TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 193, 1974

A RELATION BETWEEN K-THEORY AND COHOMOLOGY

BY

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ABSTRACT. It is well known that for X a CW-complex, $K(X)$ and $H^{\mathrm{ev}}(X)$ are isomorphic modulo finite groups, although the "isomorphism" is not natural. The purpose of this paper is to improve this result for X a finite CW-complex.

1. Preliminaries. For the basic definitions and theory[1] of λ-rings, we refer to [2], [12]. The ring Z of integers has a λ-ring structure with $\lambda^m\colon Z \to Z$ the function

$$\lambda^m(n) = \binom{n}{m} = \frac{n(n-1)\cdots(n-m+1)}{m!}.$$

Definition. An augmented λ-ring is a λ-ring R together with λ-ring homomorphisms $i\colon Z \to R$ and $\epsilon\colon R \to Z$ such that $\epsilon i = 1$.

Since the Definition implies that $i\colon Z \to R$ is a monomorphism, we think of $Z \subset R$ as the multiples of the identity. Let $\mathfrak{B}$ denote the category of augmented λ-rings and λ-ring homomorphisms which commute with the augmentation. If $B \in \mathrm{Ob}\,\mathfrak{B}$, we write B_n^γ for the nth term of the γ-filtration on B, and $\Gamma(B)$ for the associated graded ring. We note that $\Gamma(B)_0 = Z$. Let $\mathfrak{A}$ be the category of commutative graded rings A with $A_0 = Z$. We define $\widetilde{\Lambda}\colon \mathfrak{A} \to \mathfrak{B}$ as follows: If $A \in \mathrm{Ob}\,\mathfrak{A}$, then as a set $\widetilde{\Lambda}(A) = \Pi_{n>0} A_n$. If $a \in \widetilde{\Lambda}(A)$, we denote the component in A_n by a_n and for convenience of notation define $a_0 = 1 \in A_0$ and write a as either of the formal expressions

$$1 + a_1 + a_2 + \cdots + a_n + \cdots \quad \text{or} \quad \sum_{i \geq 0} a_i.$$

If $a, b \in \widetilde{\Lambda}(A)$, we define their sum, $a \oplus b$, componentwise:

$$(a \oplus b)_n = \sum_{i+j=n} a_i b_j$$

so that '$\oplus$' is analogous with multiplication of formal power series. This operation makes $\widetilde{\Lambda}(A)$ into an Abelian group, and we denote the inverse of a by $\ominus a$. In particular if a is an element with $a_n = 0$ for $n \geq 2$, then

Received by the editors December 13, 1972 and, in revised form, July 9, 1973.

AMS (MOS) subject classications (1970). Primary 55B15; Secondary 55B40, 55F40.

(1) We are indebted to Atiyah and Tall [2] for explicit proofs, in particular those based on 'formal algebra'.

$$\ominus a = 1 - a_1 + a_1^2 - a_1^3 + \cdots + (-a_1)^n + \cdots .$$

A map $f\colon A \to B$ in $\mathfrak{A}$ induces a function $\tilde{\Lambda}(f)\colon \tilde{\Lambda}(A) \to \tilde{\Lambda}(B)$ componentwise:

$$(\tilde{\Lambda}(f)(a))_n = f(a_n).$$

Since '$\oplus$' is defined in terms of the ring structure and f preserves the ring structure, clearly $\tilde{\Lambda}(f)$ is a homorphism.

Define $\Lambda(A) = Z \oplus \tilde{\Lambda}(A)$ as a abelian group, and write $[m, a]$ for the element $m \oplus a$.

Proposition 1 (Grothendieck). (a) *There is a unique multiplication on* $\tilde{\Lambda}(A)$, *denoted by* $\otimes$, *which is associative, commutative and distributive over addition such that*

(i) *for each integer* n *there is a polynomial* $P_n(X_1, \cdots, X_n, Y_1, \cdots, Y_n)$ *with integer coefficients such that*

$$(a \otimes b)_n = P_n(a_1, \cdots, a_n, b_1, \cdots, b_n);$$

(ii) *if* $a_1, b_1 \in A_1$, *then* $(1 + a_1) \otimes (1 + b_1) = (1 + a_1 + b_1) \ominus (1 + a_1) \ominus (1 + b_1)$.

