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THE GEOMETRY OF FLAT BANACH SPACES

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ABSTRACT. A Banach space is flat if the girth of its unit ball is 4 and if the girth is achieved by some curve. (Equivalently, its unit ball can be circumnavigated along a centrally symmetric path whose length is 4.) Some basic geometric properties of flat Banach spaces are given. In particular, the term *flat* is justified.

1. **Introduction.** Of interest here is the description of the basic geometric properties of flat Banach spaces. A real Banach space X is said to be *flat* if the *girth* of the unit ball of X (defined by Schäffer [9] to be the infimum of the lengths of all centrally symmetric curves which lie in the surface of the unit ball) is 4 *and* if the girth is achieved by some curve (i.e., the infimum is a minimum). This is equivalent to the statement that there exists a function g: $[0,2] \to X$ such that

$$\begin{aligned} &\|g(t)\| = 1 \text{ for each } t \in [0,2],\ g(0) = -g(2), \\ &\text{and } g \text{ is Lipschitz continuous with constant } 1, \end{aligned} \tag{1}$$

where $\|\cdot\|$ denotes the norm of X. In fact, the function g is easily seen to be the arc-length representation of a curve lying in the surface of the unit ball with antipodal endpoints and with length 2. This curve together with its reflection through the origin achieves the girth 4. Flat Banach spaces (namely, $C[0,1]$ and $L^1[0,1]$) were introduced by Harrell and Karlovitz [3], [4]. Further classes of flat Banach spaces are given in Nyikos and Schäffer [8] ($C_\sigma(K)$ for various choices of σ and K) and in Schäffer [10] ($L^1[\mu]$ is shown to be flat if and only if μ is not purely atomic). Beside the intriguing property that the unit ball can be circumnavigated along a centrally symmetric path of length 4, flat Banach spaces have further special geometric properties. Some of these, including the ones that give rise to the term *flat*, were announced in [4]. It is our purpose here to describe the geometric properties of flat Banach spaces in general and to provide details for the announcement [4] in particular. In the course of this, it will become evident that girth curves provide a useful tool for investigating various questions about Banach spaces.

2. **Statement of theorems.** If X is a flat Banach space and g: $[0,2] \to X$ satisfies (1), we say that g is a *girth curve* for X. If g is a girth curve for X, then for each

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pair of reals s,h with $s \in (0,2]$ and $h \in (0,s]$ we define the *difference quotient* $\Delta_g(s,h)$ by

$$\Delta_g(s,h) = h^{-1}(g(s-h) - g(s)). \tag{2}$$

Further, for each $s \in [0,2]$ we define the *chord set of g at s*, denoted by $\chi(g,s)$, by

$$\begin{aligned}\chi(g,s) = \text{closed convex hull } (\{-\Delta_g(t,h)\colon t \in (0,s], h \in (0,t]\} \\ \cup \{\Delta_g(t,h)\colon t \in (s,2], h \in (0,t-s]\}).\end{aligned} \tag{3}$$

If $s = 0$ ($s = 2$), the first (second) set is empty. Finally, for each $s \in [0,2]$ we choose a linear functional $f_g^*(s) \in X^*$, the conjugate space of X, which satisfies

$$\|f_g^*(s)\| = 1, \qquad \langle f_g^*(s), g(s)\rangle = 1, \tag{4}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between X and X^*. The following theorem and remark describe the special properties of $\chi(g,s)$ and justify the terms *flat Banach space* and *completely flat Banach space*, where X is said to be completely flat if X is flat and if $X =$ closed linear hull $\{g(s)\colon s \in [0,2], g$ is a girth curve for $X\}$.

Theorem 1. *Let X be a flat Banach space. Let $g\colon [0,2] \to X$ be a girth curve for X, i.e., g satisfies* (1). *Then for each $s \in [0,2]$*

$$g(s) \in \chi(g,s) \subset \{x\colon \langle f_g^*(s), x\rangle = 1\} \cap \{x\colon \|x\| = 1\}. \tag{5}$$

Moreover, for each $y \in \chi(g,s)$,

$$\sup_{x\in\chi(g,s)} \|y - x\| = \sup_{x,z\in\chi(g,s)} \|x - z\| = 2, \tag{6}$$

i.e., the diameter of $\chi(g,s)$ is 2 and each point of $\chi(g,s)$ is diametral. (*Hence $\chi(g,s)$ fails to have normal structure.*) *Finally, if $X =$ closed linear hull $\{g(t)\colon t \in [0,2]\}$, then*

$$\textit{closed affine hull } \chi(g,s) = \{x\colon \langle f_g^*(s), x\rangle = 1\}. \tag{7}$$

