SOMEWHERE LOCALLY FLAT CODIMENSION ONE MANIFOLDS WITH 1-ULC COMPLEMENTS ARE LOCALLY FLAT

BY

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ABSTRACT. The purpose of this paper is to prove a taming theorem for a co-dimension one manifold that is locally flat at some point and has 1-ULC comple-ment. We also prove that any two sufficiently close locally flat embeddings of a codimension one manifold are ambient isotopic. Since this paper was first sub-mitted, R. Daverman has shown that, given any point on a codimension one mani-fold with 1-ULC complement, some neighborhood of that point lies on a codimen-sion one sphere that is locally flat at some points and has 1-ULC complement. Hence the two papers combined prove that a codimension one manifold is locally flat if and only if its complement is 1-ULC.

Suppose M is a topological $(n - 1)$-manifold, Q is a topological n-manifold, and $h: M \to \mathrm{Int}\, Q$ is an embedding. We say that h can be locally approximated by locally flat embeddings if for each $x \in M$ there is a neighborhood U of x in M such that for each $\epsilon > 0$ there is a locally flat embedding $f: U \to Q$ such that $d(f, h \mid U) < \epsilon$. We shall show first that h can be locally approximated by locally flat embeddings if $Q - h(M)$ is 1-ULC, $n \geq 5$, and there is some open set $U \subset M$ such that $h \mid U$ can be approximated by locally flat embeddings. Then we show that $h(M)$ is locally flat. Daverman has used this fact to show that $h(M)$ is lo-cally flat if its complement is 1-ULC.([3]) We also obtain that two close locally flat approximations of a fixed embedding are ambient isotopic by an ϵ-push.

Throughout this paper we will assume that M and Q are topological $(n-1)$-and n-manifolds which are separable metric with metric d. We allow our manifolds to be possibly noncompact and with or without boundary. Throughout we assume that $n \geq 5$. Let $A \cup B \subset Q$ and $\epsilon: Q \to (0, \infty)$ be a map. Denote by $N(A, \epsilon)$ the set $\{y \in Q \mid$ for some $x \in A, \; d(x, y) < \epsilon(x)\}$. A homeomorphism H of Q is an ϵ-push of (Q, A) fixed on B if there is an isotopy H_t of Q such that $H_0 = $ Identity, for each $t \in [0, 1]$, $H_t = $ Identity on B and outside $N(A, \epsilon)$, $d(H_t(x), x) < \epsilon(x)$ for each $t \in [0, 1]$ and each $x \in Q$, and $H_1 = H$. Denote by D^k the k-cell $[-1, 1]^k$ and by aD^k the k-cell $[-a, a]^k$ for any $a > 0$. Let $U \subset Q$ be an open set and $x \in \mathrm{Cl}\; U$. U is 1-LC at x if for each $\epsilon > 0$ there is a $\delta > 0$ such that each map $f: \partial D^2 \to N(x, \delta) \cap U$ extends to a map $F: D^2 \to N(x, \epsilon) \cap U$.

For Theorem 1 we use essentially the same notation as in Bryant [1]. Let $b: D^k \times [-1, 1] \to E^n$ be an embedding. We use the following notation:

$D(a, b) = b(D^k \times [a, b])$,

$D(a) = b(D^k \times \{a\})$,

$D(x, a, b) = b(x \times [a, b])$, and

$N(a, b; \epsilon) = N(b(D^k \times [a, b]), \epsilon) = N(D(a, b), \epsilon)$.

Theorem 1. *Let $f: D^{n-2} \times [-1, 1] \to E^n$ $(n \geq 5)$ be a topological embedding with $E^n - D(-1, 1)$ 1-ULC at each point of $D(-1, 1)$. Let $\epsilon > 0$. Then there is a PL homeomorphism $h: E^n \to E^n$ such that $h \circ f \circ \pi: D^{n-2} \times [-1, 1] \to E^n$ is an ϵ-approximation of f (where $\pi: D^{n-2} \times [-1, 1] \to D^{n-2} \times [-1, 0]$ is defined by $\pi(x, t) = (x, (t-1)/2))$.*

Proof. Choose real numbers $a_0, a_1, \cdots, a_q$ such that $-1 = a_0 < a_1 < a_2 < \cdots < a_q = 1$, such that diameter $D(x, a_{i-1}, a_i) < \epsilon/3$ for each $x \in D^{n-2}$ and each i $(0 < i \leq q)$ and such that $a_{i-1} > (a_i - 1)/2$. Let $b_i = (a_i - 1)/2$. Then we have $b_{i-1} < b_i < a_{i-1} < a_i$ for each i $(0 < i \leq q)$.

Choose ϵ' $(0 < \epsilon' < \epsilon/3)$ so that for any x and $x' \in D^{n-2}$ we have distance $(D(x, -1, 1), D(x', -1, 1)) < 3\epsilon'$ implies that distance $(D(x, t), D(x', t)) < \epsilon/3$ for all $t \in [-1, 1]$. Choose η_0 $(0 < \eta_0 < \epsilon')$ so that for any $x, x' \in D^{n-2}$ we have distance $(D(x, -1, 1), D(x', -1, 1)) < \eta_0$ implies that $D(x', -1, 1) \subseteq N(D(x, -1, 1), \epsilon')$.

Choose η_1 $(0 < \eta_1 < \frac{1}{2}\eta_0)$ so that $N(b_1, a_1; 2\eta_1) \cap [D(-1) \cup D(a_2, 1)] = \emptyset$ and so that for any $x, x' \in D^{n-2}$ distance $(D(x, -1, 1), D(x', -1, 1)) < 2\eta_1$ implies that $D(x', -1, 1) \subseteq N(D(x, -1, 1), \eta_0)$. Choose $\tilde{\eta}_1$ $(0 < \tilde{\eta}_1 < \eta_1)$ so that $N(b_1, a_1; \tilde{\eta}_1) \cap N(a_1, 1; \tilde{\eta}_1) \subseteq N(a_1; \eta)$. By Bryant's Theorem 4.4, there exists a neighborhood W_1 of $D(a_1, 1)$ and a PL homeomorphism $b_1: E^n \to E^n$ satisfying

(a) $b_1 = $ identity except on $N(b_1, a_1; \tilde{\eta}_1)$,

(b) $b_1 = $ identity on W_1,

(c) $D(b_1, 1) \subseteq h_1(N(a_1, 1; \widetilde{\eta}_1))$,   and

(d) for each $y \in E^n$ either $h_1(y) = y$ or there exists an $x \in D^{n-2}$ such that both y and $h_1(y)$ lie in $N(D(x, b_1, a_1), \widetilde{\eta}_1)$.