(b) *If we give* $\Lambda(A)$ *the ring structure obtained from* A *by adjoining a unit (i.e. define* $[m, a] \otimes [n, b] = [mn, mb + na + a \otimes b]$), *then* $\Lambda(A)$ *admits a unique* λ*-ring structure such that*

(i) *for each pair of integers* m, n, *there is a polynomial* $Q_{m,n}(X_1, \cdots, X_n)$ *with integer coefficients such that*

$$\lambda^m[0, a] = [0, b] \quad \text{where } b_n = Q_{m,n}(a_1, \cdots, a_n);$$

(ii) $\lambda^m[0, 1 + a_1] = (-1)^{m-1}[0, 1 + a_1]$ *if* $a_1 \in A_1$ *and* $m \geq 1$.

The existence and uniqueness depend on formal algebra.

We define a functor $D\colon \mathfrak{A} \to \mathfrak{A}$ as follows:

If $A \in \operatorname{Ob} \mathfrak{A}$ let $\times$ denote its multiplication. Define $D(A) = A$ as a graded abelian group but with multiplication (denoted by $\cdot$) defined as follows: if $x_m \in A_m$ and $x_n \in A_n$ then

$$x_m \cdot x_n = \frac{(m+n)!}{m!n!} x_m \times x_n.$$

If $f\colon A \to B$ in $\mathfrak{A}$ define $D(f) = f\colon D(A) \to D(B)$.

Let $N_n = N_n(\sigma_1, \cdots, \sigma_r)$ be the polynomial defined inductively for $n \geq r$ by $N_1(\sigma_1, \cdots, \sigma_r) = \sigma_1$ and the formula

$$N_n - \sigma_1 N_{n-1} + \sigma_2 N_{n-2} - \cdots + (-1)^n n\sigma_n = 0.$$

Define $\sigma\colon \Lambda(A) \to D(A)$ by

$$(\sigma[m, a])_0 = m, \qquad (\sigma[m, a])_n = N_n(a_1, \cdots, a_n) \quad \text{for } n \geq 1,$$

where N_n is evaluated in A, not in $D(A)$.

Proposition 2. *$\sigma\colon \Lambda(A) \to D(A)$ is a ring homomorphism.*

Proof. That σ is additive depends on certain identities satisfied by the polynomials N_n, and is omitted.

To show σ preserves multiplication, by virtue of the universality of the definition of multiplication, it suffices to examine $\sigma(x \otimes y)$ where $x = [1, 1 + a]$ and $y = [1, 1 + b]$ with $a, b \in b_1$.

In this case $x \otimes y = [1, 1 + a + b]$ so that $(\sigma(x \otimes y))_n = (a + b)^n$. But

$$(\sigma(x)\sigma(y))_n = \sum_{r+s=n} \frac{(r+s)!}{r!s!} a^r b^s = (a + b)^n.$$

So $\sigma(x \otimes y) = \sigma(x) \cdot \sigma(y)$.

2. **The main theorem.** Let $\mathfrak{W}_*$ be the category of finite based connected CW-complexes and based maps. If $X \in \mathfrak{W}_*$ then $H^{ev}(X) = \bigoplus_{n\geq 0} H^{2n}(X, \mathbf{Z})$ is a graded commutative ring and, since $H^0(X, \mathbf{Z}) = \mathbf{Z}$, belongs to $\mathfrak{A}$. Thus $H^{ev}\colon \mathfrak{W}_* \to \mathfrak{A}$ is a functor, and we define

$$\tilde{G} = \tilde{\Lambda} H^{ev}\colon \mathfrak{W}_* \to \textit{rings},$$

$$G = \Lambda H^{ev}\colon \mathfrak{W}_* \to \mathfrak{B},$$

$$H = DH^{ev}\colon \mathfrak{W}_* \to \mathfrak{A}.$$

The internal multiplication $G(X) \otimes G(X) \to G(X)$ induces an external multiplication $G(X) \otimes G(Y) \to G(X \times Y)$ in the usual way. This in turn induces a multiplication $\tilde{G}(X) \otimes \tilde{G}(Y) \to \tilde{G}(X \wedge Y)$.

