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NORM INEQUALITIES FOR THE LITTLEWOOD-PALEY FUNCTION g_λ^* [1]

BY

BENJAMIN MUCKENHOUPT AND RICHARD L. WHEEDEN

ABSTRACT. Weighted norm inequalities for L^p and H^p are derived for the Littlewood-Paley function g_λ^*. New results concerning the boundedness of this function are obtained, by a different method of proof, even in the unweighted case. The proof exhibits a connection between g_λ^* and a maximal function for harmonic functions which was introduced by C. Fefferman and E. M. Stein. A new and simpler way to determine the behavior of this maximal function is given.

1. Introduction. The main purpose of this paper is to prove weighted norm inequalities for the Littlewood-Paley function g_λ^*, defined by

$$g_\lambda^*(u)(x) = \left(\iint_{E_+^{n+1}} \frac{y^{(\lambda-1)n+1}}{(y+|x-z|)^{\lambda n}} |\nabla u(z, y)|^2 \, dz \, dy \right)^{1/2},$$

where $\lambda > 1$, $x = (x_1, \cdots, x_n)$ and $z = (z_1, \cdots, z_n)$ are points in n-dimensional Euclidean space E^n, $u(z, y)$ is harmonic in $E_+^{n+1} = \{(z, y): z \in E^n, y > 0\}$ and $|\nabla u|^2 = (\partial u/\partial z_1)^2 + \cdots + (\partial u/\partial z_n)^2 + (\partial u/\partial y)^2$. This function plays an important role in questions related to multipliers (see Stein [14, p. 94 and p. 232]) and to Sobolov spaces (see Stein [14, p. 162], and Segovia and Wheeden [11]).

Two-sided weighted norm inequalities for the Lusin area integral of a harmonic u are derived in Gundy and Wheeden [5]. Since the g_λ^* function is a pointwise majorant of the Lusin area function, we shall be concerned only with inequalities which bound norms of g_λ^* by norms of u.

The measures with respect to which norms are taken have the form $d\mu(x) = w(x)\,dx$. A weight w is said to satisfy condition A_p for some p, $1 < p < \infty$, if w is a nonnegative, locally integrable function which satisfies

$$(A_p) \qquad \left(\frac{1}{|I|} \int_I w(x)\,dx\right)\left(\frac{1}{|I|} \int_I \{w(x)\}^{-1/(p-1)} dx\right)^{p-1} \le C,$$

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where I denotes an n-dimensional "cube" with sides parallel to the coordinate planes, $|I|$ is the volume of I, and c is a constant independent of I. When $p = 1$, w is said to satisfy condition A_1 if w is nonnegative, locally integrable and

$$w^*(x) \le cw(x), \tag{A_1}$$

where

$$w^*(x) = \sup_{h>0} \int_{|x-z|<h} |w(z)|\, dz$$

is the Hardy-Littlewood maximal function of w, and c is independent of x. We will write $w \in A_p$, or $w \in A_1$, for such w. These classes were introduced in Muckenhoupt [7] and in an equivalent form by Rosenblum [10].

For a measurable set S, we will use the notation

$$m_w(S) = \int_S w(x)\, dx$$

for the w-measure of S. A condition related to A_p, $1 \le p < \infty$, is that there exist positive constants c and ϵ such that for every cube I and every measurable subset $E \subset I$,

$$m_w(E)/m_w(I) \le c(|E|/|I|)^{\epsilon}. \tag{A_∞}$$

This condition was introduced in [2]; it was also observed in [2] that if $w \in A_p$ for any p, $1 \le p < \infty$, then $w \in A_\infty$. Conversely, Muckenhoupt [8] proved that if $w \in A_\infty$ then $w \in A_p$ for some $p < \infty$.

We list here two specific facts which we shall need about A_p. The first is that if f^* denotes the Hardy-Littlewood maximal function of f, then for $1 < p < \infty$ and $w \in A_p$

$$\int_{E^n} \{f^*(x)\}^p w(x)\, dx \le c \int_{E^n} |f(x)|^p w(x)\, dx, \tag{1.1}$$

and for $w \in A_1$

$$m_w\{f^*(x) > \alpha\} \le c\alpha^{-1} \int_{E^n} |f(x)| w(x)\, dx, \qquad \alpha > 0. \tag{1.2}$$

These facts are proved in Muckenhoupt [7]. Next, we shall need the fact that if $w \in A_p$, $1 < p < \infty$, then

$$\int_{E^n} \frac{w(z)}{1 + |z|^{np}}\, dz < \infty. \tag{1.3}$$

A more precise statement is given in Hunt, Muckenhoupt and Wheeden [6, Lemma 1], for the case $n = 1$; the proof in case $n > 1$ is similar.

Along with g_λ^* we shall consider a maximal function for harmonic functions which was introduced in Fefferman and Stein [4, p. 178]. For a harmonic function u and scalars λ and r, $\lambda > 1$, $r > 0$, let

$$T_{\lambda,r}(u)(x) = \sup_{h>0}\left(\frac{1}{h^{\lambda n}}\iint_{J(x,h)} y^{(\lambda-1)n-1}|u(z, y)|^r\,dz\,dy\right)^{1/r},$$

where $J(x, h) = \{(z, y): |x - z| < h,\ 0 < y < h\}$. This function will play an important role in our results for g_λ^*. It will therefore be necessary to prove weighted norm inequalities for $T_{\lambda,r}$ as a preliminary step. These will be obtained as a corollary of a new result for $T_{\lambda,r}$. In order to state this result, we define for $p \geq 1$,

$$f_p^*(x) = \sup_{h>0}\left(\frac{1}{h^n}\int_{|x-z|<h} |f(z)|^p\,dz\right)^{1/p}.$$

When $p = 1$, we have $f_1^* = f^*$.

