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COUNTABLE BOX PRODUCTS OF ORDINALS

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ABSTRACT. The countable box product of ordinals is examined in the paper for normality and paracompactness. The continuum hypothesis is used to prove that the box product of countably many σ-compact ordinals is paracompact and that the box product of another class of ordinals is normal. A third class trivially has a nonnormal product.

Because I have found a countable box product of ordinals useful in the past [1], this class of spaces particularly interests me. The purpose of this paper is to tell what I know about which of these spaces is paracompact or normal.

In [2] I prove that the continuum hypothesis implies the box product of countably many σ-compact, locally compact, metric spaces is paracompact. I prove here that the continuum hypothesis implies the box product of countably many σ-compact ordinals is paracompact (Theorem 1) and the box product of another class of ordinals is normal (Theorem 2). The proof of Theorems 1 and 2 is a quite messy join of the techniques of [1] and [2] which raises some doubt in my mind as to whether these theorems are worth proving. Because I care, because I think these spaces are set theoretically interesting and topologically useful, because I think these theorems are best possible, the theorems are worth the mess to me.

A. If $\{X_\lambda\}_{\lambda\in\Lambda}$ is a family of topological spaces, a *box* in $\prod_{\lambda\in\Lambda} X_\lambda$ is a set $\prod_{\lambda\in\Lambda} U_\lambda$ where each U_λ is open in X_λ. The *box product* of $\{X_\lambda\}_{\lambda\in\Lambda}$ is $\prod_{\lambda\in\Lambda} X_\lambda$ topologized by using the set of all boxes in it as a basis.

Throughout the paper the following notation is used.

An ordinal α *is* the set of all ordinals less than α and α is topologized by the interval topology. The statement that α is a cardinal means that α is an ordinal and no smaller ordinal has the same cardinality as α.

The notation $\prod_{\lambda\in\Lambda} \beta_\lambda$ is used to mean the ordinary Cartesian product of the β_λ's and never the cardinal or ordinal arithmetic product. Similarly α^β means the set of all functions from β into α rather than an arithmetic operation.

If α is an ordinal, let $\mathrm{cf}(\alpha)$ denote the cofinality of α; that is $\mathrm{cf}(\alpha)$ is the smallest ordinal δ such that there is a subset Δ of α, order isomorphic with δ, such that $\beta < \alpha$ implies there is a $\gamma \in \Delta$ with $\beta \leq \gamma$. Observe that α is a σ-compact ordinal if and only if α is compact or $\mathrm{cf}(\alpha) = \omega_0$.

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Let [CH] and [GCH] denote *the continuum hypothesis is true* and *the generalized continuum hypothesis is true*, respectively.

To avoid repetition assume that for each $n \in \omega_0$, α_n is a positive ordinal, and let X be the box product of $\{\alpha_n\}_{n \in \omega_0}$. If $x \in X$ or $U \subset X$, let $x(n)$ and $U(n)$ denote the projection of x and U, respectively, on α_n.

B. Let $S = \{n \in \omega_0 \mid \alpha_n$ is not σ-compact$\}$.

Case 0. *The trivial case. There are $n \in S$ and $m \in \omega_0 - \{n\}$ with $\alpha_m > \mathrm{cf}(\alpha_n)$. Theorem* 0 *yields X is not normal.*

Case 1. *The basic case. $S = \varnothing$. Theorem* 1 *yields* [CH] *X is paracompact.*

Case 2. *The other case. Not Cases* 0 *or* 1.

Since not Case 1, $S \neq \varnothing$. And $s \in S$ imply α_s is not paracompact; so X is not paracompact. Define $\kappa(X) = \sup\{$cardinality of $\prod_{n \in \omega_0 - \{s\}} \beta_n \mid s \in S, \beta_n \leq \alpha_n$, and β_n is compact$\}$. Since $S \neq \varnothing$ and not Case 0, there is a unique uncountable δ such that $\mathrm{cf}(\alpha_s) = \delta$ for all $s \in S$. Since not Case 0 and δ is uncountable, $\kappa(X) \leq \sup\{$cardinality of $\beta^{\omega_0} \mid \beta$ is a cardinal less than $\delta\}$. Thus [GCH] implies $\kappa(X) < \delta$ unless δ is the cardinal successor of an infinite σ-compact cardinal. Theorem 2 proves $\kappa(X) < \delta$ and [CH] implies X is normal.

Thus in Case 2, *if δ is not the cardinal successor of an infinite σ-compact cardinal, then* [GCH] *X is normal but not paracompact.*

I give some examples to show the flavor of the results.

(1) $\alpha_0 = \omega_2$ and $\alpha_n = \omega_0 + 1$ for $n > 0$; [CH]X is normal; Case 2.

(2) $\alpha_n = \omega_k$ for a finite $k > 1$ and all n; [CH]X is normal; Case 2.

(3) $\alpha_n = \omega_{\omega_0+2}$ for n even and $\alpha_n = \omega_0 + 1$ for n odd; [GCH]X is normal; Case 2.