If $E \to X$ is a complex vector bundle, let $c_i(E) \in H^{2i}(X, Z)$ denote its ith Chern class and define

$$\tilde{c}(E) = 1 + c_1(E) + c_2(E) + \cdots \in \tilde{G}(X),$$

$$c(E) = [\text{rank } E, \tilde{c}(E)] \in G(X).$$

Lemma 1. *If E, F are vector bundles over X and G over Y then*
(1) $\tilde{c}(E \oplus F) = \tilde{c}(E) \oplus \tilde{c}(F)$ *and* $c(E \oplus F) = c(E) \oplus c(F)$.

(2) $c(E \hat{\otimes} G) = c(E) \otimes c(G)$ *where* $E \hat{\otimes} G$ *is the exterior tensor product bundle over* $X \times Y$.

(3) $c(\lambda^i E) = \lambda^i c(E)$.

Proof. The formulae are the standard ones—see Hirzebruch [3].

Hence $\tilde{c}$ defines a ring homomorphism $\tilde{c}\colon \tilde{K}(X) \to \tilde{G}(X)$ and c defines a λ-ring homomorphism $c\colon K(X) \to G(X)$. Let $s\colon K(X) \to H(X)$ be the composite $K(X) \to^{c} G(X) \to^{\sigma} H(X)$.

The purpose of this section is to prove the following theorem. If n is a positive integer let l_n be the set of primes less than n.

Theorem 1. *If* X *is a finite CW-complex of dimension* $\leq 2n+1$, *then*

(i) $\tilde{c}\colon \tilde{K}(X) \to \tilde{G}(X)$ *and* $c\colon K(X) \to G(X)$ *are isomorphisms modulo* l_n*-torsion.*

(ii) $s\colon K(X) \to H(X)$ *is an isomorphism modulo* l_{n+1}*-torsion.*

The proof of Theorem 1 is an easy consequence of the following mod $\mathcal{C}$ version of a well-known theorem on half-exact functors.

Proposition 4. *Let* $\mathcal{C}$ *be a Serre class of abelian groups, and let* $\rho\colon t_1 \to t_2$ *be a map of half-exact functors, where* $t_i\colon \mathfrak{W}_* \to$ *Abelian groups, such that* $\rho\colon t_1(S^n) \to t_2(S^n)$ *is an isomorphism* mod $\mathcal{C}$ *for* $n \leq m$. *Then* $\rho\colon t_1(X) \to t_2(X)$ *is an isomorphism* mod $\mathcal{C}$ *when* X *is finite and* $\dim X \leq m$.

Thus in order to prove the theorem, we examine the maps on spheres.

Lemma 2. (i) $\tilde{K}(S^{2n+1}) \to^{\tilde{c}} \tilde{G}(S^{2n+1})$ *is an isomorphism.*

(ii) $\tilde{K}(S^{2n}) \to^{\tilde{c}} \tilde{G}(S^{2n})$ *is a monomorphism with cokernel* $\mathbf{Z}_{(n-1)!}$.

(iii) $G(S^{2n}) \to^{\sigma} H(S^{2n})$ *is a monomorphism with cokernel* $\mathbf{Z}_n$.

Proof. (i) Trivial, since both groups are zero.

(ii) The map $K(S^{2n}) \to G(S^{2n}) \cong H^{2n}(S^{2n})$ is given by the nth Chern class, and by theorems of Borel-Hirzebruch the nth Chern class of a complex vector bundle on S^{2n} is a multiple of $(n-1)!$ times the generator, and every such multiple arises.

(iii) The map $G(S^{2n}) \to^{\sigma} H(S^{2n}) \cong H^{2n}(S^{2n})$ is easily seen by calculation to be multiplication by n.

Let $\mathcal{C}_n$ be the class of abelian groups whose order is a product of primes in l_n. By Lemma 2

$$\tilde{c}\colon \tilde{K}(S^m) \to \tilde{G}(S^m) \quad \text{is an isomorphism mod } \mathcal{C}_n \text{ for } m \leq 2n+1,$$

$$s\colon K(S^m) \to H(S^m) \quad \text{is an isomorphism mod } \mathcal{C}_{n+1} \text{ for } m \leq 2n+1,$$

whence the theorem follows.