Theorem 1. *Let $1 < r < \infty$, $1 < \lambda < r$ and $p_0 = r/\lambda$. If $f_{p0}^*(x)$ is finite for some x then the Poisson integral u of f is finite in E_+^{n+1} and $T_{\lambda,r}(u)(x) \leq cf_{p0}^*(x)$, where c is independent of x and f.*

Before stating our main result, we consider two other maximal functions, both smaller than $T_{\lambda,r}(u)$, which will arise. Let $\Gamma_\nu(x)$, $\nu > 0$, denote the cone $\{(z, y): |z - x| < \nu y\}$ with vertex x, and let

$$N_\nu(u)(x) = \sup_{(z,y)\in\Gamma_\nu(x)} |u(z, y)| \quad \text{and} \quad D_\nu(u)(x) = \sup_{(z,y)\in\Gamma_\nu(x)} |y\nabla u(z, y)|$$

for harmonic u. It is known (see Stein [14, p. 207]) that if $\nu < \mu$, there is a constant c depending only on ν, μ and n so that

$$D_\nu(u)(x) \leq cN_\mu(u)(x). \tag{1.4}$$

Furthermore, there is a constant c depending only on λ, r, μ and n so that

$$N_\mu(u)(x) \leq cT_{\lambda,r}(u)(x) \tag{1.5}$$

for all x. This is an easy corollary of the mean-value property of harmonic functions and is stated in Fefferman and Stein [4, p. 178], in the case $r = 1$.

To prove (1.5) for any $r > 0$, we observe by a lemma due to Hardy and Littlewood (see Lemma 2, §9 of [4]) that there is a constant c so that

$$|u(z, y)| \leq c\left(y^{-n-1}\iint_{B_{y/2}(z,y)} |u(\xi, \eta)|^r\,d\xi\,d\eta\right)^{1/r}.$$

Here $B_{y/2}(z, y)$ denotes the ball in E_+^{n+1} with center (z, y) and radius $y/2$. Hence, since $y/2 < \eta < 3y/2$,

$$|u(z, y)| \le c_1 \left(y^{-\lambda n} \iint_{B_{y/2}(z,y)} \eta^{(\lambda-1)n-1} |u(\xi, \eta)|^r \, d\xi \, d\eta \right)^{1/r}.$$

If $(z, y) \in \Gamma_\mu(x)$ and $(\xi, \eta) \in B_{y/2}(z, y)$ then $|x - \xi| \le |x - z| + |z - \xi| \le \mu y + y/2 = (\mu + 1/2)y$. Thus

$$N_\mu(u)(x) \le c_1 \sup_{y>0} \left(y^{-\lambda n} \iint_{0<\eta<3y/2;\ |x-\xi|<(\mu+1/2)y} \eta^{(\lambda-1)n-1} |u(\xi, \eta)|^r \, d\xi \, d\eta \right)^{1/r},$$

and (1.5) follows.

If we denote as in Gundy and Wheeden [5]

$$\|u\|_{H_w^p} = \left(\int_{E^n} \{N_1(u)(x)\}^p w(x) \, dx \right)^{1/p}$$

for $0 < p < \infty$ and $w \in A_\infty$, then our main result is the following.

Theorem 2. *Let $u(x, y)$ be harmonic in E_+^{n+1} and $\lambda > 1$.*

(i) *If $2/\lambda < p < \infty$, $w \in A_{p\lambda/2}$ and $\|u\|_{H_w^p}$ is finite, then*

$$\left(\int_{E^n} \{g_\lambda^*(u)(x)\}^p w(x) \, dx \right)^{1/p} \le c \|u\|_{H_w^p}.$$

(ii) *If $w \in A_1$ and $\|u\|_{H_w^{2/\lambda}}$ is finite, then*

$$m_w\{g_\lambda^*(u)(x) > \alpha\} \le c\alpha^{-2/\lambda} \|u\|_{H_w^{2/\lambda}}^{2/\lambda} \quad \text{for } \alpha > 0.$$

The constants c are independent of u and α.

Part (ii) of Theorem 2 is new when $\lambda \ge 2$ and $n > 1$ even in the case $w \equiv 1$. When $\lambda = 2$, $n = 1$, $w \equiv 1$, it is proved in a periodic version in Stein [13, p. 157] by an indirect argument. When $\lambda \ge 2$ and $n = 1$, see Zygmund [15] for an equivalent result in the periodic case. When $1 < \lambda < 2$ and $w \equiv 1$, it is proved in Fefferman [3]. (In this case, $H^{2/\lambda}$ can be identified with $L^{2/\lambda}$ in the standard way since $2/\lambda > 1$.) For part (i) of Theorem 2 in the case $w \equiv 1$, see Zygmund [15] and Stein [14].

§2 contains the proof of Theorem 1, and the results on the norm behavior of $T_{\lambda,r}(u)$, for any harmonic u, which will be used to prove Theorem 2. Theorem 2 is proved in §3. The basic method of proof has two parts: the comparison of g_λ^* to the maximal function and the deduction of the weighted estimates from local Lebesgue estimates by use of the condition A_∞. This last step was first given in

Gundy and Wheeden [5] for the Lusin area integral. The method has also been used in Coifman [1] and Muckenhoupt and Wheeden [9] for singular and fractional integrals, respectively. At the end of §3, we list a remark about norm inequalities between g_λ^* and $T_{\lambda,2}$, and a corollary of Theorem 2 for harmonic functions which are Poisson integrals.

2. Results for $T_{\lambda,r}$.

Proof of Theorem 1. It follows easily from the assumption that $f_{p_0}^*(x)$ is finite for some x that

$$\int_{E^n} \frac{|f(x)|}{1+|x|^{n+1}}\,dx < \infty. \tag{2.1}$$

This condition is equivalent to the finiteness of the Poisson integral $u(x, y)$, $y > 0$, of f.

As in the introduction, let

$$T_{\lambda,r}(u)(x) = \sup_{h>0}\left(h^{-\lambda n} \iint_{J(x,h)} y^{(\lambda-1)n-1}|u(z, y)|^r\,dz\,dy\right)^{1/r}$$

for this u, where λ and r now satisfy the restrictions of Theorem 1—namely, $1 < r < \infty$ and $1 < \lambda < r$. Let $p_0 = e/\lambda$. We must show that

$$T_{\lambda,r}(u)(x) \le c f_{p_0}^*(x), \tag{2.2}$$

where c is independent of f and x.

