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PRODUCT OF RING VARIETIES AND ATTAINABILITY

BY

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ABSTRACT. The class of all rings that are Everett extensions of a ring in a variety $\mathfrak{U}$ by a ring in a variety $\mathfrak{V}$ is a variety $\mathfrak{U}\cdot\mathfrak{V}$. With respect to this operation the set of all ring varieties is a partially ordered groupoid (under inclusion), that is not associative. A variety is idempotent iff it is the variety of all rings, or generated by a finite number of finite fields. No families of polynomial identities other than those equivalent to $x = x$ or $x = y$ are attainable on the class of all rings or on the class of all commutative rings.

Hanna Neumann [12] introduced the notion of variety product for groups. This product turns the set of all group varieties into a free monoid with zero as shown independently by B. H. Neumann, Hanna Neumann and P. M. Neumann [11], and A. L. Šmilkin [16]. By analogue of groups, A. I. Mal'cev [9] defined the product for classes of algebraic systems and gave several applications. V. A. Parfenov [14] proved that the set of all Lie algebra varieties over a field of characteristic 0 is a free monoid with 0. However, in rings, the product of varieties is not associative. In the present paper we determine all the idempotent varieties of associative rings. We also apply the results to show that there are no nontrivial sets of identities attainable on the varieties of all associative rings or of all associative and commutative rings; this answers a question of T. Tamura [17].

For an account of the variety theory the reader may consult [1], [2], [6], [10], [13].

In this paper we will be concerned only with associative rings not necessarily with 1; the word "ring" will mean "associative ring". German letters will always denote classes or varieties of rings.

The methods of this paper could be modified for the variety of all commutative and associative rings, and the analogue of Theorem 5 holds for the variety of all commutative rings.

1. **Definition 1.** Let A and B be rings. The ring C is called an extension of A by B if C possesses an ideal isomorphic to A whose factor ring is isomorphic to B.

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This notion is due to Everett [4] and is the analogue of Schreier extension of groups [15].

Definition 2. If $\mathfrak{A}$ and $\mathfrak{B}$ are classes of rings $\mathfrak{A} \cdot \mathfrak{B}$ is the class of all rings that are extensions of a ring in $\mathfrak{A}$ by a ring in $\mathfrak{B}$.

This notion is due to Mal'cev [9] where he proved (for more general systems):

Theorem 1 (Mal'cev [9]). *The product of two ring varieties is a ring variety.*

Denote the set of all ring varieties by G, the variety of all rings by $\mathfrak{O}$ and the variety of all zero (degenerate) rings by $\mathfrak{E}$. In [9], it is shown:

$$\mathfrak{O} \cdot \mathfrak{A} = \mathfrak{O} = \mathfrak{A} \cdot \mathfrak{O}, \qquad \mathfrak{E} \cdot \mathfrak{A} = \mathfrak{A} = \mathfrak{A} \cdot \mathfrak{E},$$

$$\mathfrak{A} \leq \mathfrak{C}, \mathfrak{B} \leq \mathfrak{D} \Rightarrow \mathfrak{A} \cdot \mathfrak{B} \leq \mathfrak{C} \cdot \mathfrak{D}, \qquad (\mathfrak{A} \cdot \mathfrak{B}) \cdot \mathfrak{C} \geq \mathfrak{A} \cdot (\mathfrak{B} \cdot \mathfrak{C});$$

inclusion can be strict.

I.e., $\langle G; \cdot, \leq \rangle$ is a partially ordered nonassociative groupiod with zero ($\mathfrak{O}$) and a unit ($\mathfrak{E}$).

2. Every ring variety is determined by a set of identities of the type $p(x_1, \cdots, x_n) = 0$. Let F be the free ring over the set of free generators $\{x_1, x_2, \cdots\}$ (cf. e.g. [3], [13]). If $U \subseteq F$ and A is a ring, let

$$U(A) = \{p(a_1, \cdots, a_n) : p \in U, a_1, \cdots, a_n \in A\}.$$

It is well known that A belongs to the variety determined by U iff $U(A) = \overline{U}(A) = 0$ where $\overline{U}$ is the T-ideal of F generated by U (the smallest ideal of F closed under all endormorphisms of F and containing U).