3. Finite CW-complexes of dimension ≤ 5. In the case $n = 2$, Theorem 1 says that if $\dim X \leq 5$, $c\colon K(X) \to G(X)$ is an isomorphism of λ-rings, i.e. $K(X) \cong \Lambda H(X)$ [in particular, if $\dim X \leq 4$, we see that $K^1(X) \cong H^1(X) \oplus H^3(X)$], so that the graded ring structure of $H^{ev}(X)$ determines the λ-ring structure of $K(X)$. In this section we show that in these low dimensions the converse is true, namely $H^{ev}(X) \cong \Gamma K(X)$. Since we already know that $K(X) \cong G(X)$ as λ-rings, it suffices to show that $\Gamma G(X) \cong H^{ev}(X)$.

Let $\alpha\colon \tilde{G}(X) \to H^2(X)$ be the projection $\alpha(1 + a_1 + a_2) = a_1$ and let β: $H^4(X) \to \tilde{G}(X)$ be the inclusion $\beta(a_2) = 1 + (-a_2)$. The sequence $0 \to H^4(X) \to^{\beta} \tilde{G}(X) \to^{\alpha} H^2(X) \to 0$ is clearly exact.

Lemma 3. (i) $\operatorname{Im} \beta = G(X)^{\gamma}_2$. (ii) $G(X)^{\gamma}_n = 0$ *for* $n > 2$. (iii) *The product in* $\Gamma G(X)$ *from* $(G(X)^{\gamma}_1/G(X)^{\gamma}_2) \times G(X)^{\gamma}_1/G(X)^{\gamma}_2 \to G(X)_2$ *is (isomorphic to) the cup product* $H^2(X) \times H^2(X) \to H^4(X)$.

Proof. By formal algebra it follows that

(1) $$\gamma^n[0, a] = [0, 1 + (-1)^{n-1}(n-1)!\,a_n + \cdots],$$

(2) $$[0, 1 + a_n + \text{higher terms}] \otimes [0, 1 + b_n + \text{higher terms}] = [0, 1 + c_{m+n} + \text{higher terms}].$$

From (1), (2), it follows that $G(X)^{\gamma}_n = 0$ for $n > 2$ and since $\gamma^2[0, 1 + a_2] = [0, 1 - a_2]$, we see that $\operatorname{Im} \beta \subset G(X)^{\gamma}_2$. But $G(X)^{\gamma}_2$ in this case is generated by elements of the form $\gamma^2[0, 1 + a_1 + a_2]$ or $[0, 1 + a_1][0, 1 + b_1]$, i.e. by elements of the form $[0, 1 + a_2]$, whence $\operatorname{Im} \beta = G(X)^{\gamma}_2$. Finally a simple calculation shows

$$[0, 1 + a_1 + a_2] \otimes [0, 1 + b_1 + b_2] = [0, 1 + (-a_1 b_1)]$$

which completes the proof of the lemma.

4. A real version. We would like to prove a theorem analogous to Theorem 1 for KO-theory using the corresponding characteristic classes. Since Stiefel-Whitney classes do not carry enough information even on spheres, we try Pontryagin classes. However, there is a technical difficulty to overcome, namely that Pontryagin classes do not obey a Whitney-sum formula. However, if E, F are real vector bundles over X and if $p_i(E) \in H^{4i}(X, \mathbf{Z})$ denotes the ith Pontryagin class then $p_n(E \oplus F) - \sum_{i+j=n} p_i(E)p_j(F)$ is an element of order 2 in $H^{4n}(X, Z)$. Let $H^{4*}(X) = \bigoplus_{n \geq 0} H^{4n}(X, \mathbf{Z})$. $H^{4*}\colon \mathfrak{W}_* \to$ *graded rings*, so we can define

$$(GO)^{\sim}(X) = \tilde{\Lambda} H^{4*}(X), \qquad GO(X) = \Lambda H^{4*}(X), \qquad H^{4^*}(X) = DH^{4^*}(X).$$

(Alternatively, we can define $(GO)^{\sim}(X)$ as the subring of $\tilde{G}(X)$ consisting of elements with zero components in odd degrees:

$$(GO)^{\sim}(X) = \{a \in \tilde{G}(X) : a_n = 0 \text{ if } n \text{ is odd}\}.$$

Let q: $(KO)^{\sim}(X) \to \tilde{G}(X)$ be the composite

$$q : KO(X) \xrightarrow{\text{complexification}} K(X) \xrightarrow{c} G(X) \xrightarrow{\psi^2} G(X)$$

where ψ^2 is the Adams operation.