Remark 1. According to (5), $\chi(g,s)$ is a *flat* area in the surface of the unit ball, i.e., it belongs to the intersection of the unit ball with a supporting hyperplane. Since each subset of the unit ball has diameter at most 2, $\chi(g,s)$ is, by (6), maximal in the sense of diameter. (Moreover, each of its points is diametral.) Thus if we imagine a circumnavigation of the unit ball along the path formed by $g\colon [0,2] \to X$ and $-g\colon [0,2] \to X$, then we observe two properties: the length of the journey is only 4, and each point of it lies in a large flat area. Hence the term *flat Banach space*. If, moreover, $X =$ closed linear hull $\{g(s)\colon s \in [0,2]\}$ then, by (7), the flat area looks "locally" like a hyperplane. Hence the term *completely flat Banach space* (or the alternate *pre-Columbian Banach space* suggested by R. J. Duffin).

Example 1. For each point $y \in L^1[0,1]$, $\|y\| = 1$, define $g_y\colon [0,2] \to L^1[0,1]$ by

$$\begin{aligned} g_y(s) &= -y(t), \qquad t \in [0,\alpha], \\ &= y(t), \qquad t \in (\alpha, 1], \end{aligned} \qquad \text{where } \int_0^\alpha |y(t)|\,dt = \frac{s}{2}.$$

It is easy to see that g_y is indeed a girth curve for $L^1[0,1]$. Moreover, if $y(t) \equiv 1$, then $L^1[0,1] =$ closed linear hull $\{g_y(s)\colon s \in [0,2]\}$. In general, it can be shown that closed linear hull $\{g_y(s)\colon s \in [0,2]\}$ is isometrically isomorphic to $L^1[0,1]$. Since $g_y(0) = y$, each point y in the surface of the unit ball belongs to a girth curve. Therefore each point y in the surface of the unit ball not only fails to be an extreme point, but it belongs to a flat area $\chi(g_y, 0)$ which satisfies the conclusions of Theorem 1.

The following remark shows that the flat area $\chi(g, s)$ also leads to a characterization of flat Banach spaces.

Remark 2. Let $x((2i-1)/2^n) = \Delta_g(i/2^{n-1}, 1/2^{n-1})$, $n = 1, 2, \ldots, i = 1, \ldots, 2^{n-1}$. Then by (2),

$$x((2i-1)/2^n) = \tfrac{1}{2}x((4i-3)/2^{n+1}) + \tfrac{1}{2}x((4i-1)/2^{n+1}). \tag{8}$$

Furthermore, by (5), for each $s \in [0,2]$

$$\begin{aligned} \langle f_g^*(s), x((2i-1)/2^n)\rangle &= -1, \qquad i/2^{n-1} \le s, \\ &= 1, \qquad s \le (i-1)/2^{n-1}. \end{aligned} \tag{9}$$

It follows, in particular, that X has the *infinite tree property* of James [6] because the points $x((2i-1)/2^n)$ form an infinite tree. By (5) all of the points lie in the unit ball; they "branch" according to (8); and, as a consequence of (9),

$$\|x((4i-3)/2^{n+1}) - x((4i-1)/2^{n+1})\| \ge 2.$$

The space X also satisfies the *infinite supported tree property* of Harrell and Karlovitz [5] because for each rational $k/2^m \in [0,1]$, the infinite tree is *supported* by the hyperplanes $\{x\colon \langle f_g^*(k/2^m), x\rangle = 1\}$ and $\{x\colon \langle f_g^*(k/2^m), x\rangle = -1\}$; i.e., each point of the tree lies in one of the two hyperplanes or it is a finite convex combination of ones that do. In [5] it is shown that the converse is also true; namely, if X satisfies the infinite supported tree property then it is isomorphic to a flat Banach space. The infinite supported tree property is stronger than the infinite tree property. The former is preserved under conjugation while the latter is not.

The following theorem shows that girth curves actually occur in bundles.