(4) $\alpha_n = \omega_n$ for all n; X is not normal; Case 0.

(5) $\alpha_n = \omega_{\omega_0}$ for all n; [CH]X is paracompact; Case 1.

(6) $\alpha_n = \omega_0 + 1$ for all n; [CH]X is paracompact; Case 1.

But I conjecture there is a model of set theory in which X is not normal.

(7) $\alpha_0 = \omega_1$ and $\alpha_n = \omega_0 + 1$ for all $n > 0$; Case 2 but none of the present results apply. *I conjecture X is not normal in a model of set theory including* [CH]. *I also conjecture X is normal in a different model of set theory.*

C. Let $T = \{t \in \omega_0 \mid \alpha_t$ is σ-compact but not compact$\}$.

Lemma 0. *Without loss of generality, $T = \varnothing$.*

Proof. If $t \in T$, choose nonlimit ordinals $\alpha_{0t} < \alpha_{1t} < \cdots$ having α_t as a limit. Define $\mathcal{Q} = \{\prod_{n \in \omega_0} A_n \mid A_n = \alpha_n$ if $n \in \omega_0 - T$, and either $A_n = \alpha_{0n}$ or $A_n = \alpha_{in} - \alpha_{(i-1)n}$ for some $i > 0$ if $n \in T\}$. Clearly $\mathcal{Q}$ is a collection of disjoint open sets covering X; thus X is paracompact (normal) if and only if all members of $\mathcal{Q}$ are paracompact (normal). But $\prod_{n \in \omega_0} A_n \in \mathcal{Q}$ implies A_n is either isomorphic to a compact ordinal or is a non-σ-compact ordinal. Hence we may assume $T = \varnothing$.

D.

Theorem 0. *Suppose* γ *and* β *are ordinals*; γ *is not* σ*-compact, and* $\beta > \mathrm{cf}(\gamma)$. *Then* $\gamma \times \beta$ *is not normal.*

Proof. Let $\delta = \mathrm{cf}(\gamma)$. Since $\delta \times (\delta + 1) = D$ is homeomorphic to a closed subset of $\gamma \times \beta$, it suffices to show that D is not normal. For $\alpha < \delta$, let $h_\alpha = (\alpha, \delta)$ and $k_\alpha = (\alpha, \alpha)$. Then $H = \{h_\alpha \mid \alpha < \delta\}$ and $K = \{k_\alpha \mid \alpha < \delta\}$ are disjoint closed subsets of D. Suppose U and V are disjoint sets open in D and $U \supset H$ and $V \supset K$. Let $\Lambda = \{\lambda < \delta \mid \lambda \text{ is a limit ordinal}\}$. Since $k_\lambda \in V$ for each $\lambda \in \Lambda$, there is a $\beta_\lambda < \lambda$ with $\{(\eta, \lambda) \mid \beta_\lambda \leq \eta \leq \lambda\} \subset V$. Since $\mathrm{cf}(\delta) = \delta$, there is a $\beta < \delta$ such that $\rho < \delta$ implies $\{\lambda \in \Lambda \mid \beta_\lambda = \beta \text{ and } \lambda > \rho\} \neq \varnothing$. Since $h_\beta \in U$, there is a $\rho < \delta$ such that $\{(\beta, \lambda) \mid \rho \leq \lambda \leq \delta\} \subset U$. Choose $\lambda \in \Lambda$ such that $\beta_\lambda = \beta$ and $\lambda \geq \rho$. Then $(\beta, \lambda) \in U \cap V$. Thus X is not normal.

Conjecture based on the proof of Theorem 0. Let $\alpha_0 = \omega_1$ and $\alpha_n = \omega_0 + 1$ for $n > 0$. For $\alpha < \omega_1$ define $h_\alpha \in X$ by $h_\alpha(0) = \alpha$ and $h_\alpha(n) = \omega_0$ for $n > 0$. For $\alpha < \omega_1$ choose $k_\alpha \in X$ so $k_\alpha(0) = \alpha$ and, for $n > 0$, choose $k_\alpha(n) < \omega_0$ in such a way that $\beta < \alpha$ implies there is an $m \in \omega_0$ so $n > m$ gives $k_\alpha(n) > k_\beta(n)$. Then $H = \{h_\alpha \mid \alpha < \omega_1\}$ and $K = \{k_\alpha \mid \alpha < \omega_1\}$ are again closed and disjoint subsets of X. I conjecture that in some model of set theory including [CH], K can be chosen in such a way that H and K cannot be separated. However, in a model with no scale of cardinality $\aleph_1$ (that is, for any K there is an $x \in \omega_0 \times \omega_0 \times \cdots$ such that for all α there is an m for which $n > m$ implies $x(n) > k_\alpha(n)$) I feel X must be normal. [CH] implies there is a scale of cardinality $\aleph_1$.