Furthermore it follows that $D(b_1, a_2) \subseteq h_1(N(a_1, a_2; \eta_1))$. To prove this, see Figure 1 or note that $D(b_1, a_1) \subseteq D(b_1, 1) \subseteq h_1(N(a_1, 1; \widetilde{\eta}_1))$ and that $D(b_1, a_1) \cap [N(a_1, 1; \widetilde{\eta}_1) - N(b_1, a_1; \widetilde{\eta}_1)] = \emptyset$. Since $h_1 =$ identity except on
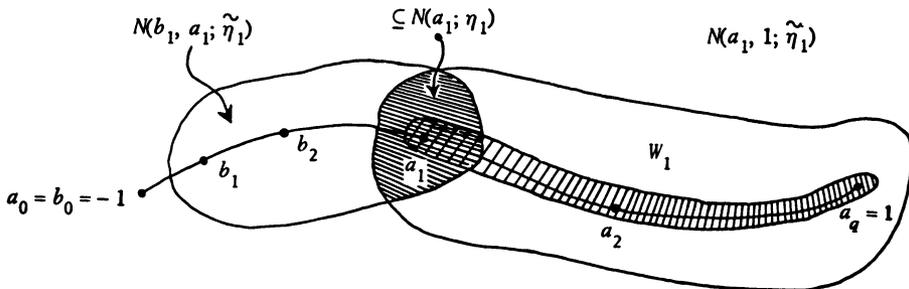

$$N(b_1, a_1; \widetilde{\eta}_1) \qquad \subseteq N(a_1; \eta_1) \qquad N(a_1, 1; \widetilde{\eta}_1)$$

Figure 1

$N(b_1, a_1; \widetilde{\eta}_1)$ it follows that $D(b_1, a_1) \subseteq h_1(N(a_1, 1; \widetilde{\eta}) \cap N(b_1, a_1; \widetilde{\eta}_1)) \subseteq h_1(N(a_1; \eta_1)) \subseteq h_1(N(a_1, a_2; \eta_1))$. Also $D(a_1, a_2) \subseteq W_1 \cap N(a_1, a_2; \eta_1) = h_1(W_1 \cap N(a_1, a_2; \eta_1)) \subseteq h_1(N(a_1, a_2; \eta_1))$. Together we have $D(b_1, a_2) = D(b_1, a_1) \cup D(a_1, a_2) \subseteq h_1(N(a_1, a_2; \eta_1))$ as claimed.

Now choose η_2 $(0 < \eta_2 < \frac{1}{2}\eta_1)$ so that $N(b_2, a_2; 2\eta_2) \cap [D(-1, b_1) \cup D(a_3, 1)] = \emptyset$, so that $N(b_2, a_2; \eta_2) \subseteq h_1(N(a_1, a_2; \eta_1))$ and so that for any $x, x' \in D^{n-2}$ distance $(D(x, -1, 1), D(x', -1, 1)) < 2\eta_2$ implies that $D(x', -1, 1) \subseteq N(D(x, -1, 1), \frac{1}{2}\eta_1)$. Then choose $\widetilde{\eta}_2$ $(0 < \widetilde{\eta}_2 < \eta_2)$ so that $N(b_2, a_2; \widetilde{\eta}_2) \cap N(a_2, 1; \widetilde{\eta}_2) \subseteq N(a_2; \eta_2)$. Then by Bryant's Theorem 4.4 there exists a neighborhood W_2 of $D(a_2, 1)$ and a PL homeomorphism $h_2: E^n \to E^n$ satisfying

(a) $h_2 =$ identity except on $N(b_2, a_2; \widetilde{\eta}_2)$,

(b) $h_2 =$ identity on W_2,

(c) $D(b_2, 1) \subseteq h_2(N(a_2, 1; \widetilde{\eta}_2))$, and

(d) for each $y \in E^n$ either $h_2(y) = y$ or there exists an $x \in D^{n-2}$ such that both y and $h_2(y)$ lie in $N(D(x, b_2, a_2), \widetilde{\eta}_2)$.

Furthermore, just as for h_1, we can prove that $D(b_2, a_3) \subseteq h_2(N(a_2, a_3; \eta_2))$.

In general, for $2 \leq k \leq q$ we choose η_k analogous to η_2 above and get W_k and h_k from Theorem 4.4. More specifically, choose η_k $(0 < \eta_k < \frac{1}{2}\eta_{k-1})$ so that $N(b_k, a_k; 2\eta_k) \cap [D(-1, b_{k-1}) \cup D(a_{k+1}, 1)] = \emptyset$, so that $N(b_k, a_k; \eta_k) \subseteq h_{k-1}(N(a_{k-1}, a_k; \eta_{k-1}))$ and so that for any $x, x' \in D^{n-2}$

$$\text{distance}(D(x, -1, 1), D(x' - 1, 1)) < 2\eta_k$$

implies that $D(x', -1, 1) \subseteq N(D(x, -1, 1), \frac{1}{2}\eta_{k-1})$. Finally we choose $\tilde{\tilde{\eta}}_k$ $(0 < \tilde{\tilde{\eta}}_k < \eta_k)$ so that $N(b_k, a_k; \tilde{\tilde{\eta}}_k) \cap N(a_k, 1; \tilde{\tilde{\eta}}_k) \subseteq N(a_k; \eta_k)$. Then by Bryant's Theorem 4.4 there exists a neighborhood W_k of $D(a_k, 1)$ and a PL homeomorphism $b_k: E^n \to E^n$ satisfying

   (a) b_k = identity except on $N(b_k, a_k; \tilde{\tilde{\eta}}_k)$,
   (b) b_k = identity on W_k,
   (c) $D(b_k, 1) \subseteq b_k(N(a_k, 1; \tilde{\tilde{\eta}}_k))$, and
   (d) for each $y \in E^n$ either $b_k(y) = y$ or there exists $x \in D^{n-2}$ such that both y and $b_k(y)$ lie in $N(D(x, b_k, a_k), \tilde{\tilde{\eta}}_k)$.

