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WARING'S PROBLEM FOR TWENTY-TWO BIQUADRATES

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ABSTRACT. That every natural number is the sum of at most twenty-two biquadrates is proven by ascent from machine results on sums of six fourth powers.

Waring's problem for fourth powers is to prove that every natural number is the sum of nineteen fourth powers. For many years, though, the best result for this problem was Dickson's [2] proof that every natural number is the sum of at most thirty-five biquadrates (a biquadrate is a nonzero fourth power). Recently, Dress [3] showed thirty fourth powers suffice to represent every number. In his dissertation [4], the author showed twenty-three suffice. We show here that twenty-two suffice.

1. Basic lemmas.

Lemma 1.1. *Every natural number exceeding* 10^{568} *is the sum of at most twenty-two biquadrates.*

Lemma 1.2. *Every number congruent to* 2, 3, *or* 4 (mod 16) *but incongruent to* 2 (mod 5) *in the interval* $W_6 = [76\ 124\ 880,\ 1\ 753\ 942\ 720]$ *is the sum of six fourth powers.*

Lemma 1.3. *Every number congruent to* 1 *or* 5 (mod 16) *but incongruent to* 2 *and* 3 (mod 5) *in* W_6 *is the sum of six fourth powers.*

Proofs. A proof of Lemma 1.1 is given on pp. 10–162 of [4]. The method used is a version of the Hardy-Littlewood (trigonometric sums) technique. Lemmas 1.2 and 1.3 required extensive machine investigation. This investigation was carried out in two phases.

The first phase employed an implementation of Algorithm 14.3 of [4] to sift out numbers in W_6 that are sums of six fourth powers. The algorithm essentially starts with a very large array A of bits, all zero. Each bit of A corresponds to

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a number in an interval I. All sums of four fourth powers that lie in I are now formed. For each sum, the corresponding bit of A is set at one. The resulting array is a sieve of sums of four fourth powers. Now the sieve is shifted by the fourth powers of integers not exceeding some constant M. The various shifts of the sieve are logically "or"ed to yield a new array B, which is a sieve of sums of five fourth powers. A bit of B equals one if and only if it corresponds to a number which is the sum of five fourth powers, the smallest not exceeding M^4. The shifting and "or"ing is repeated to obtain a sieve of sums of six fourth powers.

Because of storage limitations, we had to be satisfied with $M = 32$ in using the above algorithm. The sieve was not able to exclude a few dozen integers in W_6 meeting the congruence conditions of Lemmas 1.2 or 1.3. In the second phase of the investigation, each of these exceptions was decomposed into sums of six fourth powers by machine trial. Our decomposition program was not able to represent 76 124 868 as the sum of six fourth powers.

2. **The ascent.** Let $S = 76\,124\,880$. Both of the sets

$$\mathbf{H}_7 = W_6 \cup W_6 + 5^4 \cup W_6 + 10^4 \cup \cdots \cup W_6 + 440^4$$

and

$$\mathbf{K}_7 = W_6 + 1^4 \cup W_6 + 2^4 \cup W_6 + 6^4 \cup W_6 + 7^4 \cup W_6 + 11^4$$
$$\cup \cdots \cup W_6 + 437^4 \cup W_6 + 441^4$$

are intervals. From Lemma 1.2 we see that every natural number in $\mathbf{H}_7$ congruent to 3 or 4 (mod 16) but incongruent to 2 (mod 5) is the sum of seven fourth powers. Also, every number in $\mathbf{K}_7$ congruent to 3 or 4 (mod 16) but incongruent to 3 (mod 5) is the sum of seven fourth powers. Hence, every number in $\mathbf{H}_7 \cap \mathbf{K}_7$ congruent to 3 or 4 (mod 16) is the sum of seven fourth powers. It follows that every natural number in $W_7 = [S, 39\,234\,902\,720]$ is the sum of seven fourth powers. The above reasoning is an example of what we shall call an ascent of type C.