Proposition 2. *If U is a T-ideal of F and A is a ring, then $U(A)$ is the smallest ideal of A whose factor belongs to $\mathfrak{U}$ the variety determined by U.*

$$au(a_1, \cdots, a_n) \pm v(b_1, \cdots, b_m)b = x_1 u(x_2, \cdots, x_{n+1})$$
$$\pm v(x_{n+2}, \cdots, x_{n+m+2})x_{n+m+3}(a, a_1, \cdots, a_n, b_1, \cdots, b_m, b).$$

Proposition 3 (Mal'cev [9]). *Let U, V be two T-ideals of F and let $\mathfrak{U}$, $\mathfrak{V}$ be the ring varieties determined by U and V respectively. Then $A \in \mathfrak{U} \cdot \mathfrak{V}$ iff $V(A) \in \mathfrak{U}$.*

Thus the variety $\mathfrak{U} \cdot \mathfrak{V}$ is determined by the set of identities $U(V)$.

3. For any variety $\mathfrak{U}$ denote by $\delta(\mathfrak{U})$ the least degree [3] of polynomial identities satisfied by $\mathfrak{U}$; $\delta(\mathfrak{O}) = \infty$.

It is implicit in [5], [13] that the minimal degrees are achieved by homogeneous identities, since if p_1 is a homogeneous component of p, and p is an identity in $\mathfrak{U}$, then kp_1 is an identity in $\mathfrak{U}$ for some $k \in Z$, $k > 0$.

For any variety $\mathfrak{U}$ denote by $c(\mathfrak{U})$ the characteristic of the free $\mathfrak{U}$-ring on one generator. Then

Proposition 4. $\mathfrak{U} \leq \mathfrak{B} \Rightarrow \delta(\mathfrak{U}) \leq \delta(\mathfrak{B})$, $c(\mathfrak{U}) = 1$ *iff* $\mathfrak{U} = \mathfrak{E}, c(\mathfrak{U}) = 0$ *iff* $\delta(\mathfrak{U}) > 1$.

Denote by $GF(p, n)$ the Galois field of order p^n and by Q_p the ring $\langle\{0, a, 2a, 3a, \cdots, (p-1)a\}; a^2 = 0\rangle$, where p is a prime. The variety generated by $GF(p, n)$ is denoted by $\mathfrak{P}(p, n)$, it is the variety of all rings satisfying: $px = 0$, $x - x^{p^n} = 0$. The variety generated by Q_p is denoted by $\mathfrak{Q}(p)$, it is the variety of all rings satisfying: $px = 0$, $xy = 0$ [19].

4. We now formulate the basic theorem:

Theorem 5. *If the variety* $\mathfrak{U} \neq \mathfrak{O}$, the *following conditions are equivalent*:

(i) $\mathfrak{U}$ *does not contain any* Q_p.

(ii) $\mathfrak{U}$ *is generated by a finite number of finite fields.*

(iii) $\mathfrak{U}$ *is idempotent in* $(G, \cdot)$, *i.e.,* $\mathfrak{U} \cdot \mathfrak{U} = \mathfrak{U}$.

The proof will depend on some lemmas.

Lemma 6. *The variety* $\mathfrak{B}_n$ *defined by* $p^{n-r}x_1x_2 \cdots x_{2^r}$, $r = 0, 1, 2, \cdots, n$, *is contained in the variety*

$$\mathfrak{Q}(p) \cdot (\mathfrak{Q}(p) \cdot (\cdots (\mathfrak{Q}(p) \cdot \mathfrak{Q}(p)) \cdots)) = \mathfrak{Q}(p)^n.$$

Let

$$A \in \mathfrak{B}_{n+1}. \quad B = \sum p^{n-r} A \cdots A \ (2^r\ A\text{'s}) \text{ is an ideal in } A.$$

$$A/B \in \mathfrak{B}_n.$$

$$b \in B \Leftrightarrow b = \sum \{p^{n-r} a_{j1} \cdots a_{j2^r} : 0 \leq r \leq n,\ j \in K_r\}.$$

$$pb = \sum \{p^{n+1-r} a_{j1} \cdots a_{j2^r} : 0 \leq r \leq n,\ j \in K_r\} = 0.$$