We may assume without loss of generality that $f(x)$ is nonnegative, since replacing $f(x)$ by $|f(x)|$ only increases the left side of the inequality. Let $I(x, h) = \{z\colon |x - z| < h\}$. Then replacing $u(z, y)$ by its formula gives

$$T_{\lambda,r}(u)(x)$$
$$= \sup_{h>0}\left(h^{-\lambda n}\int_{I(x,h)}\int_0^h y^{(\lambda-1)n-1}\left\{\int_{E^n} f(t)\,\frac{y}{[y^2+(z-t)^2]^{(n+1)/2}}\,dt\right\}^r dy\,dz\right)^{1/r}.$$

Minkowski's integral inequality applied to the inner two integrals shows that $T_{\lambda,r}(u)(x)$ is bounded by a constant times

$$\sup_{h>0}\left(h^{-\lambda n}\int_{I(x,h)}\left[\int_{E^n} f(t)\left\{\int_0^h \frac{y^{(\lambda-1)n-1+r}}{(y+|z-t|)^{(n+1)r}}\,dy\right\}^{1/r} dt\right]^r dz\right)^{1/r}. \tag{2.3}$$

Splitting the middle integral into integrals over $I(x, 2h)$ and $G(x, 2h) = E^n - I(x, 2h)$, using Minkowski's inequality and some obvious simplification shows that (2.3) is bounded by the sum of

$$(2.4)\quad \sup_{h>0}\left(h^{-\lambda n}\int_{I(x,h)}\left[\int_{I(x,2h)} f(t)\left\{\int_0^h \frac{y^{(\lambda-1)n-1+r}}{(y+|z-t|)^{(n+1)r}}\,dy\right\}^{1/r} dt\right]^r dz\right)^{1/r}$$

and

$$(2.5)\quad \sup_{h>0}\left(h^{-\lambda n}\int_{I(x,h)}\left[\int_{G(x,2h)} f(t)\left\{\int_0^h \frac{y^{(\lambda-1)n-1+r}}{|z-t|^{(n+1)r}}\,dy\right\}^{1/r} dt\right]^r dz\right)^{1/r}.$$

If $z \in I(x, h)$ and $t \in I(x, 2h)$, then $|z - t| < 3h$. To estimate (2.4), split the inner integral into the integrals from 0 to $|z - t|/3$ and from $|z - t|/3$ to h. It is then easy to see that the inner integral is bounded by a constant times $|z - t|^{-nr+(\lambda-1)n}$. Consequently, (2.4) is bounded by a constant times

$$(2.6)\qquad \sup_{h>0}\left(h^{-\lambda n}\int_{E^n}\left[\int_{I(x,2h)} f(t)|z-t|^{-n+(\lambda-1)n/r}\,dt\right]^r dz\right)^{1/r}.$$

By the fractional integral theorem of Hardy-Littlewood-Sobolev (Theorem 1, p. 119 of [14]), (2.6) is bounded by a constant times

$$(2.7)\qquad \sup_{h>0} h^{-\lambda n/r}\left(\int_{I(x,2h)} \{f(t)\}^{p_0}\,dt\right)^{1/p_0},$$

since $1/r = 1/p_0 - [(\lambda - 1)n/r]/n$, $0 < (\lambda - 1)n/r < n/p_0$. Since $\lambda/r = 1/p_0$, (2.7) equals $cf^*_{p_0}(x)$ as desired.

To estimate (2.5), we first evaluate the inner integral and combine powers of h. We then see that (2.5) is bounded by a constant times

$$(2.8)\qquad \sup_{h>0}\left(\int_{I(x,h)}\left[\int_{G(x,2h)} f(t)\,\frac{h^{1-n/r}}{|z-t|^{n+1}}\,dt\right]^r dz\right)^{1/r}.$$

For z in $I(x, h)$ and t in $G(x, 2h)$, $|x - t| \le 2|z - t|$. Using this fact shows that (2.8) is bounded by a constant times

$$(2.9)\qquad \sup_{h>0}\left(\int_{I(x,h)}\left[\int_{G(x,2h)} f(t)\,\frac{h^{1-n/r}}{|x-t|^{n+1}}\,dt\right]^r dz\right)^{1/r}.$$

Since the inner integral in (2.9) is independent of z, we obtain by performing the outer integration that (2.9) equals a constant times

$$\sup_{h>0}\int_{G(x,2h)} f(t)\,\frac{h}{|x-t|^{n+1}}\,dt.$$

By Theorem 2, p. 62, of [14], this is bounded by a constant times $f^*(x)$, which is bounded by $f^*_{p_0}(x)$ by Holder's inequality since $p_0 > 1$.

This completes the proof of (2.2), and therefore of Theorem 1. As a corollary of Theorem 1, we will prove

Theorem 3. *Let* $1 < r < \infty$, $1 < \lambda < r$, $p_0 = r/\lambda$ *and let* $u(x, y)$ *be the Poisson integral of* f.

(i) *If* $p_0 < p < \infty$, $w \in A_{p/p_0}$ *and* $\int_{E^n} |f(x)|^p w(x)\,dx < \infty$ *then*

$$\left(\int_{E^n} \{T_{\lambda,r}(u)(x)\}^p w(x)\,dx\right)^{1/p} \le c\left(\int_{E^n} |f(x)|^p w(x)\,dx\right)^{1/p}.$$

(ii) *If* $w \in A_1$ *and* $\int_{E^n} |f(x)|^{p_0} w(x)\,dx < \infty$, *then for* $\alpha > 0$

$$m_w\{T_{\lambda,r}(u) > \alpha\} \le c\alpha^{-p_0} \int_{E^n} |f(x)|^{p_0} w(x)\,dx.$$

The constants c *are independent of* f *and* α.

For the case $w \equiv 1$ of Theorem 3, see [4, p. 180].

