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CONDITIONS FOR THE ABSOLUTE CONTINUITY OF TWO DIFFUSIONS

BY

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ABSTRACT. Consider two diffusion processes on the line. For each starting point x and each finite time t, consider the measures these processes induce in the space of continuous functions on $[0, t]$. Necessary and sufficient conditions on the generators are found for the induced measures to be mutually absolutely continuous for each x and t. If the first process is Brownian motion, the second one must be Brownian motion with drift $b(x)$, where $b(x)$ is locally in L_2 and satisfies a certain growth condition at $\pm\infty$.

0. Introduction. Our concern is with diffusion processes on an open one-dimensional interval I, having homogeneous transition probabilities, and possessing no singular points. We do not allow curtailment of life time (killing), and the end points of I must be inaccessible. This class of diffusions will be denoted by $\mathfrak{D}$, or $\mathfrak{D}(I)$ if the dependence on I needs to be indicated. A standard way of realizing such a diffusion is via coordinate space: C is to be the class of all continuous functions from $[0, \infty)$ into I, and for $\omega \in C$ let $X_t(\omega) = \omega(t)$. Let $\mathcal{C}_t$ be the σ-field generated by $\{X_s: s \leq t\}$, and $\mathcal{C}$ the least σ-field including all the $\mathcal{C}_t$, $0 \leq t < \infty$. A diffusion in $\mathfrak{D}$ is then given by a collection $P = (P_x)$, $x \in I$, of probability measures on $(C, \mathcal{C})$; (for details see [7, p. 84], or [4, p. 102]). We let $P_x|_t$ be the restriction of P_x to $\mathcal{C}_t$. Given two diffusions P^1 and P^2 in $\mathfrak{D}$, $P^1 \prec P^2$ is to mean that $P^1_x|_t \ll P^2_x|_t$ for each $x \in I$, $0 \leq t < \infty$, where $\ll$ means "is absolutely continuous with respect to". Now each $P \in \mathfrak{D}$ is determined by a scale function p and a speed measure m; we write $P \sim (p, m)$. In §2 we give necessary and sufficient conditions for $P^1 \prec P^2$ in terms of the associated scales and speed measures. The special case $I = (-\infty, \infty)$ and P^2 Wiener measure is discussed in §1. It turns out that in this case P^1 must correspond to Brownian motion with a suitable drift: the condition on the drift coefficient $b(x)$ is that it is locally square integrable and satisfies a certain growth condition at $\pm\infty$; the growth condition is simply the one dictated by the inaccessibility of the end points. It is also shown that the conditions on $b(x)$ are necessary and sufficient conditions

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for the process $\exp[\int_0^t b(X_u)dX_u - \frac{1}{2}\int_0^t b^2(X_u)du]$ to be a martingale under P^2, i.e. Wiener measure.

We proceed to some notational points and details. In place of the speed measure we will usually deal with the associated distribution function: $m(x) = m((-\infty, x])$. If $P \sim (p, m)$ then also $P \sim (p^*, m^*)$, where $p^*(x) = ap(x) + b$, $m^*(x) = a^{-1}m(x) + c$, where a is a positive number, b and c arbitrary; but, except for this trivial kind of nonuniqueness, (p, m) is determined by P. Conversely (p, m) determines P. We recall that any continuous strictly increasing function $p(x)$ can serve as a scale, while the speed measure is a positive measure, finite on compact sets, strictly positive on open sets. The assumption that the end points are inaccessible imposes an additional condition. For the case $I = (-\infty, \infty)$ this is

$$(0.1)\quad \int_c^\infty (p(\infty) - p(y))m(dy) = \int_{-\infty}^c (p(y) - p(-\infty))m(dy) = \infty, \qquad -\infty < c < \infty,$$

where $p(\infty)$ and $p(-\infty)$ denote the obvious limits. The condition is due to Feller [3]; in his terminology $+\infty$ and $-\infty$ are not *exit boundaries*. For details consult [1] or [7].