If $c \in B$ then

$$\begin{aligned} b \cdot c &= \left(\sum \{p^{n-r} a_{j1} \cdots a_{j2^r} : 1 \leq r \leq n,\ j \in K_r\}\right) \\ &\quad \cdot \left(\sum \{p^{n-s} c_{i1} \cdots c_{i2^s} : 0 \leq s \leq n,\ i \in L_s\}\right) \\ &= \sum \{a_{j1} \cdots a_{j2^n} c_{i1} \cdots c_{i2^n} : j \in K_n,\ i \in L_n\} \\ &\quad + \sum \{p^{2n-r-s} a_{j1} \cdots a_{j2^r} c_{i1} \cdots c_{i2^s} : 0 \leq r+s < 2n,\ j \in K_r,\ i \in L_s\} \\ &= 0, \end{aligned}$$

i.e. $B \in \mathfrak{Q}(p)$.

Hence $\mathfrak{V}_{n+1} \leq \mathfrak{Q}(p) \cdot \mathfrak{V}_n$, and by induction, the lemma is proved.

Lemma 7. *If* $\mathfrak{U}$ *contains* Q_p *for some* p, *then* $\mathfrak{U} \cdot \mathfrak{U} > \mathfrak{U}$.

If $\mathfrak{U}$ contains Q_p, then as $\mathfrak{Q}(p)$ is a minimal variety [19], we have $\mathfrak{U} \geq \mathfrak{Q}(p)$. If $\mathfrak{U} \cdot \mathfrak{U} = \mathfrak{U}$, then $\mathfrak{U} = \mathfrak{U} \cdot \mathfrak{U} \geq \mathfrak{Q}(p) \cdot \mathfrak{Q}(p)$, and by induction $\mathfrak{U} \geq \mathfrak{Q}(p)^n$. Hence by Lemma 6, $\mathfrak{U} \geq \mathfrak{V}_n$ for all $n \geq 1$.

Claim 1. $c(\mathfrak{U}) = 0$. As $c(\mathfrak{V}_n) = p^n$, $c(\mathfrak{U})$ is divisible by p^n for all $n \geq 1$, i.e., $c(\mathfrak{U}) = 0$, and $\delta(\mathfrak{U}) > 1$.

Claim 2. For any varieties $\mathfrak{V}$, $\mathfrak{W}$, $\delta(\mathfrak{V} \cdot \mathfrak{W}) = \delta(\mathfrak{V}) \cdot \delta(\mathfrak{W})$. $\mathfrak{V} \cdot \mathfrak{W}$ is determined by $V(W)$, where V and W are the T-ideals of all identities satisfied by $\mathfrak{V}$ and $\mathfrak{W}$ respectively. Claim 2 is immediate by using the comments after Proposition 3.

From Claims 1 and 2 we get a contradiction, and $\mathfrak{U} \cdot \mathfrak{U} > \mathfrak{U}$.

5. **Lemma 8.** *If* $\mathfrak{U}$ *does not contain* Q_p *for any prime* p, *then* $\delta(\mathfrak{U}) = 1$ (*i.e.*, $c(\mathfrak{U}) > 0$).

If $c(\mathfrak{U}) = 0$, the free $\mathfrak{U}$-ring on one generator contains $\{0, x, 2x, \cdots\}$ which is infinite. The factor of this ring by the ideal generated by x^2 is proper, otherwise $x = x^2 q(x)$ would be an identity in $\mathfrak{U}$, and hence $kx = 0$ would be an identity in $\mathfrak{U}$ for some $k > 0$. The factor ring satisfies $xy = 0$, and hence has a factor isomorphic to Q_p for some p.

Lemma 9. *If* $\mathfrak{U} \neq \mathfrak{S}$, $\mathfrak{U}$ *satisfies* $px = 0$, $x + x^2 g(x) = 0$, *where* $g(x) \in Z_p[x]$, *then* $\mathfrak{U}$ *is generated by a finite number of finite fields of characteristic* p.