First we note that $f_{p_0}^* = (|f|^{p_0})^{*(1/p_0)}$. To prove part (i) of Theorem 3, observe (1.1) shows that

$$\int_{E^n} \{f_{p_0}^*(x)\}^p w(x)\,dx \le c \int_{E^n} |f(x)|^p w(x)\,dx. \tag{2.10}$$

In particular, $f_{p_0}^*$ is finite for almost all x. Hence, by Theorem 1,

$$\int_{E^n} \{T_{\lambda,r}(u)(x)\}^p w(x)\,dx \le c \int_{E^n} \{f_{p_0}^*(x)\}^p w(x)\,dx.$$

Combining this with (2.10) completes the proof of part (i) of Theorem 3. The proof of part (ii) is similar, using (1.2).

We can easily obtain results for $T_{\lambda,r}(u)$ for harmonic functions u belonging to the weighted Hardy spaces referred to in the introduction. It is in this form which the results for $T_{\lambda,r}$ will be useful in obtaining results for g_λ^*. A harmonic function $u(x, y)$ belongs to H_w^p, $0 < p < \infty$, $w \in A_\infty$, if

$$\|u\|_{H_w^p} = \left(\int_{E^n} \{N_1(u)(x)\}^p w(x)\,dx\right)^{1/p} < \infty.$$

Since $w \in A_\infty$, we know from Lemma 1 of [5] that $u \in H_w^p$ if and only if the function $N_\nu(u)$ defined in the introduction satisfies

$$\left(\int_{E^n} \{N_\nu(u)(x)\}^p w(x)\,dx\right)^{1/p} < \infty.$$

Moreover, there is a constant c depending only on w, ν, and n such that

$$c^{-1}\|u\|_{H^p_w} \le \left(\int_{E^n} \{N_\nu(u)(x)\}^p w(x)\,dx\right)^{1/p} \le c\|u\|_{H^p_w}.$$

We recall some facts from [5] concerning H^p_w. If $w \in A_\infty$, the statement that a harmonic u belongs to H^p_w implies that there is a vector $F(x, y) = (u_0(x, y), u_1(x, y), \cdots, u_m(x, y))$ of length m depending on p, with harmonic components u_i satisfying appropriate differential relations, with

(i) $u_0 = u$, and

(ii) $\sup_{y>0} (\int_{E^n} |F(x, y)|^p w(x)\,dx)^{1/p} < \infty$,

where $|F| = (\Sigma_0^m u_i^2)^{1/2}$. The expression in (ii) is bounded above by a constant multiple of $\|u\|_{H^p_w}$. Moreover, the principle of harmonic majorization holds—that is, there is a number $s > 1$ and a nonnegative function $h(x)$ satisfying

(iii) $\int_{E^n} \{h(x)\}^s w(x)\,dx \le c\|u\|^p_{H^p_w}$, and

(iv) $|F(x, y)|^p \le \{h(x, y)\}^s$, where $h(x, y)$ is the Poisson integral of h.

The number s can be chosen arbitrarily large by choosing a sufficiently long vector F at the beginning. For a discussion of these facts and their relations, see the material at the end of §1 of [5]. In particular, given w, choose the r of [5] so small that $w \in A_{p/r}$. This is possible since $w \in A_\infty$. Then take $s = p/r$.

The following result will be useful in studying g^*_λ.

Theorem 4. *Let $u(x, y)$ be harmonic in E^{n+1}_+, $\lambda > 1$, $r > 0$, and $p_0 = r/\lambda$.*

(i) *If $p_0 < p < \infty$, $w \in A_{p/p_0}$ and $u \in H^p_w$, then*

$$\left(\int_{E^n} \{T_{\lambda,r}(u)(x)\}^p w(x)\,dx\right)^{1/p} \le c\|u\|_{H^p_w}.$$

(ii) *If $w \in A_1$ and $u \in H^{p_0}_w$, then for $\alpha > 0$,*

$$m_w\{T_{\lambda,r}(u)(x) > \alpha\} \le c\alpha^{-p_0}\|u\|^{p_0}_{H^{p_0}_w}.$$

The constants c are independent of u and α.

The point of Theorem 4 is that the restrictions on r have been weakened, allowing p to take values less than or equal to 1. The restriction that u be a Poisson integral is replaced by the assumption that u be any harmonic function in H^p_w.

To prove Theorem 4, we find a positive harmonic function $h(x, y)$ and a number s greater than 1 such that

$$|u(x, y)| \le \{h(x, y)\}^{s/p}, \tag{2.11}$$

where $h(x, y)$ is the Poisson integral of a nonnegative function $h(x)$ with the property

$$\int_{E^n} \{h(x)\}^s w(x)\,dx \le c\|u\|^p_{H^p_w}. \tag{2.12}$$

By (2.11), we obtain

$$T_{\lambda,r}(u)(x) \le \{T_{\lambda,rs/p}(h)(x)\}^{s/p}. \tag{2.13}$$

Raising both sides to the pth power and integrating, we obtain

$$\int_{E^n} \{T_{\lambda,r}(u)(x)\}^p w(x)\,dx \le \int_{E^n} \{T_{\lambda,rs/p}(h)(x)\}^s w(x)\,dx.$$

Theorem 3, part (i), shows that the last integral is bounded by a constant times

$$\int_{E^n} \{h(x)\}^s w(x)\,dx,$$

provided all the following conditions are met: $1 < rs/p < \infty$, $1 < \lambda < rs/p$, $rs/p\lambda < s < \infty$ and $w \in A_{s(rs/p\lambda)-1}$. The first two of these conditions can be met by choosing s sufficiently large, depending on p and r, from the beginning. The last two conditions can be written $p_0 < p$ and $w \in A_{p/p_0}$ respectively, which are the assumptions of part (i) of Theorem 4. Theorem 4 (i) now follows from (2.12).