Furthermore the same reasoning as for b_1 proves that $D(b_k, a_{k+1}) \subseteq b_k(N(a_k, a_{k+1}; \eta_k))$. Hence η_{k+1} can be chosen, as claimed, to satisfy $N(b_{k+1}, a_{k+1}; \eta_{k+1}) \subseteq b_k(a_k, a_{k+1}; \eta_k)$. This property is also used in the next paragraph.

The homeomorphism $b: E^n \to E^n$ required by the theorem is just $b = b_1^{-1} \circ b_2^{-1} \circ \ldots \circ b_q^{-1}$. We now show that b has the required properties. First note that $b_q^{-1} | D(-1, b_{q-1})$ = identity, that $D(b_{q-1}, b_q) \subseteq D(b_{q-1}, a_q) \subseteq b_{q-1}(N(a_{q-1}, a_q; \eta_{q-1})) = b_q \circ b_{q-1}(N(a_{q-1}, a_q; \eta_{q-1}))$, so $b_q^{-1}(D(b_{q-1}, b_q)) \subseteq b_{q-1}(N(a_{q-1}, a_q; \eta_{q-1}))$ and finally $b_q^{-1}(D(b_q)) \subseteq N(a_q, \eta_q)$. Continuing we get $b_{q-1}^{-1} \circ b_q^{-1} | D(-1, b_{q-2})$ = identity. From the furthermore property of b_{q-1} we get

$$D(b_{q-2}, b_{q-1}) \subseteq D(b_{q-2}, a_{q-1}) \subseteq b_{q-2}(N(a_{q-2}, a_{q-1}; \eta_{q-2}))$$

$$= b_{q-1} \circ b_{q-2}(N(a_{q-2}, a_{q-1}; \eta_{q-2}))$$

so

$$b_{q-1}^{-1} \circ b_q^{-1}(D(b_{q-2}, b_{q-1})) = b_{q-1}^{-1}(D(b_{q-2}, b_{q-1})) \subseteq b_{q-2}(N(a_{q-2}, a_{q-1}; \eta_{q-2})).$$

From the above statement about b_q^{-1} it follows immediately that $b_{q-1}^{-1} \circ b_q^{-1}(D(b_{q-1}, b_q)) \subseteq N(a_{q-1}, a_q; \eta_{q-1})$. Finally $b_{q-1}^{-1} \circ b_q^{-1}(D(b_q)) = b_q^{-1}(D(b_q)) \subseteq N(a_q, \eta_q)$. In general, for each k $(1 \leq k \leq q)$ we have $b_k^{-1} \circ \ldots \circ b_q^{-1} | D(-1, b_{k-1})$ = identity, $b_k^{-1} \circ \ldots \circ b_q^{-1}(D(b_{k-1}, b_k)) = b_k^{-1}(D(b_{k-1}, b_k)) \subseteq b_{k-1}(N(a_{k-1}, a_k; \eta_{k-1})), b_k^{-1} \circ \ldots \circ b_q^{-1}(D(b_k, b_{k+1})) \subseteq N(a_k, a_{k+1}; \eta_k)$ and for each i $(k + 1 \leq i < q)$ we have $b_k^{-1} \circ \ldots \circ b_q^{-1}(D(b_i, b_{i+1})) = b_{k+1}^{-1} \circ \ldots \circ b_q^{-1}(D(b_i, b_{i+1})) \subseteq N(a_i, a_{i+1}; \eta_i)$. Thus b has the property that for each k $(0 \leq k < q)$ $b(D(b_k, b_{k+1})) \subseteq N(a_k, a_{k+1}; \eta_k) \subseteq N(a_k, a_{k+1}; \epsilon')$. In other words, b has moved points about the correct distance lengthwise along D.

We now establish that b has not moved points too far sideways. Let $y \in E^n$. Then either $b_q^{-1}(y) = y$ or else there exists an $x_q \in D^{n-2}$ such that both $b_q^{-1}(y)$ and y lie in $N(D(x_q, -1, 1), \eta_q)$. Similarly either $b_{q-1}^{-1} \circ b_q^{-1}(y) = b_q^{-1}(y)$ or

else there exists an $x_{q-1} \in D^{n-2}$ such that both lie in $N(D(x_{q-1}, -1, 1), \eta_{q-1})$. By our choice of η_{q-1} it follows that either $h_{q-1}^{-1} \circ h_q^{-1}(y) = y$ or else both lie in $N(D(x_{q-1}, -1, 1), \eta_{q-2})$ for some $x_{q-1} \in D^{n-2}$. Continuing we get that for each k $(1 \le k \le q)$ either $h_k^{-1} \circ \cdots \circ h_q^{-1}(y) = y$ or else both lie in $N(D(x_k, -1, 1), \eta_{k-1})$ for some $x_k \in D^{n-2}$. Hence h has the property that for each $y \in E^n$ either $h(y) = y$ or else both lie in $N(D(x', -1, 1), \eta_0)$ for some $x' \in D^{n-2}$. In particular, if $y = f \circ \pi(x, t)$ (with $a_k \le t \le a_{k+1}$), then $h(y) \in N(D(x, -1, 1), 2\epsilon')$. From the last paragraph we know that there exists an (x'', t'') such that $a_k \le t'' \le a_{k+1}$ the distance from $h(y)$ to $f(x'', t'') < \epsilon'$. Hence from our choice of ϵ' we have distance from $f(x'', t'')$ to $f(x, t'')$ is less than $\epsilon/3$ and since distance from $f(x, t'')$ to $f(x, t)$ is less than $\epsilon/3$ we have that distance from $h(y) = h \circ f \circ \pi(x, t)$ to $f(x, t)$ is less than ϵ. $\square$

The following is an immediate corollary, but states the result in the form we wish to use later.