E. The theorems are proved in this section and we need more notation. Assume α_n is compact for all n.

Let $\mathfrak{L} = \{\prod_{n \in \omega_0} J_n \mid J_n \text{ is a closed subinterval of } \alpha_n\}$; we allow $J_n = \varnothing$. See that $X \in \mathfrak{L}$ and the intersection of the members of any subset of $\mathfrak{L}$ is a member of $\mathfrak{L}$. If $L \in \mathfrak{L}$, define $\prod_{n \in \omega_0} \sup\{p(n) \mid p \in L\}$ to be the *top* of L; observe that $\varnothing \neq L \in \mathfrak{L}$ implies the top of L belongs to L.

Let $\mathfrak{B} = \{U \in \mathfrak{L} \mid U(n) \text{ is both open and closed for all } n\}$; $\mathfrak{B}$ is a basis for the topology of X.

For $p \in X$ and $n \in \omega_0$, let $E^n(p) = \{x \in X \mid x(m) = p(m) \text{ for all } m \geq n\}$. Let $E(p) = \bigcup_{n \in \omega_0} E^n(p)$ and $\mathfrak{S} = \{\bigcup_{p \in U} E(p) \mid U \in \mathfrak{B}\}$. For $V \in \mathfrak{S}$ let V^* denote a particular $U \in \mathfrak{B}$ with $V = \bigcup_{p \in U} E(p)$. Observe that the relation E, where pEq means $p(m) = q(m)$ for all but finitely many $m \in \omega_0$, partitions X into equivalence classes and that $E(p)$ is the equivalence class to which p belongs. Also $V \in \mathfrak{S}$ implies

$$V = \bigcup_{n \in \omega_0} \alpha_0 \times \alpha_1 \times \cdots \times \alpha_{n-1} \times V^*(n) \times V^*(n+1) \times \cdots.$$

From this we see that $V \in \mathfrak{S}$ implies V is the union of disjoint members of $\mathfrak{B}$

because $V = \bigcup_{n\in\omega_0}\{\prod_{j\in\omega_0} J_j \mid J_j(i) = V^*(i)$ for $i \geq n$ and $J_j(i)$ is either $V^*(i)$ or a maximal subinterval of $\alpha_n - V^*(i)\}$.

Let Ω be the set of all subsets $\mathcal{Q}$ of $\mathcal{B}$ covering X such that $V \in \mathcal{Q}$ and $U \in \mathcal{B}$ and $V \supset U$ imply $U \in \mathcal{Q}$. For $\mathcal{Q} \in \Omega$ define $\mathcal{S}(\mathcal{Q}) = \{V \in \mathcal{S} \mid V$ is the union of a set of disjoint members of $\mathcal{Q}\}$.

We now prove a sequence of lemmas.

Lemma 1. *The intersection of the members of a countable subset of* $\mathcal{S}$ *is the union of a set of disjoint members of* $\mathcal{S}$.

Proof. Assume $\{V_n\}_{n\in\omega_0} \subset \mathcal{S}$.

For m and n in ω_0 let $\mathcal{I}_{mn} = \{I \mid I$ is either $V_m^*(n)$ or a maximal interval in $\alpha_n - V_m^*(n)\}$. Observe that $\mathcal{I}_{mn}$ partitions α_n into three or fewer disjoint open and closed subintervals. Define $\mathcal{I}_n = \{\bigcap_{m\leq n} I_m \mid I_m \in \mathcal{I}_{mn}\}$. Let $\mathcal{J} = \{\prod_{n\in\omega_0} J_n \mid J_n \in \mathcal{I}_n\}$. Finally let $\mathcal{K} = \{\bigcup_{p\in J} E(p) \mid J \in \mathcal{J}\}$ and $\mathcal{K} = \{V \in \mathcal{K} \mid V \subset \bigcap_{n\in\omega_0} V_n\}$. Since the terms of $\mathcal{I}_n$ are disjoint open and closed intervals of α_n whose union is α_n, the terms of $\mathcal{K}$ are disjoint members of $\mathcal{S}$ whose union is X.

Suppose $x \in \bigcap_{n\in\omega_0} V_n$; choose $k_0 < k_1 < \cdots$ in ω_0 so $x(n) \in V_m^*(n)$ for all $n \geq k_m$. Define J_n by $x(n) \in J_n \in \mathcal{I}_n$. Since $k_m \geq m$, clearly $J_n \subset V_m^*(n)$ for $n \geq k_m$. Let $J = \prod_{n\in\omega_0} J_n$ and $V = \bigcup_{p\in J} E(p)$. Then $x \in V \in \mathcal{K}$. Assume $y \in V$ and $m \in \omega_0$. There is a $k \in \omega_0$ so $y(n) \in J_n$ for all $n > k$; so $y(n) \in V_m^*(n)$ for all $n > k_m + k$. Choose a point p of V_m^* such that $p(n) = y(n)$ for $n > k_m + k$; then $y \in E(p)$ so $y \in V_m$. Thus $V \in \mathcal{K}$. Hence $\mathcal{K}$ is a set of disjoint members of $\mathcal{S}$ whose union is $\bigcap_{n\in\omega_0} V_n$.