Theorem 2. *Let X be a flat Banach space. Let $g\colon [0,2] \to X$ be a girth curve for X. Then g belongs to a "bundle" of nonintersecting girth curves each of which passes through all of the sets $\chi(g,s)$, $s \in [0,2]$. More precisely, for each $y \in$ convex hull*

$\{\Delta_g(s,h)\colon s \in (0,2], h \in (0,s]\}$ *there exists a girth curve* $g_y\colon [0,2] \to X$ *with the properties*:

$$g_y(0) = y; \tag{10}$$

$$g_y(1 - \langle f_g^*(t), y\rangle) \in \chi(g,t), \qquad t \in [0,2]; \tag{11}$$

the function $\phi_y(t) = 1 - \langle f_g^*(t), y\rangle$ *is nondecreasing and maps* $[0,2]$ *onto itself; and if* $y_1 \neq y_2$ *then* $g_{y_1}(s_1) \neq g_{y_2}(s_2)$ *for all* $s_1, s_2 \in [0,2]$.

By definition, the convex hull of $\{\Delta_g(s,h)\colon s \in (0,2], h \in (0,s]\}$ is dense in $\chi(g,0)$. However, it is not known whether the conclusion of Theorem 2 can be extended to all of $\chi(g,0)$, in general. In the following theorem, the extension is made under one additional hypothesis.

Theorem 3. *Let X be flat Banach space. Let* $g\colon [0,2] \to X$ *be a girth curve for X. Suppose that the functional*

$$|\cdot| = \sup_{s\in[0,2]} |\langle f_g^*(s), \cdot\rangle| \tag{12}$$

defines an equivalent norm on X. Then for each $y \in \chi(g,0)$ *there exists a girth curve* $g_y\colon [0,2] \to X$ *satisfying* (10) *and* (11). *It follows that* $\chi(g,0)$ *contains no extreme points.*

If X satisfies the hypotheses of Theorem 3, then X cannot be isomorphic to any subspace of any separable conjugate space. For $\chi(g,0)$ is a closed bounded convex subset of X which, by Theorem 3, has no extreme points; and a recent result of Bessaga and Pełczyński [1] asserts that the conclusion of the Kreĭn-Milman theorem also holds for the norm topology of separable conjugate spaces. By a different argument, it can be shown that, in fact, all flat Banach spaces fail to be isomorphic to subspaces of separable conjugate spaces. This is done in the next theorem.

Theorem 4. *Let X be a flat Banach space. Let* $g\colon [0,2] \to X$ *be a girth curve for X. Then g fails to be weakly differentiable for each* $s \in [0,2]$. *It follows that X is not isomorphic to any subspace of any separable conjugate space.*

Remark 3. Since every flat Banach space contains a separable flat subspace, the second part of Theorem 4 strengthens the known result that a flat space is nonreflexive and that there exist nonreflexive spaces which are not flat. It follows, for example, that l^1 is not flat. Moreover, by Remark 2, l^1 cannot even be renormed to be flat because, by James [6], it fails to have the infinite tree property.

Remark 4. The proof of the last part of Theorem 4 depends on a result of Gel'fand [2, Theorem 3] concerning the differentiability of Lipschitz continuous functions with values in a Banach space. As a corollary, Gel'fand derived the fact

that $L^1[0,1]$ is not isomorphic to any separable conjugate space. Here, this corollary follows from Example 1 and Theorem 4.

The conjugate spaces of flat Banach spaces are discussed in Karlovitz [7]. In particular, it is shown that the conjugate of a flat Banach space is again flat, and that it is nonseparable.

3. **Proofs.** Let X be a flat Banach space and g: $[0,2] \to X$ a girth curve of X. By (1), if $0 \le s \le t \le 2$ then $2 = \|g(0) - g(2)\| \le \|g(0) - g(s)\| + \|g(s) - g(t)\| + \|g(t) - g(2)\| \le s + (t - s) + (2 - t) = 2$. Consequently,

$$\|g(s) - g(t)\| = |s - t|, \quad \text{for } s, t \in [0,2]. \tag{13}$$

Hence, by (2), for $s \in (0,2]$ and $h \in (0,s]$,

$$\|\Delta_g(s,h)\| = 1. \tag{14}$$

By (1), (4) and (13), if $s, t \in [0,2]$,

$$|\langle f_g^*(t), g(s)\rangle - 1| \le \|g(s) - g(t)\| = |s - t|,$$

and

$$\begin{aligned} |\langle f_g^*(t), g(s)\rangle + 1| &\le \|g(s) + g(t)\| \\ &\le \begin{aligned} &\|g(s) - g(0)\| + \|g(2) - g(t)\| = s + (2 - t), \\ &\|g(s) - g(2)\| + \|g(0) - g(t)\| = (2 - s) + t. \end{aligned} \end{aligned}$$