The proof of Theorem 4 (ii) is similar. Take $p = p_0$ in (2.11), (2.12) and (2.13). By (2.13),

$$m_w\{T_{\lambda,r}(u)(x) > \alpha\} \le m_w\{T_{\lambda,rs/p_0}(h)(x) > \alpha^{p_0/s}\}$$

for $\alpha > 0$. Applying the appropriate version of Theorem 3 (ii), we see that the last expression is bounded by a constant times

$$(\alpha^{p_0/s})^{-rs/p_0\lambda} \int_{E^n} \{h(x)\}^{rs/\lambda p_0} w(x)\,dx = \alpha^{-p_0} \int_{E^n} \{h(x)\}^s w(x)\,dx,$$

provided $1 < rs/p_0 < \infty$, $1 < \lambda < rs/p_0$ and $w \in A_1$. But $w \in A_1$ by hypothesis, and the first two conditions can be met by choosing s large. Theorem 4 (ii) therefore follows from the last equality and (2.12) with $p = p_0$. This completes the proof of Theorem 4.

Remark. When the harmonic function $u(x, y)$ is the Poisson integral of a function $f(x)$, Theorem 4 gives an extension of Theorem 3 to values $\lambda \ge r$ and $p > 1$. However, the estimate

$$\|u\|_{H^p_w} \le c\left(\int_{E^n} |f(x)|^p w(x)\,dx\right)^{1/p} \tag{2.14}$$

for the right-hand side of the conclusion of Theorem 4 is not true in general unless $w \in A_p$. (Note that $p/p_0 \geq p$ since $p_0 = r/\lambda \leq 1$.) With this added assumption, (2.14) is true by (1.1), since $N_1(u)(x) \leq cf^*(x)$.

3. **Results for g_λ^*.** In this section we will prove our main result, Theorem 2, as stated in the introduction, and state separately some special cases of it. Fix $\lambda > 1$, $w(x)$ and the harmonic function $u(x, y)$, $y > 0$. Consider the truncated operator

$$g^*(x; R) = \left(\iint_{B_R} \frac{y^{(\lambda-1)n+1}}{(y + |x - z|)^{\lambda n}} |\nabla u(z, y)|^2 \, dz\, dy \right)^{1/2},$$

where $0 < R < \infty$ and $B_R = \{(z, y): |z| < R, 1/R < y < R\}$. Clearly, $g^*(x; R)$ increases to $g_\lambda^*(u)(x)$ as R increases to infinity. Moreover, a very simple estimate shows that

$$(3.1) \qquad g^*(x; R) \leq k/(1 + |x|)^{\lambda n/2} = k/(1 + |x|)^{n/p_0},$$

where $p_0 = 2/\lambda$ and $k = k(R, f, \lambda)$ is a constant. Let $T(x) = T_{\lambda,2}(u)(x)$ be the maximal function of §2 for $r = 2$. Since the hypotheses of parts (i) and (ii) of Theorem 2 are the same respectively as those of parts (i) and (ii) of Theorem 4 for $r = 2$, T satisfies the conclusions of Theorem 4 for $r = 2$ and $p_0 = 2/\lambda$: namely,

$$(3.2) \qquad \left(\int_{E^n} \{T(x)\}^p w(x)\, dx \right)^{1/p} \leq c\|u\|_{H_w^p}$$

if $2/\lambda < p < \infty$ and $w \in A_{p\lambda/2}$, and

$$(3.3) \qquad m_w\{T(x) > \alpha\} \leq c\alpha^{-2/\lambda} \|u\|_{H_w^{2/\lambda}}^{2/\lambda}$$

if $w \in A_1$, $\alpha > 0$.

For $0 < \alpha$, $R < \infty$ we have that

$$\{g^*(x; R) > \beta\alpha\} \subset \{g^*(x; R) > \beta\alpha,\ T(x) \leq \delta\alpha\} \cup \{T(x) > \delta\alpha\},$$

where β and δ are constants which satisfy $\beta > 1$ and $0 < \delta < 1$, and which will be chosen in the course of the proof to be independent of R, α and u. Hence for any $w(x)$,

$$(3.4) \quad m_w\{g^*(x; R) > \beta\alpha\} \leq m_w\{g^*(x; R) > \beta\alpha,\ T(x) \leq \delta\alpha\} + m_w\{T(x) > \delta\alpha\}.$$

The major part of the proof consists of showing there are positive constants c and ϵ depending only on λ, n and w such that

$$m_w\{g^*(x;\ R) > \beta\alpha,\ T(x) \le \delta\alpha\} \le c(\delta/\beta)^\epsilon m_w\{g^*(x;\ R) > \alpha\}. \tag{3.5}$$

The only assumption on w required to prove this inequality will be that $w \in A_\infty$. It is easy to check from the definition of $g^*(x;\ R)$ and (3.1) that $\{g^*(x;\ R) > \alpha\}$ is a bounded open set. We then use the Whitney lemma (see [14, p. 16]) to decompose

$$\{g^*(x;\ R) > \alpha\} = \bigcup_k I_k \tag{3.6}$$

into the union of nonoverlapping "cubes" I_k with the property that $2I_k$ intersects $\{g^*(x;\ R) \le \alpha\}$.(2) Since $\{g^*(x;\ R) > \beta\alpha\} \subset \{g^*(x;\ R) > \alpha\}$, we have

$$\{g^*(x;\ R) > \beta\alpha,\ T(x) \le \delta\alpha\} = \bigcup_k \{x \in I_k,\ g^*(x;\ R) > \beta\alpha,\ T(x) \le \delta\alpha\}.$$

We will show that there exist such c and ϵ so that

$$m_w\{x \in I_k,\ g^*(x;\ R) > \beta\alpha,\ T(x) \le \delta\alpha\} \le c(\delta/\beta)^\epsilon m_w(I_k). \tag{3.7}$$

To see that (3.7) implies (3.5), add both sides of (3.7) over k. Since c is independent of k and the I_k are nonoverlapping, we obtain

$$m_w\{g^*(x;\ R) > \beta\alpha,\ T(x) \le \delta\alpha\} \le c(\delta/\beta)^\epsilon \sum_k m_w(I_k) = c(\delta/\beta)^\epsilon m_w\{g^*(x;\ R) > \alpha\}$$

by (3.6).