Lemma 2. *If* $x \in X$ *and* $\mathcal{Q} \in \Omega$, *then* $x \in \bigcup\mathcal{S}(\mathcal{Q})$.

Proof. By induction we define for each $n \in \omega_0$ a finite subset $\mathcal{Q}_n$ of $\mathcal{Q}$ such that $V \in \mathcal{Q}_n$ implies $V \cap E^n(x) \neq \varnothing$; $\mathcal{Q}_n$ is a cover of $E^n(x)$ by disjoint sets. Also there is an open-closed interval I_n of α_n to which $x(n)$ belongs which is $V(n)$ for all $V \subset \mathcal{Q}_n$.

Choose $W \in \mathcal{Q}$ such that $x \in W$ and let $\mathcal{Q}_0 = \{W\}$ and $I_0 = W(0)$.

Assume $\mathcal{Q}_{n-1}$ has been chosen. Since $\mathcal{Q}$ is a basis for the topology of X and $\mathcal{Q}_{n-1}$ is finite and its members are closed, for each $p \notin \bigcup\mathcal{Q}_{n-1}$ there is a $V_p \in \mathcal{Q}$ such that $p \in V_p$ and $V_p \cap (\bigcup\mathcal{Q}_{n-1}) = \varnothing$. Since $E^n(x) = \alpha_0 \times \alpha_1 \times \cdots \times \alpha_{n-1} \times \{x(n)\} \times \{x(n+1)\} \times \cdots$, and each α_i is compact, $E^n(x)$ is compact. Hence $E^n(x)$ contains a finite subset $p_0, p_1, \ldots, p_k$ such that $\{V_{p_i} \mid i \leq k\}$ covers $E^n(x) - (\bigcup\mathcal{Q}_{n-1})$. For $i \leq k$ and $j < n$ let $\mathcal{J}_{ij} = \{K \mid K$ is $V_{p_i}(j)$ or a maximal subinterval of $\alpha_j - V_{p_i}(j)\}$. If $j < n$ define $\mathcal{J}_j = \{\bigcap_{i\leq k} K_i \mid K_i \in \mathcal{J}_{ij}\}$. And if $j \geq n$ define $J_j = \bigcap_{i\leq k} V_{p_i}(j)$. Then define $\mathcal{B}_n = \{J \mid J \subset V_{p_i}$ for some $i \leq k$ and $J = \prod_{j\in\omega_0} J_j$ where $J_j \in \mathcal{J}_j$ for $j < n\}$.

Let $I_n = \bigcap_{V\in\mathcal{Q}_{n-1}\cup\mathcal{B}_n} V(n)$. Since $V \in \mathcal{Q}_{n-1} \cup \mathcal{B}_n$ implies $V \cap E^n(x) \neq \varnothing$, $x(n) \in I_n$. For $V \in \mathcal{Q}_{n-1} \cup \mathcal{B}_n$, define ${}^nV \in \mathcal{B}$ by ${}^nV(i) = V(i)$ for $i \neq n$ and ${}^nV(n) = I_n$. Then define $\mathcal{Q}_n = \{{}^nV \mid V \in \mathcal{Q}_{n-1} \cup \mathcal{B}_n\}$; clearly $\mathcal{Q}_n$ has the desired properties.

For all r and n in ω_0 and $V \in \mathcal{A}_n$, define $V^{nr} = V(0) \times \cdots \times V(n-1) \times I_n \times I_{n+1} \times \cdots \times I_{n+r} \times V(n+r+1) \times \cdots$; and define $V^n = V(0) \times \cdots \times V(n-1) \times I_n \times I_{n+1} \times \cdots$. Let $\mathcal{H} = \{V^n \mid n \in \omega_0 \text{ and } V \in \mathcal{A}_n\}$. Clearly $V \supset V^{n0} \supset V^{n1} \supset \cdots \supset V^n$, so $V \in \mathcal{A}_n \subset \mathcal{A}$ implies $V^n \in \mathcal{A}$. By an easy induction on r, $V^{nr} \in \mathcal{A}_{n+r}$. Thus the fact that, for all $m \in \omega_0$, the members of each $\mathcal{A}_m$ are disjoint, yields that $\mathcal{H}$ is a collection of disjoint members of $\mathcal{A}$.

Define $Z = \{p \in X \mid \text{for some } n \in \omega_0, p(m) \in I_m \text{ for all } m > n\}$. Clearly $x \in Z \in \mathcal{S}$. We prove $Z = \bigcup \mathcal{H}$ and this proves the lemma.

Clearly each term of $\mathcal{H}$ is contained in Z, so we only need prove $Z \subset \bigcup \mathcal{H}$. Suppose $p \in Z$. There is an $n \in \omega_0$ with $p(m) \in I_m$ for all $m \geq n$. Let q be the point of $E^n(x)$ with $q(m) = p(m)$ for $m < n$ and $q(m) = x(m)$ for $m \geq n$. Then $q \in V \in \mathcal{A}_n$. But also $p \in V$ and $p \in V^n \in \mathcal{H}$. So $Z \subset \bigcup \mathcal{H}$.