By a theorem of Herstein [7], every ring satisfying $x + x^2 g(x)$ is commutative. Also the identity $x + x^2 g(x) = 0$ tells that the ring does not have any nonzero nilpotent elements. Hence [8] every such ring is a subdirect product of fields. Since rings of $\mathfrak{U}$ also satisfy $px = 0$, then all fields involved are of characteristic p. As degree $(x + x^2 q(x))$ is fixed for all elements these fields involved must be finite in number, and each of which is of finite order, since if $n_1 < n_2 < n_3 < \cdots$ $\Pi\{GF(p, n_i)\colon i \geq 1\}$ does not satisfy any identity of the type $x + x^2 g(x) = 0$ for any $g(x) \in Z_p[x]$.

6. **Lemma 10.** *If* $\mathfrak{U} \neq \mathfrak{S}$, $\mathfrak{V} \neq \mathfrak{S}$, $\mathfrak{U}$ *satisfies* px, $x + x^2 g(x)$, *and* $\mathfrak{V}$ *satisfies* $px = 0$, $x + x^2 h(x)$ *where* $g(x), h(x) \in Z_p[x]$, *then* $\mathfrak{U} \cdot \mathfrak{V} = \mathfrak{U} \vee \mathfrak{V} = \mathfrak{V} \cdot \mathfrak{U}$.

Substituting identities of $\mathfrak{V}$ in identities of $\mathfrak{U}$, we get identities of $\mathfrak{U} \cdot \mathfrak{V}$ (Proposition 3). Hence

$$p^2 x = 0, \qquad (x + x^2 h(x)) + (x + x^2 h(x))^2 g(x + x^2 h(x)) = x + x^2 f(x) = 0$$

and

$$px + p^2x^2g(px) = 0$$

are identities in $\mathfrak{U} \cdot \mathfrak{B}$, hence $\mathfrak{U} \cdot \mathfrak{B} \neq \mathfrak{S}$ satisfies $px = 0$ and $x + x^2f(x) = 0$.

Hence by Lemma 9, every ring in $\mathfrak{U} \cdot \mathfrak{B}$ is the subdirect sum of finite fields, and every finitely generated ring is finite and hence the direct product of finite fields of characteristic p, i.e. every finitely generated ring has 1. If $C \in \mathfrak{U} \cdot \mathfrak{B}$ is finitely generated, its ideal belonging to $\mathfrak{U}$ is finite, and hence has a unit that is a central idempotent (since C is commutative). Hence $C \cong A \times B$ where $A \in \mathfrak{U}$, $B \in \mathfrak{B}$. As every variety is generated by its finitely generated members, the lemma is proved.

Lemma 11. *If* $c(\mathfrak{U}) = p^k$, $\mathfrak{U} \neq \mathfrak{S}$ *and* $\mathfrak{U}$ *does not contain* Q_p, *then* $k = 1$ *and* $\mathfrak{U}$ *satisfies* $px = 0$ *and* $x + x^2g(x) = 0$ *for some* $g(x) \in Z_p[x]$.

Let $\mathfrak{B}$ be the variety of all rings in $\mathfrak{U}$ satisfying $px = 0$. As $\mathfrak{U}$ does not contain Q_p, $\mathfrak{B}$ does not contain Q_p. $\mathfrak{B}$ satisfies $px = 0$ and $x + x^2g(x) = 0$ for some $g(x) \in Z_p[x]$, $g(x) \neq 0$. Indeed, the free ring of $\mathfrak{B}$ in one generator is $xZ_p[x]/xq(x)Z_p[x]$, degree $q(x) \in Z_p[x]$ is ≥ 1, otherwise the free $\mathfrak{B}$-ring on one generator is either 0 or $xZ_p[x]$ contradicting $\mathfrak{U} \neq \mathfrak{S}$ and $\mathfrak{U}$ does not contain Q_p. Hence $xq(x) = 0$ is an identity in $\mathfrak{B}$, i.e., $\alpha x + x^2g(x) = 0$ is an identity in $\mathfrak{B}$, $\alpha \neq 0$, otherwise $xZ_p[x]/x^2Z_p[x] \in \mathfrak{B}$, that is $Q_p \in \mathfrak{B}$.