To prove (3.7), it is enough by (A_∞) to prove the following analogue for Lebesgue measure:

$$|\{x \in I_k,\ g^*(x;\ R) \ge \beta\alpha,\ T(x) \le \delta\alpha\}| \le c(\delta/\beta)^2 |I_k| \tag{3.8}$$

where c depends only on λ and n. This local Lebesgue estimate is the heart of the matter. The weighted version (3.7) follows immediately from (3.8) and (A_∞). Write $I_k = I$ and $E = \{x \in I,\ g^*(x;\ R) \ge \beta\alpha,\ T(x) \le \delta\alpha\}$. We may assume E is not empty, since otherwise (3.8) is trivial. Here I is a cube contained in $\{g^*(x;\ R) > \alpha\}$ with the property that $2I$ intersects $\{g^*(x;\ R) \le \alpha\}$, E is a subset of I and E is closed. If J denotes the subset $\{(z,\ y):\ z \in 4I,\ 0 < y < |I|\}$ of E_+^{n+1} then

$$\begin{aligned}[g^*(x;\ R)]^2 &= \iint_{B_R} \frac{y^{(\lambda-1)n+1}}{(y + |x-z|)^{\lambda n}} |\nabla u(z,\ y)|^2\, dz\, dy \\ &\le \iint_J + \iint_{B_R - J} \frac{y^{(\lambda-1)n+1}}{(y + |x-z|)^{\lambda n}} |\nabla u(z,\ y)|^2\, dz\, dy.\end{aligned} \tag{3.9}$$

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(2) Here and later, if I is a cube and c is a positive constant, we use cI to denote the cube concentric with I whose edges are c times as long as those of I.

Let a be any fixed point of $2I \cap \{g^*(x; R) \le \alpha\}$. If $x \in I$ and $(z, y) \notin J$, a simple computation shows that $y + |a - z| \le c(y + |x - z|)$, for a constant c which depends only on n. Thus, for the second integral on the right in (3.9) when $x \in I$ we have

$$\iint_{B_R - J} \frac{y^{(\lambda-1)n+1}}{(y + |x - z|)^{\lambda n}} |\nabla u(z, y)|^2 \, dz \, dy \le c \iint_{B_R} \frac{y^{(\lambda-1)n+1}}{(y + |a - z|)^{\lambda n}} |\nabla u(z, y)|^2 \, dz \, dy$$

$$= c[g^*(a; R)]^2 \le c\alpha^2,$$

by the choice of a. Here c depends only on λ and n. This inequality holds for $x \in E$, since E is a subset of I. Using this estimate in (3.9) and recalling the definition of E, we see that if we choose β sufficiently large, the choice depending only on λ and n, then for all $x \in E$

$$\frac{\beta^2 \alpha^2}{2} \le \iint_J \frac{y^{(\lambda-1)n+1}}{(y + |x - z|)^{\lambda n}} |\nabla u(z, y)|^2 \, dz \, dy.$$

Integrating both sides of this inequality over E, we obtain

$$\frac{\beta^2 \alpha^2}{2} |E| \le \iint_J y \left(\int_E \frac{y^{(\lambda-1)n}}{(y + |x - z|)^{\lambda n}} \, dx \right) |\nabla u(z, y)|^2 \, dz \, dy. \tag{3.10}$$

We now decompose J. Let $\Gamma_1(x)$ be the cone $\{(z, y): |x - z| < y\}$ and let $W = \bigcup_{x \in E} \Gamma_1(x)$. Then we write $J = (J \cap W) + (J - W)$ and consider separately the integrals

$$\iint_{J \cap W} y \left(\int_E \frac{y^{(\lambda-1)n}}{(y + |x - z|)^{\lambda n}} \, dx \right) |\nabla u(z, y)|^2 \, dz \, dy \tag{3.11}$$

and

$$\iint_{J - W} y \left(\int_E \frac{y^{(\lambda-1)n}}{(y + |x - z|)^{\lambda n}} \, dx \right) |\nabla u(z, y)|^2 \, dz \, dy, \tag{3.12}$$

whose sum is the expression on the right of (3.10).

To estimate (3.11), we use a standard argument. Since

$$\int_E \frac{y^{(\lambda-1)n}}{(y + |x - z|)^{\lambda n}} \, dx \le \int_{E^n} \frac{y^{(\lambda-1)n}}{(y + |x|)^{\lambda n}} dx,$$

and the last integral is a constant depending only on λ and n, (3.11) is majorized by this constant times

$$\iint_{J \cap W} y |\nabla u(z, y)|^2 \, dz \, dy.$$

Since $|\nabla u|^2 = \frac{1}{2}\Delta(u^2)$ for harmonic u, this equals half of

$$\iint_{J\cap W} y\Delta[u^2(z,y)]\,dz\,dy. \tag{3.13}$$

Applying Green's theorem, we find (3.13) equals

$$\int_{\partial(J\cap W)}\left[y\frac{\partial u^2}{\partial\eta} - u^2\frac{\partial y}{\partial\eta}\right]d\sigma,$$

where $\partial(J\cap W)$ denotes the boundary of $J\cap W$, $\partial/\partial\eta$ denotes differentiation with respect to the outer normal of $\partial(J\cap W)$ and $d\sigma$ denotes the differential surface area on $\partial(J\cap W)$. Since $|\partial u^2/\partial\eta| \le 2|u|\,|\nabla u|$ and $|\partial y/\partial\eta| \le 1$, (3.13) is bounded by twice

$$\int_{\partial(J\cap W)}[|u|(y|\nabla u|) + u^2]\,d\sigma.$$

If $(z, y)\in\partial(J\cap W)$ then (z, y) belongs to the closure of a cone $\Gamma_1(x)$ with $x\in E$. Since $T(x)\le\delta\alpha$ for $x\in E$, it follows from (1.4) and (1.5) that the last integral is bounded by a constant times

$$(\delta\alpha)^2\int_{\partial(J\cap W)}d\sigma \le c(\delta\alpha)^2|I|.$$

This is our estimate for (3.11).