From these inequalities we immediately derive

$$\langle f_g^*(t), g(s)\rangle = 1 - |s - t|, \quad \text{for } s, t \in [0,2]. \tag{15}$$

Hence we compute from (2),

$$\begin{aligned} \langle f_g^*(t), \Delta_g(s,h)\rangle &= 1, \qquad t \le s - h, \\ &= -1, \qquad s \le t. \end{aligned} \tag{16}$$

We also note that, by virtue of (2), for each h', $0 < h' < h$,

$$\Delta_g(s,h) = (h - h')h^{-1}\Delta_g(s - h', h - h') + h'h^{-1}\Delta_g(s, h'). \tag{17}$$

Proof of Theorem 1. From (2) and $g(0) = -g(2)$ it follows directly that if $s \in (0,2)$ then

$$((2 - s)/2)\Delta_g(2, 2 - s) + (s/2)(-\Delta_g(s,s)) = g(s).$$

Hence, by (3), $g(s) \in \chi(g,s)$. Similarly, $g(0) \in \chi(g,0)$ and $g(2) \in \chi(g,2)$. It follows directly from (16) and (3) that, for each $s \in [0,2]$,

$$\chi(g,s) \subset \{x: \langle f_g^*(s), x\rangle = 1\}. \tag{18}$$

Moreover, by (3) and (14), $\chi(g, s)$ belongs to the unit ball. Therefore, since $\|f_g^*(s)\| = 1$, it follows from (18) that if $x \in \chi(g,s)$ then $\|x\| = 1$. This finishes the proof of (5).

We now prove (6). Note that by (16) and (17), for each $s \in (0,2]$ and $h \in (0,s]$,

$$\begin{aligned}\langle f_g^*(s-h'), -\Delta_g(s,h)\rangle &= (h-h')h^{-1}\langle f_g^*(s-h'), -\Delta_g(s-h', h-h')\rangle \\ &\quad + h'h^{-1}\langle f_g^*(s-h'), -\Delta_g(s,h')\rangle \\ &= (h-h')h^{-1} - h'h^{-1} \to 1\end{aligned}$$

as $h' \to 0$. This argument is easily extended to all $y \in$ convex hull $(\{-\Delta_g(t,h): t \in (0,s], h \in (0,t]\} \cup \{\Delta_g(t,h): t \in (s,2], h \in (0,t-s]\})$. Hence, by (3), $\langle f_g^*(s-h'), y\rangle \to 1$ as $h' \to 0$ for each $y \in \chi(g,s)$. Hence, for each $y \in \chi(g,s)$,

$$\begin{aligned}\|y - (-\Delta_g(s,h'))\| &\geq \langle f_g^*(s-h'), y\rangle + \langle f_g^*(s-h'), \Delta_g(s,h')\rangle \\ &= \langle f_g^*(s-h'), y\rangle + 1 \to 2 \quad \text{as } h' \to 0.\end{aligned}$$

Since $-\Delta_g(s,h') \in \chi(g,s)$ for $h' \in (0,s]$, this shows that $\sup\{\|y-x\|: x \in \chi(g,s)\} \geq 2$. On the other hand, $\chi(g, s)$ is contained in the unit ball, and hence the supremum is not more than 2. Since y was arbitrarily chosen in $\chi(g, s)$, this proves (6) for $s \in (0,2]$. Since, by definition, $\chi(g,0) = -\chi(g,2)$, this finishes the proof of (6).

Finally, assuming $X =$ closed linear hull $\{g(t): t \in [0,2]\}$, we prove (7). From this assumption, by (5), we have

$$X = \text{closed linear hull } \{\Delta_g(t,h): t \in (0,2], h \in (0,t]\}.$$

Using (16) and (17), we readily find that $x \in X$ and $\langle f_g^*(s), x\rangle = 1$ if and only if $x \in$ closed affine hull $(\{-\Delta_g(t,h): t \in (0,s], h \in (0,t]\} \cup \{\Delta_g(t,h): t \in (s,2], h \in (0,t-s]\})$. Hence, by (3), $\langle f_g^*(s), x\rangle = 1$ if and only if $x \in$ closed affine hull $\chi(g, s)$. This finishes the proof of Theorem 1.