Corollary. *Let M be a connected topological $(n-1)$-manifold and let Q be a topological n-manifold $(n \ge 5)$. Let $g: M \to \text{Int } Q$ be a topological embedding with $Q - g(M)$ 1-LC at each point of $g(M)$. Suppose that g can be locally approximated by locally flat embeddings at some point of M (for example, if g is locally flat at some point of M), then g can be locally approximated by locally flat embeddings at each point of M.*

Proof. For each $x \in M$ there exists an open n-cell $\mathcal{O} \subseteq Q$ and an embedding $f: D^{n-2} \times [-1, 1] \to M$ such that $x \in \text{Int } f(D^{n-2} \times [-1, 1])$, $g \circ f(D^{n-2} \times [-1, 1]) \subseteq \mathcal{O}$ and $g \circ f | D^{n-2} \times [-1, 0]$ can be ϵ-approximated by locally flat embeddings for every $\epsilon > 0$. This provides us with a close locally flat approximation G of $g \circ f \circ \pi: D^{n-2} \times [-1, 1] \to \mathcal{O}$ and hence Theorem 1 gives us a homeomorphism $h: \mathcal{O} \to \mathcal{O}$ so that $h \circ G$ is a close locally flat approximation of $h \circ g \circ f \circ \pi$ which is in turn close to $g \circ f$.

Lemma 1. *Suppose $h: D^{n-1} \to E^n$ is an embedding, $0 < a < b < 1$, N is a neighborhood of $h(bD^{n-1})$, and $\epsilon > 0$. Then there is a $\delta > 0$ and neighborhoods $U' \subset V'$ of $h(aD^{n-1})$ such that if $g: (D^k - \text{Int } aD^k) \times D^{n-k-1} \to E^n$ is an embedding such that $d(g, h | (D^k - \text{Int } aD^k) \times D^{n-k-1}) < \delta$ then there is a neighborhood W of $g(\text{Int } bD^k - aD^k \times \text{Int } D^{n-k-1})$ such that if $f: D^{n-1} \to E^n$ is an embedding such that $d(f, h) < \delta$ and $f | (D^k - \text{Int } aD^k) \times D^{n-k-1} = g$ then $U = U' \cup W$ is a neighborhood of $f(\text{Int } bD^k \times aD^{n-k-1})$ in N that is separated by $f(D^{n-1})$ into two sides U_1 and U_2 and $V = V' \cup W$ is a neighborhood of $f(\text{Int } bD^k \times aD^{n-k-1})$ that is separated by $f(D^{n-1})$ into two sides V_1 and V_2, $U_1 \subset V_1$, $U_2 \subset V_2$, $V \cap N(h(D^k - bD^{n-k-1}), \epsilon) \subset W$, $V \cap f(D^k - bD^k) \times bD^{n-k-1} = \emptyset$, and there are homotopies $f_t: \text{Cl } U_1 \cup \text{Cl } V_2 \to \text{Cl } V$ and $g_t: \text{Cl } U_2 \cup \text{Cl } V_1$*

→ Cl V *satisfying the following conditions*:

1. $f_0 = Identity$ and $g_0 = Identity$,
2. $f_t | \text{Cl } V_2 = Identity$ and $g_t | \text{Cl } V_1 = Identity$ *for all* $t \in [0, 1]$.
3. $d(f_t, \text{Id}) < \epsilon$ and $d(g_t, \text{Id}) < \epsilon$ *for all* $t \in [0, 1]$,
4. $f_1(U_1) \subset \text{Cl } V_2$ and $g_1(U_2) \subset \text{Cl } V_1$, *and*
5. $f_t(U_1) \cap V_2 = \emptyset = g_t(U_2) \cap V_1$ *for all* $t \in [0, 1]$.

Proof. First choose $a < m_1 < m_2 < m_3 < m_4 < b$. Let N_1 be a neighborhood of $h(bD^{n-1})$ in N and δ be so small that N_1 $\epsilon/2$-retracts onto $f(bD^{n-1})$ for any δ-approximation f of h. Let $N_2 \subset N_1$ be a neighborhood of $h(M_4 D^{n-1})$ so small that $N_2 \cap f(D^{n-1}) \subset f(bD^{n-1})$ for any δ-approximation f of h. Next we claim that there is a neighborhood $V' \subset N_2$ of $h(m_3 D^{n-1})$ and $\delta > 0$ so small that $f(D^{n-1})$ separates V' for any embedding f such that $d(f, h) < \delta$. Let $V' \subset N_2$ and δ be chosen so that the inclusion $i: V' \to E^n - f(\partial D^{n-1})$ is homotopic to a retraction onto $f(m_3 D^{n-1})$ for any embedding $f: D^{n-1} \to E^n$ such that $d(f, h) < \delta$. Now pick points x and y in $V' - h(D^{n-1})$ which can be joined by an arc α from x to y that pierces $h(m_3 D^{n-1})$ once. Let β be an arc in $E^n - h(D^{n-1})$ such that $\alpha \cup \beta$ is a simple closed curve and suppose δ is less than