To estimate (3.12), we must first decompose $J - W$. Using Whitney's lemma (see [14, p. 16]) on the open set $(8I)^\circ - E$, we can cover $(4I) - E$ by nonoverlapping "cubes" Q_k so that no Q_k intersects E and $c^{-1}\delta_k \le d_k \le c\delta_k$, where d_k is the edge length of Q_k, δ_k is the distance from Q_k to E (not to the complement of $(4I) - E$) and $c > 1$ is a constant. Let $H_k = \{(z, y)\colon z\in Q_k,\ (z, y)\notin W\}$ denote the points of E_+^{n+1} above Q_k not in W. Then the H_k are nonoverlapping and $J - W\subset\bigcup_k H_k$. Hence, (3.12) is majorized by

$$\sum_k\iint_{H_k} y\left(\int_E\frac{y^{(\lambda-1)n}}{(y+|x-z|)^{\lambda n}}\,dx\right)|\nabla u(z,y)|^2\,dz\,dy.$$

If $x\in E$ and $(z, y)\in H_k$ then $|x - z|\ge d_k$. Therefore, (3.12) is majorized by

$$\sum_k\iint_{H_k} y^{(\lambda-1)n+1}\left(\int_{|x-z|\ge d_k}\frac{dx}{|x-z|^{\lambda n}}\right)|\nabla u(z,y)|^2\,dz\,dy,$$

which equals a constant depending on λ and n times

$$\sum_k d_k^{-(\lambda-1)n}\iint_{H_k} y^{(\lambda-1)n+1}|\nabla u(z,y)|^2\,dz\,dy.$$

We claim that

$$(3.14)\qquad d_k^{-(\lambda-1)n} \iint_{H_k} y^{(\lambda-1)n+1}|\nabla u(z,y)|^2\,dz\,dy \le c(\delta\alpha)^2|Q_k|,$$

where c depends only on n and λ. If this is true, then adding these estimates, we see that (3.12) is majorized by

$$c(\delta\alpha)^2 \sum_k |Q_k| \le c(\delta\alpha)^2|I|.$$

This is our estimate for (3.12). Since the estimate for (3.11) is the same, we obtain from (3.10) that $|E| \le c(\delta/\beta)^2|I|$, which is what was claimed in (3.8).

To show (3.14), we note by [14, p. 275], that for each ν, $0<\nu<1$, there is a constant c depending only on ν and n such that

$$(3.15)\qquad |\nabla u(z,y)|^2 \le cy^{-2}\frac{1}{|B_{\nu y}(z,y)|}\iint_{B_{\nu y}(z,y)} u^2(\xi,\eta)\,d\xi\,d\eta,$$

where $B_{\nu y}(z,y)$ is the ball with center (z,y) and radius νy. If $(\xi,\eta)\in B_{\nu y}(z,y)$ then $(1-\nu)y \le \eta \le (1+\nu)y$, and therefore $B_{\nu y}(z,y) \subset \Gamma_{\nu/(1-\nu)}(z)$. If we substitute (3.15) for $|\nabla u|^2$ in (3.14), change the order of integration and use the estimates above, we see that the expression on the left in (3.14) is bounded by

$$cd_k^{-(\lambda-1)n}\iint_{H_k^*} \eta^{(\lambda-1)n-1}u^2(\xi,\eta)\,d\xi\,d\eta,$$

where $H_k^* = (\bigcup_{z\in Q_k}\Gamma_{\nu/(1-\nu)}(z)) - W$ and $c=c(\nu,\lambda,n)$. Thus for small ν, H_k^* is a slightly expanded version of H_k with the same base Q_k. Choose $x_k\in E$ so that the distance from x_k to Q_k is δ_k. Clearly, H_k^* is contained in the complement of $\Gamma_1(x_k)$. Moreover, $H_k^*\subset J(x_k, cd_k)$ where $J(x_k, cd_k) = \{(z,y)\colon |x_k - z| < cd_k,\ 0<y<cd_k\}$, $c=c(\nu,n)$. Hence

$$d_k^{-(\lambda-1)n}\iint_{H_k^*}\eta^{(\lambda-1)n-1}u^2(\xi,\eta)\,d\xi\,d\eta \le cd_k^n\{T(x_k)\}^2 \le c|Q_k|(\delta\alpha)^2,$$

$c=c(\nu,n,\lambda)$, as claimed. This proves (3.14); hence (3.5) is proved.

To prove Theorem 2, we combine (3.4) and (3.5) to obtain

$$(3.16)\qquad m_w\{g^*(x;R)>\beta\alpha\} \le c(\delta/\beta)^\epsilon m_w\{g^*(x;R)>\alpha\} + m_w\{T(x)>\delta\alpha\},$$

with $w\in A_\infty$ and c and ϵ depending only on λ, n and w. Multiplying both sides by α^{p-1}, and integrating with respect to α over $(0,\infty)$, we obtain

$$(3.17)\qquad \begin{aligned}\frac{1}{\beta^p}\int_{E^n}\{g^*(x;R)\}^p w(x)\,dx &\le c(\delta/\beta)^\epsilon\int_{E^n}\{g^*(x;R)\}^p w(x)\,dx\\ &\quad+\frac{1}{\delta^p}\int_{E^n}\{T(x)\}^p w(x)\,dx.\end{aligned}$$

The estimate (3.1) implies that

$$\int_{E^n} \{g^*(x;\ R)\}^p w(x)\, dx \le k^p \int \frac{w(x)}{(1+|x|)^{np/p_0}}\, dx,$$

$p_0 = 2/\lambda$, $k = k(R, f, \lambda)$. If we now assume that $w \in A_{p/p_0}$, $p > p_0$, it follows from (1.3) that the last integral is finite. Hence the first two integrals in (3.17) are finite. We recall that β has already been chosen in the discussion after (3.9), and that β depends only on λ and n. Now choosing δ sufficiently small, the choice depending only on β and the c in (3.17), we obtain

$$\int_{E^n} \{g^*(x;\ R)\}^p w(x)\, dx \le c \int_{E^n} \{T(x)\}^p w(x)\, dx,$$

with c independent of u and R. Hence, letting $R \to \infty$, we have

$$\int_{E^n} \{g_\lambda^*(u)(x)\}^p w(x)\, dx \le c \int_{E^n} \{T(x)\}^p w(x)\, dx \tag{3.18}$$

by the monotone convergence theorem. Theorem 2, part (i), now follows by applying (3.2) to the right side of (3.18).