On the space C shift operators θ_t are defined in the natural way: $(\theta_t\omega)(s) = \omega(t+s)$. If X is a diffusion in $\mathfrak{D}_I$, B a Borel subset of I, we write $\mu(t, B; X) = \int_0^t \chi_B(X_s)ds$. Thus $\mu(t, B; X)$ is the sojourn time of X in B up to time t. For fixed t this is a measure on the Borel sets of I. It is known to be absolutely continuous with respect to the speed measure m of X, and there exists a nice version of the Radon-Nikodym derivative, known as the *local time*: thus $\mu(t, B; X) = \int_B L(t, x; X)m(dx)$. Here for fixed t and x, $L(t, x; X)$ is a random variable; and for fixed ω, $(t, x) \to L(t, x; X)$ is, with probability one, continuous. Further details and references about this and other matters needed in the body of the paper are collected in the appendix.

Whenever dealing with P, possibly with affixes, we will use E with the same affixes to denote the expectation operator corresponding to the probability measure denoted by P.

1. Brownian motion with drift. Throughout this section $\mathfrak{D} = \mathfrak{D}_{(-\infty,\infty)}$, and $P^0 = (P_x^0)$, $-\infty < x < \infty$, is the element of $\mathfrak{D}$ corresponding to the Wiener distribution; so the coordinate process (X_t) is Brownian motion under P^0. Somewhat more generally, if $Z = (Z_t)$, $0 \le t < \infty$, is a real-valued stochastic process on C, and $(P_x) \in \mathfrak{D}$, and (a) Z_t is $\mathcal{C}_t$-measurable for each t, (b) the finite-dimensional distributions of the Z process under P_x agree with the finite-dimensional distributions of the coordinate process X under P_x^0 for each x, (c) $E[Z_t|\mathcal{C}_s] = Z_s$

P_x-a.s. for $0 \le s < t$, $-\infty < x < \infty$, then Z will be said to be a *Brownian motion* with respect to (P_x).

Suppose $P' \in \mathfrak{D}$, $P' \prec P^0$. For each x and t let $L_t^{(x)}$ be the Radon-Nikodym derivative of $P'_x|_t$ with respect to $P^0_x|_t$. Then, as discussed in III of the appendix, $(L_t^{(x)}, \mathcal{C}_t, t \ge 0)$ is a martingale under P^0_x, and we can choose a right continuous version. Note that $L_0^{(x)} = 1$ P^0_x-a.s. by the zero-one law. $L_t^{(x)}$ is Borel measurable in x; this can be seen by expressing the Radon-Nikodym derivative explicitly as a limit of difference quotients. We may then define: $L_t = L_t^{(X_0)}$. Thus

$$dP'_x|_t/dP^0_x|_t = L_t, \qquad 0 \le t < \infty, \ -\infty < x < \infty. \tag{1.1}$$

By L^1_{loc} (L^2_{loc}) we mean the class of Borel measurable functions $b(x)$ defined on $(-\infty, \infty)$ which are integrable (square integrable) over compact intervals. We will use the notation

$$L_t[b] = \exp\left\{\int_0^t b(X_u)\, dX_u - \frac{1}{2}\int_0^t b^2(X_u)\, du\right\}, \qquad 0 \le t < \infty, \ b \in L^2_{\text{loc}}; \tag{1.2}$$

here X will be coordinate process under Wiener measure. For the existence of the integrals see Appendix (I.C). We also will use

$$Y_t[b] = X_t - \int_0^t b(X_u)\, du, \qquad 0 \le t < \infty, \ b \in L^1_{\text{loc}}. \tag{1.3}$$

We write $L[b]$ or $Y[b]$ for the process $(L_t[b], \mathcal{C}_t, t \ge 0)$, respectively $(Y_t[b], \mathcal{C}_t, t \ge 0)$.

Our first proposition is a Markov process variant of a result of Kailath and Zakai [10]; parts of the argument trace back to Hitsuda [6].

Proposition 1. *Let* $P' \in \mathfrak{D}$, $P' \prec P^0$. *Then there exists* $b \in L^2_{\text{loc}}$ *such that*

$$dP'_x|_t/dP^0_x|_t = L_t[b], \qquad 0 \le t < \infty, \ -\infty < x < \infty. \tag{1.4}$$

It follows that

$$E_x L_t[b] = 1, \qquad 0 \le t < \infty, \ -\infty < x < \infty. \tag{1.5}$$

(1.6) $\quad Y[b]$ *is a Brownian motion under* P'.