$$\min\{d(h(\partial D^{n-1}), \alpha[0, 1]), \ d(h(D^{n-1}), \beta[0, 1])\}.$$

Now suppose that $f: D^{n-1} \to E^n$ is an embedding such that $d(f, h) < \delta$ and such that there is an arc, say γ, from x to y in $V' - f(D^{n-1})$. Then $\gamma \cup \beta$, we can assume, is a simple closed curve in $E^n - f(D^{n-1})$ so $\gamma \cup \beta$ does not link $f(\partial D^{n-1})$. But $\gamma \cup \beta$ is homotopic to $\alpha \cup \beta$ in $E^n - f(\partial D^{n-1})$ so $\alpha \cup \beta$ does not link $f(\partial D^{n-1})$. Also $f | \partial D^{n-1}$ is homotopic to $h | \partial D^{n-1}$ in $E^n - \alpha \cup \beta$ so $\alpha \cup \beta$ does not link $h(\partial D^{n-1})$. But $\alpha \cup \beta$ pierces $h(D^{n-1})$ once so $\alpha \cup \beta$ does link $h(\partial D^{n-1})$. Therefore we have a contradiction and so V' is separated by $f(D^{n-1})$. Furthermore no simple closed curve in $V' - f(D^{n-1})$ links $f(\partial D^{n-1})$ so no simple closed curve in $V' - f(D^{n-1})$ can pierce $f(D^{n-1})$ once. Thus $V' - f(D^{n-1})$ can be written as the union of V'_1 and V'_2 and each set approaches $f(D^{n-1})$ from one side, i.e. there is a neighborhood V'' of $f(\text{Int } D^{n-1})$ that is separated by $f(\text{Int } D^{n-1})$ onto components V''_1 and V''_2 and $V'_1 \cap V''_2 = \emptyset = V'_2 \cap V''_1$. Now let $N_3 \subset V'$ be a neighborhood of $h(m_2 D^{n-1})$ and let δ be so small that N_3 deforms by an $\epsilon/2$-homotopy in V' to a retraction onto $f(m_2 D^{n-1})$ for any embedding f such that $d(f, h) < \delta$. Suppose further that N_4, a neighborhood of $h(m_1 D^{n-1})$, $N_4 \subset N_3$, and δ are such that $N_4 \cap f(D^{n-1}) \subset f(m_2 D^{n-1})$ for any δ-approximation f of h. Last we choose $U' \subset N_4$, U' a neighborhood of $h(aD^{n-1})$, and $\delta > 0$ so that U' is separated by $f(D^{n-1})$ for any embedding f such that $d(f, h) < \delta$.

Now let g: $(D^k - aD^k) \times D^{n-k-1} \to E^n$ be an embedding such that $d(g, h|(D^k - aD^k) \times D^{n-k-1}) < \delta$ and we pick a small neighborhood W of $g((\text{Int } bD^k - aD^k) \times \text{Int } bD^{n-k-1})$ that is separated by the image of g into two components W_1 and W_2 and suppose that

$$W \cap \text{Image of } g = g((\text{Int } bD^k - aD^k) \times \text{Int } bD^{n-k-1}).$$

Suppose that f: $D^{n-1} \to E^n$ is an embedding such that f extends g and $d(f, h)$ $< \delta$. Then V' and U' are separated by $f(D^{n-1})$ into sides $U_1' \subset V_1'$ and $U_2' \subset V_2'$. Suppose that W_1 and W_2 are indexed so that $W_1 \cap V_2' = \varnothing = W_2 \cap V_1'$. Then $U = U' \cup W$ and $V = V' \cap W$ are separated by $h(D^{n-1})$ into components $U_1 = U_1' \cup W_1$, $U_2 = U_2' \cup W_2$, $V_1 = V_1' \cup W_1$, and $V_2 = V_2' \cup W_2$ and $U_1 \subset V_1$ and $U_2 \subset V_2$ and $\text{Cl } V_1 \cap \text{Cl } V_2 \subset f(D^{n-1})$. Therefore it follows from the fact that $V' \subset N_2 \subset N_1$ that there are $\epsilon/2$-retractions r_1', r_2': $\text{Cl } V' \to \text{Cl } V_1'$, $\text{Cl } V_2'$, respectively. If W is chosen sufficiently close to the image of g then r_1' and r_2' can be extended to $\epsilon/2$-retractions r_1, r_2: $\text{Cl } V \to \text{Cl } V_1$, $\text{Cl } V_2$, respectively. Similarly it is possible to define homotopies f_t': $\text{Cl } U_1 \to \text{Cl } V$ and g_t': $\text{Cl } U_2 \to \text{Cl } V$ satisfying the conditions 1'. $f_0' = \text{Id}$, $g_0' = \text{Id}$; 2'. $f_t'|\text{Cl } U_1 \cap \text{Cl } V_2 = \text{Id} = g_t'|\text{Cl } U_2 \cap V_1$; 3'. $d(f_t', \text{Id}) < \epsilon/2$ and $d(g_t', \text{Id}) < \epsilon/2$; 4'. $f_1'(\text{Cl } U_1) \subset f(m_2 D^{n-1})$ and $g_1'(\text{Cl } U_1)$ $\subset f(m_2 D^{n-1})$. Now define f_t: $\text{Cl } U_1 \cup \text{Cl } V_2 \to \text{Cl } V_1$ by $f_t = r_1 \circ f_t'$ on $\text{Cl } U_1$ and $f_t = \text{Id}$ on $\text{Cl } V_2$. Analogously set $g_t = r_2 \circ g_t'$ on $\text{Cl } U_2$ and $g_t = \text{Id}$ on $\text{Cl } V_1$. Then conditions 1–4 follow in turn from conditions 1'–4' and condition 5 follows from the definitions of r_1 and r_2.

Lemma 2. *Suppose* h: $D^{n-1} \to E^n$ *is a topological embedding,* $0 < a < 1$, *and* $\epsilon > 0$. *Then there is a* $\delta > 0$ *such that if* h_0, h_1: $D^{n-1} \to E^n$ *are locally flat embeddings such that* $d(h_i, h) < \delta$ *and* $h_0 | (D^k - aD^k) \times D^{n-k-1} = h_1 | (D^k - aD^k) \times D^{n-k-1}$ *then for any neighborhood N of* $h_1(D^k \times aD^{n-k-1})$ *there is an ϵ-push H of* $(E^n, h(aD^{n-1}))$ *such that* $Hh_0(D^k \times aD^{n-k-1}) \subset N$.