Clearly q is a ring homorphism.

Lemma 4. *If $F \to X$ is a real vector bundle, then*

$$(q(E))_{2n+1} = 0, \qquad (q(E))_{2n} = (-4)^n p_n(E).$$

Proof. If F is the complexification of E, with Chern classes $c_1, \cdots, c_n$, then $p_i(E) = (-1)^i c_{2i}$. If F were a sum of line bundles $F = L_1 \oplus \cdots \oplus L_n$ with $c_1(L_j) = \alpha_j$ then

$$\begin{aligned}\psi^2 c(F) = c\psi^2(F) &= c(L_1^2 \oplus \cdots \oplus L_n^2)\\ &= [1, 1 + 2\alpha] \oplus [1, 1 + 2\alpha_2] \oplus \cdots \oplus [1, 1 + 2\alpha_n]\\ &= [n, 1 + 2c_1 + 2^2 c_2 + 2^3 c_3 + \cdots]\end{aligned}$$

so by the splitting principle for complex vector bundles, we see that $(q(E))_i = (\psi^2 c(F))_n = 2^n c_n(F)$.

If i is odd, then $2c_i(F) = 0$, so $(q(E))_i = 0$. If $i = 2n$, then $(q(E))_{2n} = 2^{2n} c_{2n}(E) = 2^{2n}(-1)^n p_n(E)$.

Thus the image of q is contained in $GO(X)$. By naturality it induces a map $\tilde{q} : (KO)^{\sim}(X) \to (GO)^{\sim}(X)$.

Theorem 2. *If X is a finite CW-complex of dimension $\leq 4n + 3$, then*

(i) $\tilde{q}$: $(KO)^{\sim}(X) \to (GO)^{\sim}(X)$ *is an isomorphism modulo* $(l_{2n} \cup \{2\})$*-torsion.*

(ii) q: $KO(X) \to GO(X)$ *is an isomorphism modulo* $(l_{2n+1} \cup \{2\})$*-torsion.*

(iii) σq: $KO(X) \to DH^{4*}(X)$ *is an isomorphism modulo* $(l_{2n+1} \cup \{2\})$*-torsion.*

Proof. As before it suffices to examine the maps on spheres. On S^t, where $t \not\equiv 0 \bmod 4$, $(KO)^{\sim}(S^t)$ is either $\mathbf{Z}_2$ or 0, but $(GO)^{\sim}(S^t)$ and $H^{4*}(S^t)$ are zero. The map from $(KO)^{\sim}(S^{4n}) \to (GO)^{\sim}(S^{4n}) \cong H^{4n}(S^{4n}, \mathbf{Z})$ is 4^n times the nth Pontryagin class and again by theorems of Borel-Hirzebruch the nth Pontryagin class is a multiple of $(2n-1)!\, GCD(n+1, 2)$, and moreover every such multiple arises. Thus this map from $\mathbf{Z}$ to $\mathbf{Z}$ is multiplication by some power of 2 times $(2n-1)!$

whose cokernel is thus a $(l_{2n} \cup \{2\})$-torsion group. The result now follows.

We briefly state two corollaries of Theorems 1 and 2.

Corollary 1. (1) *If* X *is a finite CW-complex of dimension* $\leq 2n+1$ *and* $H^{ev}(X)$ *has no* l_{n+1}*-torsion then*

$$K(X) \cong H^{ev}(X) \quad \textit{as abelian groups.}$$

(2) *If* X *is a finite CW-complex of dimension* $\leq 4n+3$ *and* $H^{4*}(X)$ *has no* l_{2n+1} *torsion then*

$$KO(X) \cong H^{4^*}(X) \quad \textit{modulo 2 torsion, as abelian groups.}$$

Proof. (1) $K(X)$ and $H^{ev}(X)$ have the same ranks and the same p-torsion for $p \notin l_n$. From [6] by a simple spectral sequence argument we see that if $H^{ev}(X)$ has no p-torsion then $K(X)$ has no p-torsion.