To prove part (ii) of Theorem 2, we multiply both sides of (3.16) by $\alpha^{2/\lambda}$, take the supremum over $\alpha > 0$, and adjust the constants to obtain

$$\begin{aligned}\frac{1}{\beta^{2/\lambda}} \sup_{\alpha>0} \alpha^{2/\lambda} m_w\{g^*(x;\ R) > \alpha\} &\le c(\delta/\beta)^\epsilon \sup_{\alpha>0} \alpha^{2/\lambda} m_w\{g^*(x;\ R) > \alpha\} \\ &\quad + \frac{1}{\delta^{2/\lambda}} \sup_{\alpha>0} \alpha^{2/\lambda} m_w\{T(x) > \alpha\}.\end{aligned} \tag{3.19}$$

The estimate (3.1) implies that $\{g^*(x;\ R) > \alpha\}$ is empty for large α, say for $\alpha \ge N$. For $0 < \alpha < N$, it implies that $\{g^*(x;\ R) > \alpha\} \subset \{|x| < (k/\alpha)^{2/\lambda n}\}$, and therefore

$$\alpha^{2/\lambda} m_w\{g^*(x; R) > \alpha\} \le \alpha^{2/\lambda} \int_{|x|<(k/\alpha)^{2/\lambda n}} w(x)\, dx.$$

The last expression is bounded for all α satisfying $0 < \alpha < N$ by a fixed constant times $w^*(x_0)$ for any x_0 in a sufficiently small neighborhood of the origin. Therefore, if $w \in A_1$,

$$\sup_{\alpha>0} \alpha^{2/\lambda} m_w\{g^*(x;\ R) > \alpha\} \le c w^*(x_0)$$

for any x_0 near 0. In particular, $\sup_{\alpha>0} \alpha^{2/\lambda} m_w\{g^*(x;\ R) > \alpha\}$ is finite if $w \in A_1$, and choosing δ sufficiently small we obtain from (3.19) that

$$\sup_{\alpha>0} \alpha^{2/\lambda} m_w\{g^*(x;\ R) > \alpha\} \le c \sup_{\alpha > 0} \alpha^{2/\lambda} m_w\{T(x) > \alpha\},$$

with c independent of u and R. Letting $R \to \infty$, we obtain

$$(3.20) \qquad \sup_{\alpha>0} \alpha^{2/\lambda} m_w\{g_\lambda^*(u)(x) > \alpha\} \le c \sup_{\alpha>0} \alpha^{2/\lambda} m_w\{T(x) > \alpha\}.$$

Theorem 2 (ii) follows by applying (3.3) to the right side of (3.20).

Remark. If one is interested only in deriving inequality (3.18), or its weak-type analogue (3.20), a wider class of weight functions w can be used. Let us consider (3.18) for $p > 0$. An examination of the proof shows that the only assumption needed to derive (3.18) from (3.16) is that

$$(3.21) \qquad \int_{E^n} \frac{w(x)\,dx}{(1+|x|)^{np/p_0}} < \infty,$$

$p_0 = 2/\lambda$, $p > 0$, this assumption being made in order to insure that

$$\int_{E^n} \{g^*(x;\, R)\}^p w(x)\,dy < \infty.$$

Of course, the assumption that $w \in A_\infty$ was used to derive (3.16). Condition (3.21) is also necessary for (3.18) if the right side of (3.18) is finite for some u. To see this, let u be such that the right side of (3.18) is finite. By very simple computations, we see

$$g_\lambda^*(u)(x) \ge g^*(x;\, R) \ge k(1 + |x|)^{-n\lambda/2},$$

$k = k(R, u, \lambda, n)$. Hence, if (3.18) holds, we must have

$$\int_{E^n} \frac{w(x)}{(1+|x|^{n\lambda/2})^p}\,dx < \infty.$$

We list as a corollary some results which can be proved from special cases of Theorem 2.

Corollary. *Let $u(x, y)$ denote the Poisson integral of $f(x)$.*

(i) *Let $1 < \lambda < 2$. If $2/\lambda < p < \infty$, $w \in A_{p\lambda/2}$, and $\int_{E^n} |f(x)|^p w(x)\,dx < \infty$, then*

$$\int_{E^n} \{g_\lambda^*(u)(x)\}^p w(x)\,dx \le c \int_{E^n} |f(x)|^p w(x)\,dx.$$

Moreover, if $w \in A_1$ and $\int_{E^n} |f(x)|^{p_0} w(x)\,dx < \infty$ then

$$m_w\{g_\lambda^*(u)(x) > \alpha\} \le c\alpha^{-2/\lambda} \int_{E^n} |f(x)|^{2/\lambda} w(x)\,dx.$$

(ii) *Let $\lambda \ge 2$. If $1 < p < \infty$, $w \in A_p$, and $\int_{E^n} |f(x)|^p w(x)\,dx < \infty$ then*

$$\int_{E^n} \{g_\lambda^*(u)(x)\}^p w(x)\,dx \le c \int_{E^n} |f(x)|^p w(x)\,dx.$$

The constants c are independent of f and α.

If $1 < \lambda < 2$, then $p\lambda/2 < p$ and therefore, by Holder's inequality, $w \in A_p$ if $w \in A_{p\lambda/2}$. If $\lambda \geq 2$, however, the assumption that $w \in A_{p\lambda/2}$ is weaker than $w \in A_p$ since $p\lambda/2 \geq p$. Since $N_\nu(u)(x) \leq cf^*(x)$,

$$\|u\|_{H^p_w} \leq c\left(\int_{E^n} |f^*(x)|^p w(x)\,dx\right)^{1/p},$$

which is majorized by a constant times $(\int_{E^n}|f(x)|^p w(x)\,dx)^{1/p}$ if $w \in A_p$ (see (1.1)). The corollary follows easily from this and Theorem 2.

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903