Proof. Pick a' and b so that $a < a' < b$ and $h(bD^{n-1}) \subset N(h(aD^{n-1}), \epsilon)$. Then apply Lemma 1 with ϵ replaced by $\epsilon/3$ and a replaced by a'. Thus we obtain neighborhoods $U \subset V \subset N(h(aD^{n-1}), \epsilon)$ such that the conclusions of Lemma 1 are satisfied. Let $U_1(h_0)$ and $U_2(h_0)$ be the two sides of $h(D^{n-1})$ in U. Similarly define $U_i(h_j)$ and $V_i(h_j)$ so that $U_1(h_0)$ and $U_1(h_1)$ are on the "same side". Since $V_2(h_0)$ is 1-LC at each point of $h_0(D^{n-1}) \cap V_2(h_0)$ and $V_1(h_1)$ is 1-LC at each point of $h_1(D^{n-1}) \cap V_1(h_1)$ it is possible to modify the homotopies f_t and g_t given in Lemma 1 which retract $\text{Cl } U_1(h_0) \cup \text{Cl } V_2(h_0)$ onto $\text{Cl } V_2(h_0)$ and $\text{Cl } U_2(h_1) \cup \text{Cl } V_1(h_1)$ onto $\text{Cl } V_1(h_1)$, respectively, and to use radial engulfing (see Proposition 7 and Lemma 2 of [6]). The result is that for any closed set C_1 of $\text{Cl } V_2(h_0)$ and C_2 of $\text{Cl } V_1(h_1)$ such that $C_1 \cap V \cap h_0(D^{n-1}) = \varnothing =$

$C_2 \cap V \cap h_1(D^{n-1})$ there are $\epsilon/2$-pushes F and G of $(E^n, h(aD^{n-1}))$ fixed on C_1 and C_2, respectively such that $F(V_2(h_0)) \cup G(V_1(h_1)) \supset U$. However, we can choose C_1 and C_2 so that $V \subset C_1 \cup C_2 \cup U \cup N(h(D^k \times (D^{n-k-1} - a'D^{n-k-1})), \epsilon)$ $\subset F(V_2(h_0)) \cup G(V_1(h_1)) \cup N(h(D^k \times (D^{n-k-1} - a'D^{n-k-1})), \epsilon)$. Since F and G can be chosen to be fixed outside V, ϵ can be chosen so small that $G^{-1}Fh_0(\text{Int } bD^k \times aD^{n-k-1}) \subset V_1(h_1)$. Thus without loss of generality we may assume that $h_0 | (D^k - aD^k) \times D^{n-k-1} = h_1 | (D^k - aD^k) \times D^{n-k-1}$, $h_0(\text{Int } bD^k \times bD^{n-k-1}) \subset V_1(h_1)$, and $d(h_i, h) < \delta$. It is possible now to modify the homotopies constructed in Lemma 1 and use radial engulfing to construct $\epsilon/2$-pushes F' and G' of $(E^n, h(aD^{n-1}))$ fixed on $h_0((D^k - bD^k) \times aD^{n-k-1})$ and on $V_2(h_1)$ such that $F'(V_2(h_1) \cup N) \cup G'(V_1(h_0)) \cup N(h(D^k \times (D^{n-k-1} - a'D^{n-k-1}), \epsilon))$ $\supset V$. Thus $F'^{-1}G'h_0(bD^k \times aD^{n-k-1}) \subset N$ if ϵ is sufficiently small. But $F'^{-1}G'$ is fixed on $h_1((D^k - bD^k) \times aD^{n-k-1})$ so $F'^{-1}G'h_0(D^k \times aD^{n-k-1}) \subset N$.

Addendum to Lemma 2. In the case that $h_0(D^k \times bD^{n-k-1}) \cap V_2(h_1) = \varnothing$ the ϵ-push H can be constructed to be fixed on $\text{Cl } V_2(h_1)$.

Theorem 2. *Suppose* $h: D^{n-1} \to E^n$ *is an embedding and* $\epsilon > 0$. *Then there is a* $\delta > 0$ *such that if* $h_0, h_1: D^{n-1} \to E^n$ *are locally flat embeddings such that* $d(h_i, h) < \delta$ *and* h_0 *agrees with* h_1 *on* $(D^k - \frac{1}{2}D^k) \times D^{n-k-1}$ *then there is an* ϵ-*push* H *of* $(E^n, h(\frac{1}{2}D^{n-1}))$ *such that* $Hh_0 | D^k \times \frac{1}{2}D^{n-k-1} = h_1 | D^k \times \frac{1}{2}D^{n-k-1}$

Proof. The proof is a slight modification of [7] or Lemma 5 of [6]. Let h_0, $h_1: D^{n-1} \to E^n$ be close locally flat approximations of h and $\frac{1}{2} < a < 1$. Then h_0 and h_1 can be extended to embeddings of $D^n = D^{n-1} \times [-1, 1]$ into E^n such that h_0 and h_1 agree on $(D^k - aD^k) \times D^{n-k}$. By first applying Lemma 2 and then modifying h_0 by squeezing toward the set $h_0(D^k \times D^{n-k-1})$ it follows that we can assume that $h_0(D^k \times \frac{1}{2}D^{n-k}) \subset h_1(D^n)$. Let $-1 = t_0 < t_1 < \cdots < t_{2m} = 1$. We can use Lemma 2 and the Addendum to Lemma 2 to modify h_0 by an ϵ'-push so that for $0 \leq i \leq (m-1)$, $h_1^{-1}h_0(D^k \times aD^{n-k-1} \times t_{2i+1}) \subset D^{n-1} \times (2i, 2i+2)$. Thus for ϵ' small enough and the t_i chosen sufficiently dense

$$d(h_1^{-1}h_0 | D^k \times aD^{n-k-1} \times [-1, 1], \text{Id})$$

is small. Thus it follows from the Main Lemma of [4] that there is an ϵ'-push H' of (D^n, aD^n) fixed on ∂D^n such that $H'h_1^{-1}h_0 | D^k \times \frac{1}{2}D^{n-k} = \text{Id} | D^k \times \frac{1}{2}D^{n-k}$. Thus we can define H to be fixed outside $h_1(D^n)$ and equal $h_1H'h_1^{-1}$ on $h_1(D^n)$. Then $Hh_0 | D^k \times \frac{1}{2}D^{n-k-1} = h_1H'h_1^{-1}h_0 | D^k \times \frac{1}{2}D^{n-k-1} = h_1 | D^k \times \frac{1}{2}D^{n-k-1}$. For ϵ' sufficiently small and a chosen close enough to $\frac{1}{2}$, H will be an ϵ-push of $h(\frac{1}{2}D^{n-1})$.