If $A \in \mathfrak{U}$ then p^rA is an ideal of A, $r = 0, 1, 2, \cdots, k$, and $p^rA/p^{r+1}A \in \mathfrak{B}$, $r = 0, 1, \cdots, k-1$. Thus $\mathfrak{B} \leq \mathfrak{U} \leq \mathfrak{B}^k = \mathfrak{B}$ by Lemma 10.

Lemma 12. *If* $c(\mathfrak{U}_1) = p^k$, $c(\mathfrak{U}_2) = p^l$, $c(\mathfrak{B}_1) = q^m$, $c(\mathfrak{B}_2) = q^n$, *where* p, q *are distinct primes, then*

$$(\mathfrak{U}_1 \cdot \mathfrak{B}_1) \cdot (\mathfrak{U}_2 \cdot \mathfrak{B}_2) = (\mathfrak{U}_1 \cdot \mathfrak{U}_2) \cdot (\mathfrak{B}_1 \cdot \mathfrak{B}_2).$$

Let $C \in (\mathfrak{U}_1 \cdot \mathfrak{B}_1) \cdot (\mathfrak{U}_2 \cdot \mathfrak{B}_2)$. There is A, an ideal of C, such that $A \in \mathfrak{U}_1 \cdot \mathfrak{B}_1$ and $B = C/A \in \mathfrak{U}_2 \cdot \mathfrak{B}_2$.

$$\text{ch } A = p^{\alpha}q^{\beta}, \quad \text{ch } B = p^{\gamma}q^{\delta}, \quad \text{ch } C = p^{\lambda}q^{\mu}.$$

Thus

$$A = A_1 \times A_2, \quad A_1 \in \mathfrak{U}_1, \quad A_2 \in \mathfrak{B}_1,$$

$$B = B_1 \times B_2, \quad B_1 \in \mathfrak{U}_2, \quad B_2 \in \mathfrak{B}_2, \quad C = C_1 \times C_2.$$

C_1 contains all elements of C whose order is a power of p. Thus $C_1 \supseteq A_1$, $C_2 \supseteq A_2$, as C_2 contains all elements of C of order a power of q.

$$(C_1/A_1) \times (C_2/A_2) \cong (C_1 \times C_2)/(A_1 \times A_2) \cong B_1 \times B_2$$

as every element of C_1/A_1 or of B_1 is of order a power of p and every element of C_2/A_2 or B_2 is of order a power of q. Hence $C_1/A_1 \cong B_1$, $C_2/A_2 \cong B_2$, i.e., $C \in (\mathfrak{U}_1 \cdot \mathfrak{U}_2) \cdot (\mathfrak{B}_1 \cdot \mathfrak{B}_2)$.

Conversely if $C \in (\mathfrak{U}_1 \cdot \mathfrak{U}_2) \cdot (\mathfrak{B}_1 \cdot \mathfrak{B}_2)$, there is an ideal A of C such that $A \in \mathfrak{U}_1 \cdot \mathfrak{U}_2$ whose factor $C/A \in \mathfrak{B}_1 \cdot \mathfrak{B}_2$. But ch $A = p^{\alpha}$, ch $C/A = q^{\beta}$, hence $C \cong A \times (C/A)$, there are ideals A_1 of A and A_2 of $B = C/A$ such that $A/A_1 \in \mathfrak{U}_2$, $A_1 \in \mathfrak{U}_1$, $B/A_2 \in \mathfrak{B}_2$, $A_2 \in \mathfrak{B}_1$. Hence $A_1 \times A_2$ is an ideal of C, belonging to $\mathfrak{U}_1 \cdot \mathfrak{B}_1$ and $C/(A_1 \times A_2) \cong (A \times B)/(A_1 \times A_2) \cong (A/A_1) \times (B/A_2) \in \mathfrak{U}_2 \cdot \mathfrak{B}_2$.