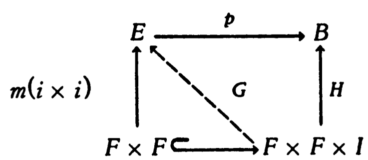
(2) Proof similar.

Corollary 2. *If* X *is a finite CW-complex of* dim $\leq 2n$ *and* $H^{2k}(X)$ *has no* l_k*-torsion then* c: $K(X) \to G(X)$ *is an isomorphism on non-*l_n*-torsion and a monomorphism on* l_n*-torsion.*

Proof. By Corollary 1, it suffices to show that $\tilde{c}$ is a monomorphism. Suppose $\tilde{c}(x) = 0$. Then since dim $X \leq 2n$, we can represent x by $E - n$ for some n-dimensional complex vector bundle E. Then $\tilde{c}(E) = 0$, so by a theorem of Peterson [7] E is trivial whence $\tilde{c}$ is a monomorphism.

5. **Bott periodicity and an exact sequence.** By Brown's theorem, $\tilde{K}$ and $\tilde{G}$ are representable functors. Let $\tilde{K}(\ ) = [\ , E]$ and $\tilde{G} = [\ , B]$ where E, B are H-spaces with multiplication m. Then $\tilde{c}$: $\tilde{K} \to \tilde{G}$ defines a map p: $E \to B$ which we can assume without loss of generality to be a fibration, and which is an H-map, since $\tilde{c}$ is a homomorphism.

Let i: $F \to E$ be the fibre of p: $E \to B$. The composite $F \times F \to^{i \times i} E \times E \to^m E \to^p B$ is homotopic to the composite $F \times F \to_{i \times i} E \times E \to_{p \times p} B \times B \to_m B$ which is the constant map. Let H: $* \simeq pm(i \times i)$.



There exists a map G: $F \times F \times I \to E$ making the diagram commute. Let $m = G_1$: $F \times F \to F$. Thus m defines an H-space structure on F such that i: $F \to E$ is an H-map.

Define $\tilde{U} = [\ , F]$: $\mathfrak{W}_* \to$ *Abelian groups.* The Puppe sequence

$$\cdots \to \Omega^n F \xrightarrow{\Omega^n i} \Omega^n E \xrightarrow{\Omega^n p} \Omega^n B \to \cdots \to \Omega B \to F \xrightarrow{i} E \xrightarrow{p} B$$

induces a long exact sequence

$$\cdots \to [X, \Omega^n F] \to [X, \Omega^n E] \to [X, \Omega^n B] \to \cdots$$
$$\to [X, \Omega B] \to [X, F] \to [X, E] \to [X, B]$$

which can be rewritten as

$$\text{(S)} \qquad \begin{aligned} \cdots \to \widetilde{U}(\Sigma^n X) \to \widetilde{K}(\Sigma^n X) \xrightarrow{\widetilde{c}(\Sigma^n X)} \widetilde{G}(\Sigma^n X) \to \cdots \\ \to \widetilde{U}(X) \to \widetilde{K}(X) \xrightarrow{\widetilde{c}(X)} \widetilde{G}(X). \end{aligned}$$

A simple calculation when $X = S^k$ gives the following lemma.

Lemma 5. (i) $\widetilde{U}(S^{2n}) = 0$.

(ii) $\widetilde{U}(S^{2n+1}) = Z_{n!}$.

(iii) *If* X *is a finite CW-complex of dimension* $\leq 2n+1$ *with* ν_r *r-cells then* $|\widetilde{U}(X)|$ *divides* $\Pi_{r<n} (r!)^{\nu_{2r+1}}$.

The purpose of this section is to obtain an exact sequence

$$\text{(E)} \qquad \begin{aligned} \widetilde{U}(\Sigma^2 X) \to K(X) \xrightarrow{s} H(X) \to \widetilde{U}(\Sigma X) \to K^1(X) \\ \xrightarrow{t} H^{\text{odd}}(X) \to \widetilde{U}(X) \to K(X) \xrightarrow{c} G(X) \end{aligned}$$

where t is defined as follows: If $E \to SX$ is an n-dimensional vector bundle, then it is determined by a map $X \to^f U(n)$. The cohomology of $U(n)$ is an exterior algebra on generators $x_i \in H^{2i-1}(U(n))$ where $1 \leq i \leq n$. Define $t(E) = (f^*x_1, f^*x_2, \cdots, f^*x_n)$.