Theorem 3. *Suppose M is a topological $(n-1)$-manifold, Q is a topological n-manifold, $h: M \to \text{Int } Q$ is an embedding and $\epsilon: Q \to R^+$ is a map. Then there is a positive function δ on M such that if h_0, $h_1: M \to Q$ are locally flat embeddings such that $d(h_i(x), h(x)) < \delta(x)$ for all $x \in M$ then there is an ϵ-push H of $(Q, h(M))$ such that $Hh_0 = h_1$.*

Proof. The proof is virtually identical to the proof of Theorem 5.1 of [4] using our Theorem 2 instead of their Main Lemma to get h_0 and h_1 to agree on a closed subset in Int M. Then a little push-pull near the boundary does the rest.

Theorem 4. *Suppose M and Q are topological $(n-1)$- and n-manifolds, $h: M \to \text{Int } Q$ is an embedding that can be locally approximated (see introduction) by locally flat embeddings, and ϵ is a nonnegative map on M. Then there is an embedding $f: M \to Q$ such that $d(f(x), h(x)) < \epsilon(x)$ for all $x \in M$ and f is locally flat at any point x for which $\epsilon(x) > 0$.*

Proof. It is sufficient to prove the following statement. Suppose A_1 is closed in M, A_2 is compact, U_1 and U_2 are open sets such that $U_1 \supset A_1$, $U_2 \supset A_2$ and $h_i = h \mid U_i$ can be approximated by locally flat embeddings for $i = 1, 2$. Then there is an open set $U \subset M$ such that $A_1 \cup A_2 \subset U$ and $h \mid U$ can be approximated by locally flat embeddings. Let P_1, P_2 and P_3 be polyhedral neighborhoods of A_1, A_2, and $A_1 \cap A_2$ such that $P_1 \cap P_2 \subset P_3 \subset U_1 \cap U_2$, $P_1 \subset U_1$, and $P_2 \subset U_2$. Choose $\epsilon < \frac{1}{2} d(Q(P_2 - P_3), P_1)$. Let δ be given by Theorem 3 depending on ϵ for $h \mid P_3$. Then there is an ϵ-push H of (Q, P_3) such that $Hh_2 = h_1$ on P_3. Now define f to be h_1 on P_1 and Hh_2 on P_2. It is clear from $\epsilon < d(Q(P_2 - P_3), P_1)$ that f is an embedding.

Theorem 5. *Suppose $h: D^{n-1} \to \text{Int } Q$ is a topological embedding, h can be locally approximated by locally flat embeddings, U is a connected open subset of Q such that $h(\text{Int } D^{n-1}) \subset U$ and U is separated by $h(D^{n-1})$ into components U_1 and U_2, and U_1 is 1-LC at each point of $h_1(\text{Int } D^{n-1})$. Then $h(\text{Int } D^{n-1})$ has a collar in $\text{Cl } U_1$.*

Proof. Suppose $\epsilon: \text{Int } aD^{n-1} \to R^+$ is a map. It follows from Theorem 4 that there is a locally flat $\epsilon(x)$-approximation f of $h \mid \text{Int } aD^{n-1}$. But from the proof of Lemma 2 we can push $f(\text{Int } aD^{n-1})$ into U_1. Thus it suffices to show that there is a $\delta: \text{Int } aD^{n-1} \to R^+$ such that if f_0, $f_1: \text{Int } aD^{n-1} \to U_1$ are disjoint locally flat $\delta(x)$-approximations of $h \mid \text{Int } aD^{n-1}$, then there is an embedding $F: \text{Int } aD^{n-1} \times [0, 1] \to U_1$ such that $F(x, i) = f_i(x)$ for $i = 0, 1$ and $\text{diam}(F(x \times [0, 1])) < \epsilon(x)$ for all $x \in \text{Int } aD^{n-1}$. Choose $\epsilon': \text{Int } aD^{n-1} \to R^+$ so small that if $x, y \in \text{Int } aD^{n-1}$, $d(x, y) < \epsilon'(x)$, and $d(f_0, h) < \epsilon'$, then $d(f_0(x), f_0(y)) < \epsilon(x)$. Now we can extend such an f_0 to $\overline{f}_0: \text{Int } aD^{n-1} \times [0, 1]$ so that $d(\overline{f}_0(x), \overline{f}_0(y)) < \epsilon(x)$

wherever $x, y \in M \times [0, 1]$ and $d(x, y) < \epsilon'(x)$. From Lemma 2 it follows that we can engulf $f_1(\text{Int } aD^{n-1})$ and so we assume that $f_1(\text{Int } aD^{n-1}) \subset \overline{f}_0(M \times [t_0, 1])$ for some $t_0 > 0$ and that $d(\overline{f}_0^{-1}f_1, \text{ incl.} \mid M \times 0) < \delta(x)$ where $\delta(x)$ is chosen from Theorem 5 depending on $\min\{\frac{1}{2}, \epsilon'\}$. Thus there is an ϵ'-push H of $(M \times [-1, 1], M \times 0)$ such that $H(x, t_1) = f_0^{-1}f_1(x)$ for all $x \in M$ and for some fixed $t_1 > t_0$. Let $S_2: [-1, 1] \to [-1, 1]$ be defined by setting