Lemma 3. *If* $\mathcal{A} \in \Omega$, *then* [CH] *there is a set of disjoint members of* $\mathcal{S}(\mathcal{A})$ *covering* X.

Proof. Define a one-to-one function f: $\omega_1 \times \omega_1 \to \omega_1$ such that $f(\beta, \alpha) > \beta$ for all β and α; f need not be onto.

For each countable ordinal β we define sets $\mathcal{H}_\beta$ and $\mathcal{K}_\beta$ by transfinite induction. Our induction hypotheses are:

(1) $\mathcal{H}_\beta \subset \mathcal{S}(\mathcal{A})$ and $\mathcal{K}_\beta \subset \mathcal{S}$.

(2) $\mathcal{H}_\beta \cup \mathcal{K}_\beta$ is a disjoint cover of X and no term of $\mathcal{H}_\beta$ intersects a term of $\mathcal{K}_\beta$.

(3) $\rho < \beta$ and $V \in \mathcal{H}_\rho$ implies $V \in \mathcal{H}_\beta$.

(4) $\rho < \beta$ and $V \in \mathcal{K}_\beta$ implies there is a $U \in \mathcal{K}_\rho$ with $U \supset V$.

We use some functions in the definitions and we define these before beginning the induction. Suppose $\beta < \omega_1$ and suppose $\{\mathcal{H}_\rho \mid \rho \leq \beta\}$ and $\{\mathcal{K}_\rho \mid \rho \leq \beta\}$ have been defined satisfying the induction hypotheses. Then define a function g_β: $(\mathcal{K}_\beta - \{\varnothing\}) \times \omega_1 \to \mathcal{L}$ as follows. Suppose $\varnothing \neq W \in \mathcal{K}_\beta$. If $\rho \leq \beta$, by (4), there is a $W(\rho) \in \mathcal{K}_\rho$ such that $W(\rho) \supset W$. By (2), the terms of $\mathcal{K}_\rho$ are disjoint so $W(\rho)$ is uniquely determined. For $\rho \leq \beta$ and $n \in \omega_0$, define $\mathcal{I}_{\rho n} = \{I \mid I \text{ is either } W(\rho)^*(n) \text{ or a maximal interval of } \alpha_n - W(\rho)^*(n)\}$. Let $\mathcal{I}_n = \{\bigcap_{\rho \leq \beta} I \mid I \in \mathcal{I}_{\rho n}\}$ and $\mathcal{I} = \{\prod_{n \in \omega_0} I_n \mid I_n \in \mathcal{I}_n\}$. Since $\{\rho \leq \beta\}$ is countable, the cardinality of $\mathcal{I}$ is at most that of the continuum. Hence [CH] there is a function g_W from ω_1 *onto* $\mathcal{I}$. For $\alpha \in \omega_1$ define $g_\beta(W, \alpha) = g_W(\alpha)$.

We are now ready to begin our induction. Define $\mathcal{H}_0 = \varnothing$ and $\mathcal{K}_0 = \{X\}$.

Assume $\mathcal{H}_\beta$ and $\mathcal{K}_\beta$ satisfying the induction hypotheses have been defined for all $\beta < \gamma$ where $0 < \gamma < \omega_1$.

We first define $\mathcal{H}_\gamma$ and $\mathcal{K}_\gamma$ in the case $\gamma = \delta + 1$ for some $\delta < \omega_1$. Observe that part of our assumption in this case is that g_β has been defined for all $\beta \leq \delta$. Let $Q_\delta = \{U \in \mathcal{K}_\delta \mid \text{there are } \alpha \in \omega_1, \beta < \delta, \text{ and } W \in \mathcal{K}_\beta \text{ such that } f(\beta, \alpha) = \delta, U \subset W, \text{ and the top of } g_\beta(W, \alpha) \text{ belongs to } U\}$.

Fix $U \in Q_\delta$. Since f is one-to-one, α and β are uniquely determined. By (2) there can be at most one $W \in \mathcal{K}_\beta$ such that $U \subset W$. So $g_\beta(W, \alpha)$ and the top t_U of $g_\beta(W, \alpha)$ are uniquely determined by U.

Choose $Z \in \mathcal{S}(\mathcal{Q})$ with $t_U \in Z$ as guaranteed by Lemma 2; keep in mind that Z is a function of U. Define $H_U = H \cap Z$. For $n \in \omega_0$ define $\mathcal{F}_n = \{I \cap J \mid I$ is either $U^*(n)$ or a maximal subinterval of $\alpha_n - U^*(n)$ and J is either $Z^*(n)$ or a maximal subinterval of $\alpha_n - Z^*(n)\}$. Then define $\mathcal{F} = \{\prod_{n \in \omega_0} F_n \mid F_n \in \mathcal{F}_n$ and, for infinitely many n, $F_n \not\subset Z^*(n)\}$. Let $\mathcal{K}_U = \{\bigcup_{p \in F} E(p) \mid F \in \mathcal{F}$ and $F \subset U\}$. The terms $\mathcal{K}_U$ are disjoint and their union is $U - Z$.