Since the set $W_8 = W_7 \cup W_7 + 1^4 \cup W_7 + 2^4 \cup \cdots \cup W_7 + 2139^4$ is the interval $[S, 20\,972\,797\,155\,760]$, we see that all numbers congruent to 4 (mod 16) in W_8 are sums of eight fourth powers. This illustrates an ascent which we shall call type A.

Both of the sets

$$\mathbf{H}_9 = W_8 \cup W_8 + 2^4 \cup W_8 + 4^4 \cup \cdots \cup W_8 + 13788^4$$

and

$$\mathbf{K}_9 = W_8 + 1^4 \cup W_8 + 3^4 \cup \cdots \cup W_8 + 13789^4$$

are intervals. Since numbers in $\mathbf{H}_9$ congruent to 4 (mod 16) are sums of nine fourth powers and numbers in K_9 congruent to 5 (mod 16) are sums of nine fourth powers, every number in $H_9 \cap K_9$ and therefore in $W_9 = [S, 3.6162383 \times 10^{16}]$ that are congruent to 4 or 5 (mod 16) are sums of nine fourth powers. We shall call this an ascent of type B.

An ascent of type A, followed by ascents of types B, A, B, B, B and B respectively, shows that every natural number congruent to one of 6, 7, 8, 9 or 10 (mod 16) in $W_{16} = [S, 10^{104.1189}]$ is the sum of sixteen fourth powers.

Another ascent of type B gives that every number congruent to 6, 7, $\cdots$, or 11 (mod 16) in $W'_{17} = [S, 10^{137.6209}]$ is the sum of seventeen fourth powers. However, by looking at numbers of the form $16w + x^4$, where w is congruent to 6, 7, 8, 9 or 10 (mod 16) and in W_{16} and x is odd, we see that every number congruent to 1, 6, 7, $\cdots$, 11 (mod 16) in $W_{17} = [16S, 10^{137.6209}]$ is the sum of seventeen fourth powers. Details of this step can be found in Lemma 13.3 of [4]. The passage from W_{16} to W_{17} will be called an ascent of type B$'$.

Four more ascents of type B get us to

Lemma 2.1. *Every natural number not divisible by* 16 *in* $W'_{21} = [16S, 10^{427.1452}]$ *is the sum of twenty-one fourth powers.*

According to Chandler [1], every number in $[1, 10^{10}]$ is the sum of nineteen fourth powers. Hence, from Lemma 2.1, every number not divisible by 16 in $W_{21} = [1, 10^{427.1452}]$ is the sum of at most twenty-one biquadrates. By removing all factors of 16 from the numbers in W_{21} that are congruent to 0 (mod 16), we have

Lemma 2.2. *Every natural number in* W_{21} *is the sum of twenty-one fourth powers.*

3. **The result.** An ascent of type A from Lemma 2.2 yields

Lemma 3.1. *Every natural number not exceeding* $10^{568.7}$ *is the sum of twenty-two fourth powers.*

An immediate corollary of Lemmas 1.1 and 3.1 is

Theorem 3.2. *Every natural number is the sum of twenty-two fourth powers.*

Concerning the status of the original Waring problem, we have

Theorem 3.3. *Every natural number less than* 10^{310} *or greater than* 10^{1409} *is the sum of noneteen fourth powers.*

Proof. In [4], it is shown that numbers beyond 10^{1409} are sums of nineteen fourth powers. To get the first part of the theorem we made the following ascents from the interval W_6 in Lemma 1.2: an ascent of type C, and ascent of type A, five ascents of type B, an ascent of type B$'$ and five more ascents of type B.

Concerning the Waring problem for sufficiently large numbers as sums of fourth powers:

Theorem 3.4. *Every natural number in* $[13\,793, 10^{143}]$ *is the sum of sixteen fourth powers.*

Proof. An ascent can be made which combines the parity considerations of type B with the modulo 5 considerations of type C. We shall call this an ascent of type D. After making two ascents of type D from the information of Lemmas 1.2 and 1.3, we make eight ascents of type B. The small end of the interval of the theorem is taken care of by Theorem 16.2 of [4].

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