First we observe that since cup products vanish on suspensions the natural bijection $\widetilde{H}^{\text{ev}}(\Sigma X) \to \widetilde{G}(\Sigma X)$ is an isomorphism of abelian groups, so that $\widetilde{G}(\Sigma X) \to H^{\text{odd}}(X)$, and $\widetilde{G}(\Sigma^2 X) \cong \widetilde{H}^{\text{ev}}(X)$, so we can insert these groups into the sequence (S). That the map

$$t: K^1(X) = \widetilde{K}(\Sigma X) \xrightarrow{\widetilde{c}} \widetilde{G}(\Sigma X) \cong H^{\text{odd}}(X)$$

is given by the above construction we leave to the reader. Essentially it remains to prove the following lemma.

Lemma 6. *The following diagram commutes*:

$$\begin{array}{ccccc} & & \tilde K(X) & & \\ & \swarrow^{\beta} & & \nwarrow^{*} & \\ \tilde K(\Sigma^2 X) & & & & \tilde K(X) \\ \downarrow{\tilde c} & & & & \uparrow{s} \\ \tilde G(\Sigma^2 X) & & \xrightarrow[\cong]{\qquad\qquad} & & \tilde H(X) \end{array}$$

where β is Bott periodicity and $*$: $\tilde K(X) \to \tilde K(X)$ *is conjugation.*

Proof. Let $L \to X$ be a line bundle with $c_1(L) = l \in H^2(X)$ and let $H \to S^2$ be the Hopf bundle with $c_1(H) = h$. From the commutative diagram

$$\begin{array}{ccc} \tilde K(X) \otimes \tilde K(S^2) & \xrightarrow[\tilde c \otimes \tilde c]{} & \tilde G(X) \otimes \tilde G(S^2) \\ \downarrow & & \downarrow \\ \tilde K(\Sigma^2 X) & \xrightarrow[\tilde c]{} & \tilde G(\Sigma^2 X) \end{array}$$

we see that

$$\begin{aligned} \tilde c\beta(L-1) &= \tilde c(L-1) \otimes \tilde c(H-1) \\ &= (1+l) \otimes (1+h) \\ &= (1+\hat h+\hat l) \ominus (1+\hat h) \ominus (1+\hat l) \end{aligned}$$

where $\hat h$, $\hat l$ are the images of h, l in $H^2(S^2 \times X)$. That is, $\tilde c\beta(L-1) = (1+\hat h+\hat l) \oplus (1-\hat h) \oplus (1-\hat l+\hat l^2-\hat l^3+\cdots)$ since $\hat h^n = 0$ for $n > 1$, i.e. $\tilde c\beta(L-1) = (1+\hat l-\hat h\hat l) \oplus (1-\hat l+\hat l^2-\cdots)$. The component in dimension n, that is in $H^{2n}(S^2 \times X)$, is

$$(-1)^{n-1}\hat l^{n-1}\hat l - (\hat h\hat l)(-1)^{n-2}\hat l^{n-2}.$$

The term $(-1)^{n-1}\hat l^n$ lies in $H^{2n}(S^2 v X)$ and so contributes nothing in $H^{2n}(\Sigma^2 X)$ and the term $(-1)^{n-1}\hat h\hat l^{n-1}$ on desuspending maps to $(-1)^{n-1} l^{n-1} \in H^{2n-2}(X)$, so that

$$\tilde c\beta(L-1) = (-l, l^2, -l^3, \ldots) = s(L-1)$$

and so by the splitting principle and the universal definition of s, we see that $\tilde c\beta * = s$.

The exact sequence (E) is now obtained from (S) by replacing $\tilde K(\Sigma^2 X) \to^{\tilde c} \tilde G(\Sigma^2 X)$ by $K(X) \to^{s} H^{ev}(X)$ which is isomorphic to $\tilde K(\Sigma^2 X) \oplus Z \to^{\tilde c \oplus 1} \tilde G(\Sigma^2 X) \oplus Z$ and so preserves exactness.

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