Now define $\mathcal{H}_\gamma = \mathcal{H}_\delta \cup \{H_U \mid U \in Q_\delta\}$ and $\mathcal{K}_\gamma = (\mathcal{K}_\delta - Q_\delta) \cup \bigcup\{\mathcal{K}_U \mid U \in Q_\delta\}$. Using only the preceding paragraph and the facts that $\delta + 1 = \gamma$, the induction hypotheses are satisfied for $\beta < \gamma$, $Q_\delta \subset \mathcal{K}_\delta$, and that for each $U \in Q_\delta$, a unique term t_U of U has been chosen, it is easy to check that the induction hypotheses hold for γ. The messy definitions of g_β and Q_δ are only used later. But we need $\mathcal{H}_\gamma$ and $\mathcal{K}_\gamma$ chosen in this complicated way in order to prove the lemma.

If γ is a limit ordinal, define $\mathcal{H}_\gamma = \bigcup_{\beta<\gamma}\mathcal{H}_\beta$ and $\mathcal{L}_\gamma = \{\bigcap_{\beta<\gamma} V_\beta \mid V_\beta \in \mathcal{K}_\beta\}$. By Lemma 1, $\varnothing \neq V \in \mathcal{L}_\gamma$ implies V is the union of a set $\mathcal{K}_V$ of disjoint members of $\mathcal{S}$. Define $\mathcal{K}_\gamma = \bigcup\{\mathcal{K}_V \mid V \in \mathcal{L}_\gamma\}$. The induction hypotheses clearly thus hold for γ.

Define $\mathcal{H} = \bigcup_{\beta<\omega_1}\mathcal{H}_\beta$. The members of $\mathcal{H}$ are certainly disjoint terms of $\mathcal{S}(\mathcal{Q})$ so the lemma is proved if $\mathcal{H}$ covers X.

Assume $p \in X - \bigcup\mathcal{H}$. Then for each $\beta < \omega_1$, there is a unique $U_\beta \in \mathcal{K}_\beta$ with $p \in U_\beta$. For $n \in \omega_0$ and $\beta < \omega_1$ let $\mathcal{I}_{\beta n} = \{I \subset \alpha_m \mid I$ is either $U_\beta^*(n)$ or a maximal subinterval of $\alpha_n - U_\beta^*(n)\}$. For $n \in \omega_0$ and $\beta < \omega_1$, let $I_{\beta n}$ be the term of $\mathcal{I}_{\beta n}$ to which $p(n)$ belongs, and for each $\delta < \omega_1$, let $J_{\delta n} = \bigcap_{\beta \le \delta} I_{\beta n}$. Define $J_\delta = \prod_{n\in\omega_0} J_{\delta n}$. Clearly $p \in J_\delta \subset U_\delta$, and $\beta < \delta$ implies $J_\delta \subset J_\beta$. Let t_δ be the top of J_δ; that is, $t_\delta = \prod_{n\in\omega_0} \sup\{x(n) \mid x \in J_\delta\}$. Clearly $t_\delta \in J_\delta$, and $\beta < \delta$ implies $t_\delta(n) \le t_\beta(n)$ for all n. Since ω_1 and α_n are well ordered, there is, for each $n \in \omega_0$, a smallest $\beta_n < \omega_1$ such that $t_{\beta_n}(n) = \inf\{t_\beta(n) \mid \beta < \omega_1\}$. Define $\beta = \sup\{\beta_n \mid n \in \omega_0\}$. Then $t_\beta = t_\delta$ for all $\delta > \beta$. Look again at the definition of g_β. If $U_\beta = W$, then $\rho \le \beta$ yields $U_\rho = W(\rho)$ and $\mathcal{I}_{\rho n} = \mathcal{J}_{\rho n}$. So $J_{\beta n} \in \mathcal{J}_n$ and $J_\beta \in \mathcal{J}$. Hence, $J_\beta = g_\beta(U_\beta, \alpha)$ for some $\alpha < \omega_1$. Let $f(\beta, \alpha) = \delta$; look at the definition of $\mathcal{H}_\gamma$ and $\mathcal{K}_\gamma$ in the case $\gamma = \delta + 1$. Clearly $U_\delta \in Q_\delta$ and $t_{U_\delta} = t_\delta$. Thus t_δ belongs to a term of $\mathcal{H}_\gamma$; but this contradicts $t_\delta = t_\gamma \in U_\gamma \in \mathcal{K}_\gamma$. Hence $\mathcal{H}$ covers X.