$$S_2(x) = \begin{cases} -1, & x = -1, \\ 0, & x = -1/2, \\ t_0, & x = t_0, \\ 1, & x = 1 \end{cases}$$

and extending linearly. Set $S = \text{Id} \times S_2: \text{Int } aD^{n-1} \times [-1, 1] \to \text{Int } aD^{n-1} \times [-1, 1]$. Then set $F = \overline{f}_0 SHS^{-1}: M \times [0, t_1] \to U_1$. $F(x, 0) = \overline{f}_0(x, 0) = f(x)$, $F(x, t_1) = \overline{f}_0 S\overline{f}_0^{-1}f_1(x) = f_1(x)$. But t_1 can be chosen so small that $\text{diam } SHS^{-1}(x \times [0, t_1]) < \epsilon'(x)$ for all $x \in \text{Int } aD^{n-1}$ so $\text{diam } F(x \times [0, t_1]) < \epsilon(x)$ for all x. Using a sequence of such F's it can be shown that $b(aD^{n-1}) \cup f_0(\text{Int } aD^{n-1})$ is the boundary of an n-cell; so $b(\text{Int } D^{n-1})$ is locally flat in U_1 at each point.

Corollary 6. *Suppose M and Q are topological $(n-1)$- and n-manifolds, respectively, and $h: M \to \text{Int } Q$ is an embedding such that $Q - b(M)$ is 1-LC at each point of $b(M)$ and for some open set $U \subset M$ there are locally flat approximations of $b \mid U$. Then $b(M)$ is locally flat.*

Proof. It follows from Corollary to Theorem 1 that b can be locally approximated by locally flat embeddings and thus we can apply Theorem 5 on a neighborhood of any point of M. Thus $b(M)$ is locally flat at each point of $b(\text{Int } M)$. However, from [3] we then obtain that $b(M)$ is locally flat.

Theorem 7. *Suppose $h: D^{n-1} \to E^n$ is an embedding such that $b \mid D^{n-2} \times \{-1\}$ is locally flat and $E^n - b(D^{n-1})$ is 1-ULC. Then $b(D^{n-1})$ is locally flat.*

Proof. We adapt a covering argument due to Černavskiĭ [Theorems 1, 2]. Let $D_+ = D^{n-2} \times [0, 1]$ and $D_- = D^{n-2} \times [-1, 0]$. First we may assume that $b: D_+ \to E^n$ and $b \mid D^{n-2} \times \{0\} = \text{Identity}$. Extend b to $\partial D_- \cup D_+$, then restrict b to ∂D-union a small enough neighborhood of $\text{Int } D^{n-2} \times \{0\}$ in D_+ that the restriction is an embedding. Thus we can find an embedding $b': \partial D_- \cup D_+ \to E^n \subset S_n$ such that $b'(D_+) \subset b(D^{n-1})$ and $b' \mid \partial D_- = \text{Identity}$. Let $p: E^n \to S^n - \partial D_-$ be a covering projection such that $p(E^{n-1} \times \{0\}) = \text{Int } D_-^{n-1}$. There is a lifting $\widetilde{b}': D_+ - (D^{n-2} - \{0\}) \to E^n$ such that $\widetilde{b}'((0, \cdots, 0, 1)) \in E^n - p^{-1}(\text{Int } D_-)$. Let

$H: E^n \to E^n$ be an embedding that moves points vertically, is fixed near $\widetilde{b}'((0, \cdots, 0, 1))$, and satisfies $H\widetilde{b}'(D_+ - (D^{n-2} \times 0)) \subset p^{-1}(E^n - D_-)$. Define $f: D_+ \to E^n$ by $f = pH\widetilde{b}'$. Then f can be extended by the identity to an embedding $\overline{f}: D^{n-1} \to E^n$. Also $E^n - \overline{f}(D^{n-1})$ is 1-LC at such point of $\overline{f}(D_+ - (D^{n-2} \times 0))$ and so $E^n - \overline{f}(D^{n-1})$ is 1-ULC. Thus $\overline{f}(D^{n-1})$ is locally flat by Corollary 6. But H was fixed near $\widetilde{b}'((0, \cdots, 0, 1))$ thus $\overline{f}(D^{n-1})$ has an $(n-1)$-cell in common with $b(D^{n-1})$ and so again by Corollary 7 $b(D^{n-1})$ is locally flat.

Corollary 8. *Suppose M is a compact PL $(n-1)$-manifold, Q is a PL n-manifold, and $h: M \to \mathrm{Int}\, Q$ is an embedding that can be locally approximated by locally flat embeddings. Then there is an obstruction in $H^3(M, Z_2)$ which vanishes if and only if h can be approximated by PL locally flat embeddings.*

Proof. It follows from Theorem 5 that h can be approximated by locally flat embeddings and that, for some $\delta > 0$, and two locally flat approximations that are δ-close to h are ambient isotopic by a 1-push of $(Q, h(M))$. Let h_1 be a locally flat δ-approximation of h and let ν_1 be a topological tubular neighborhood of $h_1(M)$ (i.e., an open bicollar). Then ν_1 has two PL structures, namely the structure induced by the PL structure on Q and the product structure induced by $M \times E^1$. Let $\alpha_1 \in H^3(M, Z_2) \approx H^3(M \times E^1, Z_2)$ be the obstruction to isotoping the product structure on ν_1 to the induced structure on ν_1. If h_2 is another locally flat δ-approximation of h and $\alpha_2 \in H^3(M, Z_2)$ is defined analogous to α_1, then, using the fact that h_1 and h_2 are ambient isotopic, it is easy to prove that $\alpha_1 = \alpha_2$. Hence we have a well-defined element α of $H^3(M, Z_2)$. If $\alpha = 0$, then any such h_1 can be isotoped slightly to be made PL and hence h can be approximated by PL locally flat homeomorphisms. On the other hand, if h can be δ-approximated by a PL locally flat homeomorphism h', then we can use h' for determining α and since h' is PL locally flat the two structures on the tubular neighborhood of $h'(M)$ are the same and hence $\alpha = 0$.

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