Remark 1. For any $P' \in \mathfrak{D}$ condition (1.6) can be satisfied for at most one function b, where we identify two functions which are equal a.e. For otherwise, one would obtain the difference of two Brownian motions, i.e. two continuous martingales, represented as a function of bounded variation, which is impossible except in the trivial case where the function of bounded variation vanishes identically.

Proof. Obtain a right continuous martingale (L_t) satisfying (1.1) as above. It must be shown that (L_t) is a multiplicative functional of Brownian motion. Let H be a bounded $\mathcal{C}_t$-measurable random variable. One obtains easily (this is an instance of (3.1)) that

$$(1.7) \qquad E'_x[H \circ \theta_s | \mathcal{C}_s] = E_x[(H \circ \theta_s \cdot L_{t+s}/L_s) | \mathcal{C}_s], \quad P'_x\text{-a.s.}$$

Also, using the Markov property of P^0,

$$(1.8) \qquad E'_{X_s}[H] = E^0_{X_s}[HL_t] = E_x[H \circ \theta_s \cdot L_t \circ \theta_s | \mathcal{C}_s], \quad P^0_x\text{-a.s.}$$

By the Markov property of P', the first members of (1.7) and (1.8) agree P'_x-a.s. Since $P' \prec P^0$ we can conclude that the last members of (1.7) and (1.8) agree P'_x-a.s.; and the exceptional set Λ on which agreement fails belongs to $\mathcal{C}_s$, $P'_x[\Lambda] = 0$. Throughout this discussion x is arbitrary but fixed. Keeping (1.1) in mind, we may infer that $P^0_x[\Lambda \cap [L_s > 0]] = 0$. Since every $\mathcal{C}_{s+t}$-measurable random variable is of the form $H \circ \theta_s$ for some $\mathcal{C}_t$-measurable H, it follows that $L_{t+s} = L_s \cdot L_t \circ \theta_s$ P^0_x-a.s., for, by what has been said already, the equality holds P^0_x-a.s. on the set $[L_s > 0]$; and, as already remarked, if $L_s = 0$, then P^0_x-a.s. also $L_{t+s} = 0$. So (L_t) is a multiplicative functional of Brownian motion. As already noted, $L_0 = 1$, and (L_t) is a martingale with respect to the σ-fields $\mathcal{C}_t$ generated by our Brownian motion (coordinate process). It follows that one has a representation $L_t - 1 = \int_0^t H_u \, dX_u$ where H_u is $\mathcal{C}_u$ measurable and $\int_0^t H_u^2 \, du < \infty$ P^0_x-a.s. Indeed, if L_t is square integrable, such a representation is known to hold with $E[\int_0^t H_u^2 \, du] < \infty$ (see Kunita-Watanabe [11] or Meyer [12]) and, as pointed out by Hitsuda [6], an easy argument using stopping times gives the result needed here. In particular, then, L_t is continuous. Let $A_t = -\log L_t$. This gives rise to an additive functional, with $A_0 = 0$ P_x-a.s. The values of A_t lie in $(-\infty, \infty]$, but A_t is continuous in the topology of the extended line. It follows (see Appendix II) that A must actually be finite valued. That is, $L_t > 0$ and we may apply Ito's formula to obtain

$$\log L_t = \int_0^t \frac{1}{L_s} \, dL_s - \frac{1}{2} \int_0^t \frac{H_s^2}{L_s^2} \, ds.$$