Proof of Theorem 2. Choose arbitiary $y \in$ convex hull $\{\Delta_g(s,h): s \in (0,2], h \in (0,s]\}$. By virtue of (17), for each $t \in (0,2)$, y can be expressed as

$$y = u(t) + w(t),$$

$$u(t) \in \text{nonnegative hull } \{\Delta_g(s,h): s \in (0,t], h \in (0,s]\}, \tag{19}$$

$$w(t) \in \text{nonnegative hull } \{\Delta_g(s,h): s \in (t,2], h \in (0,s-t]\},$$

where by the nonnegative hull we mean the set of finite linear combinations with nonnegative coefficients. Moreover, $u(t)$ and $w(t)$ are uniquely determined by (19) because, using (16) and (17), it is easily shown that the intersection of the linear hulls of the two sets in (19) contains only the zero element. By this uniqueness, using (17) and (19), we readily find that

$$u(t_2) - u(t_1) \in \text{nonnegative hull } \{\Delta_g(s,h)\colon s \in (t_1,t_2], h \in (0, s - t_1]\}, \quad \text{for } 0 \le t_1 < t_2 \le 2, \tag{20}$$

where we also let

$$u(0) = w(2) = 0, \qquad u(2) = w(0) = y. \tag{21}$$

Using (16), we readily find that if $u(t) \neq 0$ then $u(t)/\langle -f_g^*(s), u(t)\rangle \in \chi(g,0)$, for $t \le s$. Hence, using (5), we find

$$\|u(t)\| = \langle -f_g^*(s), u(t)\rangle, \qquad 0 \le t \le s \le 2. \tag{22}$$

Similarly, by (20), if $t_1 < t_2$ and $u(t_1) \neq u(t_2)$ then

$$(u(t_2) - u(t_1))/\langle -f_g^*(t_2), u(t_2) - u(t_1)\rangle \in \chi(g,0),$$

and hence

$$\|u(t_2) - u(t_1)\| = \langle -f_g^*(t_2), u(t_2) - u(t_1)\rangle, \qquad 0 \le t_1 \le t_2 \le 2.$$

Consequently, by (22),

$$\|u(t_2) - u(t_1)\| = \|u(t_2)\| - \|u(t_1)\|, \qquad 0 \le t_1 \le t_2 \le 2. \tag{23}$$

Since $y \in$ convex hull $\{\Delta_g(s,h)\colon s \in (0,2], h \in (0,s]\}$, it follows from (19) and (21) that

$$y - 2u(t) = -u(t) + w(t) \in \chi(g,t), \qquad t \in [0,2]. \tag{24}$$

Hence, by (5),

$$\langle f_g^*(t), y\rangle - 2\langle f_g^*(t), u(t)\rangle = 1. \tag{25}$$

We define the function $\phi_y\colon [0,2] \to [0,2]$ by $\phi_y(t) = 1 - \langle f_g^*(t), y\rangle$. Using (16) and (17) we see that ϕ_y is continuous. By (22) and (25), $\phi_y(t) = 2\|u(t)\|$, $t \in [0,2]$. Consequently, by (23), ϕ_y is nondecreasing; and since $\|u(0)\| = 0$ and $\|u(2)\| = \|y\| = 1$, it maps [0,2] onto itself. Moreover, if $\phi_y(t_1) = \phi_y(t_2)$ then, by (23), $u(t_1) = u(t_2)$. Therefore we can define a function $g_y\colon [0,2] \to X$ by

$$\begin{aligned} g_y(s) &= -u(t) + w(t) \\ &= y - 2u(t), \quad \text{where } \phi_y(t) = 2\|u(t)\| = s. \end{aligned} \tag{26}$$

By (24) and (5), $\|g_y(s)\| = 1$ for each $s \in [0,2]$, and, by (21), $g_y(0) = -g_y(2) = y$. Furthermore, since $\|g_y(s_1) - g_y(s_2)\| = \|(y - u(t_1)) - (y - u(t_2))\|$, where $2\|u(t_1)\| = s_1$ and $2\|u(t_2)\| = s_2$, it follows by (23) that $\|g_y(s_1) - g_y(s_2)\| = |s_1 - s_2|$. Thus g_y is Lipschitz continuous with constant 1. Altogether, g_y is a girth curve for X. By (21) and (26), it satisfies (10); and, by (24) and (26), it satisfies (11).