Lemma 13. *If $\mathfrak{U}$ does not contain Q_p for any prime p and $\mathfrak{U} \neq \mathfrak{G}$, then $\mathfrak{U} = \mathfrak{U}_1 \times \mathfrak{U}_2 \times \cdots \times \mathfrak{U}_k$, where $\mathfrak{U}_i \neq \mathfrak{G}$ satisfies $p_i x = 0$, $x + x^2 g_i(x) = 0$ for some $g_i(x) \in Z_{p_i}(x)$, where $p_1, \cdots, p_k$ are k distinct primes, $k \geq 1$, $c(\mathfrak{U}) = p_1 p_2 \cdots p_k$.*

If $\mathfrak{U}$ does not contain Q_p for any p, $c(\mathfrak{U}) = n > 0$ (by Lemma 8). Let $n = p_1^{r_1} \cdots p_k^{r_k}$ be the prime decomposition of n, $r_1, \cdots, r_k > 0$. Hence, $\mathfrak{U} = \mathfrak{U}_1 \times \mathfrak{U}_2 \times \cdots \times \mathfrak{U}_k$, where $\mathfrak{U}_i$ is the family of rings of $\mathfrak{U}$ satisfying $p_i^{r_i} x = 0$. Thus $\mathfrak{U}_i$ does not contain Q_{p_i}, and hence (by Lemma 11) $r_i = 1$ and $\mathfrak{U}_i$ satisfies $x + x^2 g_i(x)$ for some $g_i(x) \in Z_p[x]$, $g_i(x) \neq 0$.

7. By the previous lemmas, if $\mathfrak{U}$ does not contain Q_p for any prime p, $\mathfrak{U}$ is the join of a finite number of varieties of the type $px = 0$, $x + x^2 g(x) = 0$, $g(x) \neq 0$, $g(x) \in Z_p[x]$. Hence, $\mathfrak{U}$ does not contain any Q_p implies that $\mathfrak{U}$ is generated by a finite number of finite fields, i.e., $\mathfrak{U} = \bigvee \{\mathfrak{P}(p, n)\colon p \in K, n \in L\}$, where K is a finite set of primes and L is a finite set of positive integers.

If $\mathfrak{U} = \bigvee \{\mathfrak{P}(p, n)\colon p \in K, n \in L\}$, then $\mathfrak{U} = \mathfrak{U}_1 \cdot (\mathfrak{U}_2 \cdot (\cdots (\mathfrak{U}_k))$, where $\mathfrak{U}_i$ is the variety of all rings of $\mathfrak{U}$ of characteristic p_i; they satisfy also $x + x^2 g_i(x) = 0$, and hence $\mathfrak{U}_i \cdot \mathfrak{U}_i = \mathfrak{U}_i \vee \mathfrak{U}_i = \mathfrak{U}_i$ (by Lemma 10), By Lemma 12 and induction on k, we get $\mathfrak{U} \cdot \mathfrak{U} = \mathfrak{U}$.

If $\mathfrak{U} \neq \mathfrak{O}$ and $\mathfrak{U} \cdot \mathfrak{U} = \mathfrak{U}$, then $\mathfrak{U}$ does not contain Q_p for any p (by Lemma 7). This concludes the proof of Theorem 5.

Theorem 14. *Let H be the set of all idempotent varieties distinct from $\mathfrak{O}$. H is a subalgebra of $\langle G; \cdot, \wedge, \vee \rangle$, and on H, the product coincides with the join. H is an ideal of the lattice $\langle G; \wedge, \vee \rangle$, isomorphic to the lattice of all finite ideals of the poset $\{(p, n)\colon p$ is prime, $n \geq 1\}$, $(p, n) \leq (q, m) \Leftrightarrow p = q$ and $n | m$. The complement of H in G is an ideal of the groupoid $\langle G, \cdot \rangle$. $\langle G, \cdot \rangle$ is an ideal extension of the complement of H by the lattice H.*

8. As an application of Theorem 5, we will determine all attainable identities on $\mathfrak{O}$.

Let $U \subseteq F$, and let $\overline{U}$ be the T-ideal of F generated by U.

Definition 3 [17], [18]. A ring R is said to be U-decomposable if $\overline{U}(R) \neq R$ and U-indecomposable if $\overline{U}(R) = R$. If $\mathfrak{K}$ is a class of rings such that every $R \in \mathfrak{K}$ is U-decomposable and $\overline{U}(R)$ is U-indecomposable, U is said to be attainable on $\mathfrak{K}$.