Theorem 1. *The continuum hypothesis implies the box product of countably many σ-compact ordinals is paracompact. In fact every open cover of such a product has a refinement consisting of disjoint open-closed sets.*

Proof. By Lemma 0 we assume a space $X = \prod_{n\in\omega_0} \alpha_n$ where each α_n is compact. Let $\mathcal{G}$ be an open cover of X. Define $\mathcal{Q} = \{V \in \mathcal{B} \mid$ for some $G \in \mathcal{G}$, $V \subset G\}$. Obviously $\mathcal{Q} \in \Omega$. So, by Lemma 3, [CH] there is a set $\mathcal{H}$ of disjoint members of $\mathcal{S}(\mathcal{Q})$ covering X. For $H \in \mathcal{H}$, let $\mathcal{Q}_H$ denote a set of disjoint members of $\mathcal{Q}$ whose union is H. Then $\bigcup_{H\in\mathcal{H}}\mathcal{Q}_H$ is a set of disjoint open sets refining $\mathcal{G}$ and covering X.

Theorem 2. *Suppose that for each $n \in \omega_0$, γ_n is an ordinal and Y is the box product of $\{\gamma_n \mid n \in \omega_0\}$. Suppose δ is an uncountable ordinal and $S = \{n \in \omega_0 \mid$ $\mathrm{cf}(\gamma_n) = \delta\} \neq \varnothing$. Suppose also that $n \in \omega_0 - S$ implies γ_n is σ-compact. Define $\kappa = \sup\{$cardinality of $\prod_{n \in \omega_0 - \{s\}} \beta_n \mid s \in S, \beta_n \leq \gamma_n$, and β_n is compact$\}$. Then* [CH] *and $\delta > \kappa$ imply Y is normal.*

An analogous proof shows Y is collectionwise normal.

Proof. Using Lemma 0, we assume $n \in \omega_0 - S$ implies γ_n is compact. Suppose A and B are disjoint closed subsets of Y. Define X to be the box product of $\{\alpha_n\}_{n \in \omega_0}$ where $\alpha_n = \gamma_n$ when $n \in \omega_0 - S$ and $\alpha_n = \gamma_n + 1$ when $n \in S$. Observe that α_n is compact for each n, and Y is a subspace of X. We now use the notation set up at the beginning of §E for X; recall $\mathfrak{B}$ is a basis for X. Define $\mathfrak{A} = \{W \in \mathfrak{B} \mid$ either $W \cap A = \varnothing$ or $W \cap B = \varnothing\}$. Lemma 4 below proves $\mathfrak{A} \in \Omega$. Then Lemma 3 proves [CH] there is a set $\mathcal{H}$ of disjoint members of $\mathcal{E}(\mathfrak{A})$ covering X. For $H \in \mathcal{H}$ let $\mathfrak{A}_H$ denote a set of disjoint members of $\mathfrak{A}$ whose union is H. Let $U = \bigcup\{W \in \mathfrak{A}_H \mid H \in \mathcal{H}$ and $W \cap A \neq \varnothing\}$ and $V = \bigcup\{W \in \mathfrak{A}_H \mid$ $H \in \mathcal{H}$ and $W \cap B \neq \varnothing\}$. Then $U \supset A$, $V \supset B$, and $U \cap V = \varnothing$. Thus Y is proved normal.

Lemma 4. *Assume y, S, δ, κ, A, B, X, and $\mathfrak{A}$ as above. Suppose $x \in X$ and $\kappa < \delta$. Then $x \in \bigcup \mathfrak{A}$.*

Proof. Let $R = \{n \in S \mid x(n) = \gamma_n\}$, $Z = \prod_{n \in R} \gamma_n$, $W = \prod_{n \in \omega_0 - R} (x(n) + 1)$; then $(W \times Z) \subset Y$.

If $R = \varnothing$ then $x \in Y$ and, since A and B are closed and disjoint in Y, the lemma is true. Assume $R \neq \varnothing$ for the rest of the proof. Observe that $R \neq \varnothing$ and $\kappa < \delta$ imply the cardinality of W is less than δ. We have two similar major cases.

Case (1). *R has more than one member.* In this case, by the definition of κ and $\kappa < \delta$, $\gamma_n = \delta$ for all $n \in R$. For $\sigma < \delta$, define z_σ to be the point of Z all of whose coordinates are σ and define Z_σ to be the set of all points of Z all of whose coordinates are greater than σ. We use $\{z_\sigma \mid \sigma < \delta\}$ and $\{Z_\sigma \mid \sigma < \delta\}$ to help us choose a special ordinal $\lambda < \delta$.

Case (1a). $R \neq \omega_0$. In this case we wish to choose $\lambda < \delta$ such that, for all $p \in W$, one of the following hold:

(i) $(p, Z_\lambda) \cap B = \varnothing$ and $(p, z_\lambda) \in A$,

(ii) $(p, Z_\lambda) \cap A = \varnothing$ and $(p, z_\lambda) \in B$, or

(iii) $(p, Z_\lambda) \cap (A \cup B) = \varnothing$.