Now suppose that $g_{y_1}(s_1) = g_{y_2}(s_2)$ for some $y_1, y_2 \in$ convex hull $\{\Delta_g(s,h)\colon s \in (0,2], h \in (0,s]\}$ and $s_1, s_2 \in [0,2]$. We first note that $s_1 = s_2 = s'$. For if $s_1 < s_2$ then $2 = \|y_1 + y_2\| \leq \|g_{y_1}(0) - g_{y_1}(s_1)\| + \|g_{y_2}(s_2) - g_{y_2}(2)\| \leq s_1 + (2 - s_2) < 2$, which is a contradiction. We can express y_1 and y_2 by

$$y_1 = \sum_{i=1}^{n} \mu_i \Delta_g(s_i, h_i), \qquad y_2 = \sum_{i=1}^{n} \nu_i \Delta_g(s_i, h_i).$$

$s_i \in (0,2]$, $h_i \in (0,s_i]$, $\mu_i, \nu_i \geq 0$, $i = 1, \ldots, n$ and $\sum_i \mu_i = \sum_i \nu_i = 1$. By definition (26) we have

$$g_{y_1}(s') = y_1 - 2u_1(t_1), \qquad \text{where } 2\|u_1(t_1)\| = s',$$

$$g_{y_2}(s') = y_2 - 2u_2(t_2), \qquad \text{where } 2\|u_2(t_2)\| = s'.$$

So if $s' = 0$ then $y_1 = y_2$. If $s' \neq 0$ then by virtue of (17) we may assume without loss of generality that

$$u_1(t_1) = \sum_{i=1}^{k_1} \mu_i \Delta_g(s_i, h_i), \qquad u_2(t_2) = \sum_{i=1}^{k_2} \nu_i \Delta_g(s_i, h_i)$$

for some k_1 and k_2 with $1 \leq k_1 \leq k_2 \leq n$. We may also assume that $(s_j - h_j, s_j] \cap (s_k - h_k, s_k] = \varnothing$ for $j \neq k$. Using (16), it easily follows that $\{\Delta_g(s_i, h_i)\colon i = 1, \ldots, n\}$ is a linearly independent set. Consequently, we readily deduce from $g_{y_1}(s') = g_{y_2}(s')$ that $\mu_i = \nu_i$, $i = 1, \ldots, n$. Hence $y_1 = y_2$. This finishes the proof of Theorem 2.

Proof of Theorem 3. Choose arbitrary $y \in \chi(g, 0)$. By definition, there exists a sequence $\{y_n\}$ so that $\|y_n - y\| \to 0$ and so that for each n, $y_n \in$ convex hull $\{\Delta_g(s,h)\colon s \in (0,2], h \in (0,s]\}$. For each n and each $t \in [0,2]$, we define $u_n(t)$ and $w_n(t)$ according to (19) and (21). By the proof of Theorem 2, for each n, the function $g_{y_n}\colon [0,2] \to X$ defined by

$$g_{y_n}(s) = y_n - 2u_n(t), \quad \text{where } 2\|u_n(t)\| = s,$$

is a girth curve for X. Moreover, by (25), $\langle f_g^*(t), y_n\rangle = 1 + 2\langle f_g^*(t), u_n(t)\rangle$. By (22), $\langle f_g^*(s), u_n(t)\rangle = -\|u_n(t)\|$ if $s \geq t$, and, by a completely analogous argument, $\langle f_g^*(s), u_n(t)\rangle = \|u_n(t)\| - 2\|u_n(s)\|$ for $s \leq t$. Combining these relations, we find:

$$\begin{aligned} \langle f_g^*(s), u_n(t)\rangle &= \langle -\tfrac{1}{2} f_g^*(0) + \tfrac{1}{2} f_g^*(t), y_n\rangle, && 0 \leq t \leq s \leq 2, \\ &= \langle f_g^*(S) - \tfrac{1}{2} f_g^*(0) - \tfrac{1}{2} f_g^*(t), y_n\rangle, && 0 \leq s \leq t \leq 2, \end{aligned}$$

