)$ satisfying $\phi = 1$ on $\Delta(w_0, \delta/2)$, $\|\partial \phi / \partial z\| \leq 4/\delta$ and $\|\phi\| \leq 1$. Define $H = \phi b + R\phi b$. As above $H \in B(Q_1 \times Q_2)$, and $H - b$ extends analytically in $w$ to $w_0$ for each $z \in Q_1$.

We may define

$$G_2(z, w) = \frac{(H - b)(z, w) - (H - b)(z, w_0)}{w - w_0}$$

and we get $G_2 \in B(Q_1 \times Q_2)$. Now we may write

$$(F - f)(z, w) = (z - z_0)G_1(z, w) + (w - w_0)G_2(z, w) + H(z, w) + (b - H)(z, w_0).$$

The last term here is in fact a constant, and when $\Psi \in \mathcal{M}(z_0, w_0)$, then

$$(ii) \quad |\Psi(F - f)| \leq |\Psi(H)| + |(b - H)(z, w_0)|.$$

Here

$$(iii) \quad (b - H)(z, w_0) = -\frac{1}{\pi} \int \frac{(F - f)(z_0, \xi)}{\xi - w_0} \frac{\partial \phi}{\partial \xi} d\sigma_2(\xi).$$

From the integral formulas (i) and (iii) we get the estimates

$$\|F - f\|_{x_0 \times (Q_2 \cap \Delta(w_0, \delta))} \leq A_1 \|f\|_{Q_1 \times Q_2 \cap \Delta(w_0, \delta)}$$

and by (i) we get

$$\|(b - H)(z, w_0)\| \leq A_1 \|F - f\|_{x_0 \times (Q_2 \cap \Delta(w_0, \delta))} \leq A_2 \|f\|_{Q_1 \times Q_2 \cap \Delta(w_0, \delta)}.$$

Adding up all these estimates in (ii), we conclude
(iv) \[ |\Psi(F - f)| \leq A_5 \|f\|_{(Q_1 \times Q_2)^nW_8} \]

Next we go through the whole argument all over again with \( F \) instead of \( f \), but reversing the order of the \( R_\phi \)-operations, i.e. first we choose smooth \( \phi \) with compact support in \( \Delta(w_0, \delta) \), \( \phi = 1 \) on \( \Delta(w_0, \delta/2) \), \( \|\nabla \phi / \nabla \bar{x}\| \leq 4/\delta \) and \( \|\phi\| \leq 1 \), and we define \( G = \phi F + R_\phi F \in B(Q_1 \times Q_2) \). Next we proceed as above, and finally we reach the conclusion

(v) \[ |\Psi(G - F)| \leq A_5 \|F\|_{(Q_1 \times Q_2)^nW_8} \]

Looking at the definition of \( F \) we have

(vi) \[ \|F\|_{Q_1 \times Q_2^nW_8} \leq \|F\|_{Q_1 \times (Q_2 \cap \Delta(w_0, \delta))} \leq A_6 \|f\|_{(Q_1 \times Q_2)^nW_8} \]

Furthermore

(vii) \[ \|G\|_{Q_1 \times Q_2} \leq A_7 \|F\|_{Q_1 \times (Q_2 \cap \Delta(w_0, \delta))} \]

When we put together the estimates (iv), (v), (vi), and (vii), and by breaking up \( \Psi(f) = \Psi(f - F) + \Psi(F - G) + \Psi(G) \), we conclude

(viii) \[ |\Psi(f)| \leq A_8 \|f\|_{(Q_1 \times Q_2)^nW_8} \]

\( A_1 \) to \( A_8 \) are universal constants, and since (viii) holds for all \( f \in B(Q_1 \times Q_2) \), we get

\[ |\Psi(f)| = |\Psi(f^n)|^{1/n} \leq A_8^{1/n} \|f\|_{(Q_1 \times Q_2)^nW_8} \]

for each \( n = 1, 2, \ldots \) which finally proves

\[ |\Psi(f)| \leq \|f\|_{(Q_1 \times Q_2)^nW_8} \]

and the proof is complete.

This completes the proofs of our theorems. However, let us note the following simple example to the effect that the conditions on \( Q_i \) being dense in \( K_i \) \((i = 1, 2)\) are really necessary.

Let \( K_1 = \{z; |z| \leq 1\} \). Let \( z_0 \notin K_1 \), and let \( K_2 = K_1 \cup \{z_0\} \). Let \( u \) be continuous on \( K_1 \times K_2 \) such that

\[ u|_{K_1 \times K_1} \in R(K_1 \times K_1), \quad u|_{K_1 \times \{z_0\}} \notin R(K_1). \]
Then dist\( (u, H^\infty(\sigma)) = 0 \), but dist\( (u, R(K_1 \times K_2)) > 0 \).

Finally, we should also note that similar results to those presented here hold for \( R(K_1 \times K_2 \times \cdots \times K_N) \) for compacts \( K_1, K_2, \cdots, K_N \) in \( \mathbb{C} \). More details on this are in [1].

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