Let $P = \{p \in W \mid$ there is a $\sigma < \delta$ such that $q \in Z_\sigma$ implies $(p,q) \notin A \cup B\}$. Since the cardinality of W is less than δ, there is a $\beta < \delta$ such that $p \in P$ and $q \in Z_\beta$ implies $(p,q) \notin A \cup B$. Any λ chosen with $\beta \leq \lambda < \delta$ yields (iii) for all $p \in P$. If $W \subset P$, define $\lambda = \beta$ and (iii) holds for $p \in W$. Otherwise we have (1a*) or (1a**).

Case (1a*). *$W - P \neq \varnothing$ and $\delta \neq \omega_1$.*

Suppose $p \in W - P$ and define $\Delta_p = \{\sigma \in \delta \mid (p, z_\sigma) \in (A \cup B)\}$.

Suppose $\{\sigma_\eta \mid \eta \in \omega_1\}$ is a monotone subset of δ, and $q_\eta \in Z_{\sigma_\eta} - Z_{\sigma_{\eta+1}}$ and $(p, q_\eta) \in A \cup B$ for all $\eta \in \omega_1$. Let $\sigma = \sup\{\sigma_\eta \mid \eta \in \omega_1\}$. Since R is countable, (p, q_σ) is a limit point of $\{(p, q_{\sigma_\eta} \mid \eta \in \omega_1\}$. So $(p, q_\sigma) \in A \cup B$ and $\sigma \in \Delta_p$ and only countably many (p, q_{σ_η}) can belong to the one of A and B to which (p, q_σ) does not belong.

Thus by the preceding paragraph there is a $\sigma_p < \delta$ such that either $\{(p,q) \mid q \in Z_{\sigma_p}\} \cap A = \varnothing$ or $\{(p,q) \mid q \in Z_{\sigma_p}\} \cap B = \varnothing$. Let $\delta^* = \{\sigma \in \delta \mid \mathrm{cf}(\sigma) \geq \omega_1\}$. By the preceding paragraph and the definition of P, since $p \in W - P$, $\Delta_p \cap \delta^*$ is a cardinality δ closed subset of δ^*. Thus $\Delta_p' = \{\rho \in \Delta_p \cap \delta^* \mid \rho > \sigma_p$ and $\rho > \beta\}$ is a cardinality δ closed subset of δ^*. Recall that δ is an uncountable ordinal and $\mathrm{cf}(\delta) = \delta$. Thus it is standard set theory that the intersection of any family of cardinality less than δ of closed subsets of δ^*, each of cardinality δ, is nonempty. Therefore, since the cardinality of W is less than δ, there is a $\lambda \in \bigcap_{p \in W-P} \Delta_p'$. Clearly $p \in W - P$ implies (i) or (ii) holds for λ; and, since $\lambda > \beta$, $p \in P$ implies (iii) holds.

Case (1a**). *$W - P \neq \varnothing$ and $\delta = \omega_1$*. Observe that *cardinality less than δ* means *countable* here. Thus R is finite. So we can prove the existence of a δ with the desired properties using exactly the proof given in Case (1a*) if we replace ω_1 in the proof by ω_0 and *countable* by *finite*. In this case $\delta = \delta^*$.

Case (1b). $R = \omega_0$. In this case we wish to choose $\lambda < \delta$ such that, for all $p \in W$, either $Z_\lambda \cap A = \varnothing$ or $Z_\lambda \cap B = \varnothing$. Such a λ can be shown to exist using a simplified version of the argument given in Case (1a) where all references to P, β, and p are omitted.

Having chosen λ, we are now ready to prove the lemma in Case 1. Define $A' = \{p \in W - P \mid (p, z_\lambda) \in A\}$ and $B' = \{p \in W - P \mid (p, z_\lambda) \in B\}$. Clearly A' and B' are closed and disjoint in W. If x' is the point of W such that $x'(n) = x(n)$ for all $n \in \omega_0 - R$, then $x' \notin A' \cap B'$. Hence, for each $n \in \omega_0 - R$, there is an open and closed subinterval I_n of $x(n) + 1$ containing $x'(n)$ such that either $A' \cap \prod_{n \in \omega_0 - R} I_n = \varnothing$ or $B' \cap \prod_{n \in \omega_0 - R} I_n = \varnothing$. For $n \in R$ define $I_n = Z_\lambda$. Then $x \in \prod_{n \in \omega_0} I_n \in \mathcal{Q}$.

Case 2. R has only one member. Say $R = \{r\}$.

Since $\mathrm{cf}(\gamma_r) = \delta$, there is a closed subset $\{z_\sigma \mid \sigma < \delta\}$ of γ_r such that $\sigma < \eta < \delta$ implies $z_\sigma < z_\eta$ and $\beta < \gamma_r$ implies $\beta < z_\sigma$ for some $\sigma < \delta$. Since $R = \{r\}$, $z_\sigma \in Z$. Define $Z_\sigma = \{\beta \in \gamma_r \mid \beta > z_\sigma\}$. Then using precisely the same argument given in Case 1 after the definitions of z_σ and Z_σ, one shows $x \in \bigcup \mathcal{Q}$.

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