Consequently, $|u_n(t) - u_m(t)| \leq 2|y_n - y_m|$, where $|\cdot|$ is defined by (12). Therefore, by the hypothesis that $|\cdot|$ is equivalent to $\|\cdot\|$,

$$c_1 \|u_n(t) - u_m(t)\| \leq |u_n(t) - u_m(t)| \leq 2|y_n - y_m| \leq 2c_2 \|y_n - y_m\|,$$

for each $t \in [0,2]$, for each pair of positive integers n, m, and for some pair $c_1, c_2 > 0$. Since $\|y_n - y\| \to 0$, it follows from this inequality that for each $t \in [0,2]$ there exists $u(t)$ so that $\|u_n(t) - u(t)\| \to 0$, *uniformly in t*. By the proof of Theorem 2, for each n, the function $\phi_{y_n}\colon [0,2] \to [0,2]$ defined by $\phi_{y_n}(t) = 1 - \langle f_g^*(t), y_n\rangle = 2\|u_n(t)\|$, is nondecreasing and onto. Consequently, by the uniform convergence, the function $\phi_y(t) = 1 - \langle f_g^*(t), y\rangle = 2\|u(t)\|$ is also nondecreasing and maps [0, 2] onto itself. Furthermore, since $u_n(t)$ satisfies (23) for each n, it follows that if $\phi_y(t_1) = \phi_y(t_2)$ then $u(t_1) = u(t_2)$. Thus we can define $g_y\colon [0,2] \to X$ by $g_y(s) = y - 2u(t)$, where $\phi_y(t) = 2\|u(t)\| = s$. Thus, for each $s \in [0,2]$, $\|g_{y_n}(s) - g_y(s)\| = \|(y_n - 2u_n(t_n)) - (y - 2u(t))\|$, where $2\|u_n(t_n)\| = 2\|u(t)\| = s$. Hence, using (23),

$$\begin{aligned}\|g_{y_n}(s) - g_y(s)\| &\leq \|y_n - y\| + 2\|u_n(t_n) - u_n(t)\| + 2\|u_n(t) - u(t)\| \\ &= \|y_n - y\| + 2|\,\|u_n(t_n)\| - \|u_n(t)\|\,| + 2\|u_n(t) - u(t)\| \\ &= \|y_n - y\| + 2|\,\|u(t)\| - \|u_n(t)\|\,| + 2\|u_n(t) - u(t)\|.\end{aligned}$$

Since $\|y_n - y\| \to 0$ and $\|u_n(t) - u(t)\| \to 0$, uniformly in t, it follows that $\|g_{y_n}(s) - g_y(s)\| \to 0$, uniformly in s. Consequently, g_y is Lipschitz continuous with constant 1. Moreover, $g_y(0) = \lim g_{y_n}(0) = \lim y_n = y = -g_y(2)$, and $\|g_y(s)\| = \lim \|g_{y_n}(s)\| = 1$ for each $s \in [0,2]$. Altogether, g_y is a girth curve for X which satisfies (10). By Theorem 2, $g_{y_n}(s_n) \in \chi(g,t)$, whenever $\phi_{y_n}(t) = 2\|u_n(t)\| = s_n$. From this it follows easily that $\phi_y(s) \in \chi(g,t)$, whenever $\phi_y(t) = 2\|u(t)\| = s$. Thus g_y satisfies (11).

Finally we note that, by the definition of g_{y_n}, $\Delta_{g_{y_n}}(1,1)$ and $\Delta_{g_{y_n}}(2,1) \in \chi(g,0)$. Using (16), we find $\|\Delta_{g_{y_n}}(1,1) - \Delta_{g_{y_n}}(2,1)\| = 2$. Hence, in the limit, $\Delta_{g_y}(1,1)$, $\Delta_{g_y}(2,1) \in \chi(g,0)$ and $\|\Delta_{g_y}(1,1) - \Delta_{g_y}(2,1)\| = 2$. Since, by (2), $g_y(0) = \frac{1}{2}\Delta_{g_y}(1,1) + \frac{1}{2}\Delta_{g_y}(2,1)$, it follows that $g_y(0) = y$ is not an extreme point of $\chi(g,0)$. Since $y \in \chi(g,0)$ was arbitrarily chosen, $\chi(g,0)$ has no extreme points. This finishes the proof of Theorem 3.

Proof of Theorem 4. The fact that g fails to be weakly differentiable for each $s \in [0,2]$ follows from formula (13) and from the proof of Theorem 1 of [3]. We note, however, that for each $s \in (0,2)$, this can be immediately derived from formula (15).

To prove the second part of the theorem, we note that by Gel'fand [2, Theorem 3], a Lipschitz continuous function with values in a separable conjugate space is strongly differentiable almost everywhere. Since g is not even weakly differentiable, it follows that X cannot be isomorphic to any subspace of any separable conjugate space. This finishes the proof of Theorem 4.

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