In [18] it is shown that $\{xy = yx\}$, $\{x = x^p, \ px = 0\}$ are not attainable on $\mathfrak{D}$.

Theorem 15. *No family of identities is attainable on $\mathfrak{D}$ unless it is equivalent to $x = x$ or $x = y$.*

If $\mathfrak{U}$ is attainable on $\mathfrak{D}$, then $\mathfrak{U}$ is idempotent [9], and if U does not imply $x = y$, $\mathfrak{U} \neq \mathfrak{D}$, then $\mathfrak{U}$ is the product of a finite number of $\mathfrak{P}(p, n)$, i.e., $\mathfrak{U} = \mathfrak{U}_1 \cdot (\mathfrak{U}_2(\cdots(\mathfrak{U}_k)\cdots))$, where $\mathfrak{U}_i$ is the product of $\mathfrak{P}(p_i, n_{ij})$ for a fixed p_i, $p_1, \cdots, p_k$ are distinct primes.

$$U(Z_p[x]) = U_1(U_2(\cdots(U_k(Z_{p_1}[x]))\cdots)) = U_1(Z_{p_1}[x]),$$

since $U_2(\cdots(U_k(Z_p[x]))\cdots) \supseteq p_2 \cdots p_k Z_{p_1}[x] = Z_{p_1}[x]$, and the same is true for any ring of characteristic p_1^r. Thus

$$U(U(Z_p[x])) = U_1(U_2(\cdots(U_k(U_1(Z_p[x]))\cdots))) = U_1(U_1(Z_p[x]))$$

i.e., the proof is reduced to the case $\mathfrak{U} = \mathfrak{P}(p, n_1) \cdot (\mathfrak{P}(p, n_2)(\cdots(\mathfrak{P}(p, n_k))\cdots))$.

$U(Z_p[x])$ is principal, and hence generated by $q(x)$, where $q(x)$ is divisible by all $x - x^{p^{n_i}}$, $1 \leq i \leq k$, i.e., $\deg q(x) > 1$, i.e., $K = U(Z_p[x]) = q(x)Z_p[x]$. $U(K)$ is the ideal of K generated by the values of polynomials of U in K. $U(K) \subseteq P(p, n_1)(K) = L$. $f \in L$ iff

$$f = \sum \{[q(x)f_n(x) - (q(x)f_n(x))^N]q(x)^r g_{nr}(x) :$$
$$f_n(x), g_{nr}(x) \in Z_p[x], r \geq 0, r = 0 \Rightarrow g_{nr} = 1\}, \qquad N = p^{n_1}.$$

Thus $f = f_1 + f_2 + f_3$;

$$f_1 = a_0(q(x) - q(x)^N),$$

$$f_2 = \sum \{a_n[q(x)x^n - q(x)^N x^{nN}] : n \geq 1\},$$

$$f_3 = \sum \{b_{nrs}[q(x)^{1+r}x^{n+s} - q(x)^{N+r}x^{nN+s}], n \geq 0, r > 0, s > 0\}, \qquad a_0, a_n, b_{nrs} \in Z_p.$$

$f_2 = q(x)x^{\alpha}g(x)$, $g(x)$ is not divisible by $q(x)$, $f_3 = q(x)^{\beta}x^{\gamma}h(x)$, $h(x)$ is not divisible by $q(x)$, $\alpha > 0$, $\beta > 1$, $\gamma > 0$. If $q(x) \in L$ then

$$q(x) = a_0(q(x) - q(x)^N) + q(x)x^{\alpha}g(x) + q(x)^{\beta}x^{\gamma}h(x);$$

hence $a_0 = 1$ and $q(x)x^{\alpha}g(x) = q(x)^N - q(x)^{\beta}x^{\gamma}b(x)$ i.e., $g(x)$ is divisible by $q(x)$, i.e., $L \neq K$.

This proves that K is not U-indecomposable, and hence $\mathfrak{U}$ is not attainable on $\mathfrak{D}$ which concludes the proof of Theorem 15. The methods of this paper can be modified to get the same attainability result for commutative rings.

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