The first term on the right is a continuous local martingale; it is also an additive functional, because (L_t) is a multiplicative functional. We now apply Tanaka's representation theorem to this term (see Appendix II), obtaining

$$(1.9)\qquad \int_0^t \frac{1}{L_s}\,dL_s = \int_0^t k(X_s)\,dX_s + g(X_t) - g(X_0)$$

with $k \in L^2_{\mathrm{loc}}$, g a continuous function. According to Tanaka, if J is any compact interval, $\tau = \inf\{t: X_t \notin J\}$, each of the terms of (1.9) when evaluated at $t \wedge \tau$ has finite moments of all orders. Therefore, the two stochastic integrals evaluated at $t \wedge \tau$ define martingales; then $g(X_{t\wedge\tau})$ is a martingale. As a consequence (see Dynkin [2, Theorem 13.10]) g is harmonic, i.e. $g(x) = ax + c$. Obviously we may set $c = 0$, and letting $b(x) = k(x) + a$ gives $\int_0^t (1/L_s)dL_s = \int_0^t b(X_s)dX_s$. Let M_t denote the first term of (1.9). The continuous local martingale (M_t) has an associated continuous increasing process $\langle M, M\rangle$ (notation as in [11] or [12]) satisfying

$$\langle M, M\rangle_t = \int_0^t \frac{H_s^2}{L_s^2}\,ds = \int_0^t b^2(X_s)\,ds$$

and (1.4) is established. Thus $L_t = L_t[b]$ P_x^0-a.s., and (1.5) follows immediately. Finally (1.6) is an instance of Girsanov's theorem (see Appendix III).

For a diffusion belonging to $\mathfrak{D}$ with differential generator $\frac{1}{2}(d^2/dx^2) + d\,b(x)/dx$ with b bounded and continuous, one checks easily that the scale and speed are given by

$$p_b(x) = \int_0^x \exp\left\{-2\int_0^y b(z)\,dz\right\}dy, \qquad m_b(x) = 2\int_0^x \exp\left\{2\int_0^y b(z)\,dz\right\}dy.$$

These expressions make sense whenever $b \in L^1_{\mathrm{loc}}$. For (p_b, m_b) to correspond to some diffusion in $\mathfrak{D}$ one needs, in addition, the inaccessibility condition (0.1), which now takes the form

$$(1.10)\quad \int_c^\infty \left(\frac{1}{\beta(y)}\int_y^\infty \beta(u)\,du\right)dy = \int_{-\infty}^c \left(\frac{1}{\beta(y)}\int_{-\infty}^y \beta(u)\,du\right)dy = \infty, \qquad -\infty < c < \infty,$$

where $\beta(y) = \exp\{-2\int_0^y b(z)dz\}$.

Proposition 2. *Let $P' \in \mathfrak{D}$, $P' \sim (p_b, m_b)$, where $b \in L^1_{\mathrm{loc}}$. Then* (1.6) *holds.*

Proof. If b is bounded and Lipschitz continuous this is known. Indeed P' is then determined by its differential generator $d^2/dx^2 + db(x)/dx$. On the other hand, a diffusion with this differential generator can be obtained as a solution of the Ito stochastic integral equation $Z_t = X_t + \int_0^t b(Z_u)du$, P_x^0-a.s. and (1.6) follows.

In the general case choose a sequence b_n of bounded Lipschitz continuous functions converging to b in the L^1-sense on compact intervals. Write b_∞ for b, and p_n, m_n for p_{b_n}, m_{b_n} respectively, $n = 1, 2, \cdots, \infty$. Now, under P^0, (X_t) is

Brownian motion, and diffusions $Z^{(n)}$ with scale p_n and speed m_n can be realized (see Appendix I) as

$$Z_t^{(n)} = p_n^{\vee}(X_{A_t^{(n)}}), \qquad n = 1, 2, \cdots, \infty,$$

where $p_n^{\vee}$ is the inverse function of p_n, and $A_t^{(n)}$, as a function of t, is the inverse of $\tau_t^{(n)}$, where $\tau_t^{(n)} = \int_{-\infty}^{\infty} L(t, x; X) m_n(dx)$. Writing $Y_t^{(n)} = Z_t^{(n)} - \int_0^t b_n(Z_s^{(n)})ds$, we know $(Y_t^{(n)})$ is Brownian motion under P^0 for $n = 1, 2, \cdots$. We wish to prove the same assertion for $n = \infty$ by a limiting argument. Indeed $Y_t^{(n)}$ approaches $Y_t^{(\infty)}$ in a very strong sense: With probability one convergence holds for all t, uniformly for t in any compact interval. To see this recall some properties of $L(t, x; X)$: It is continuous in (t, x), nondecreasing in t, and, for fixed t, vanishes outside some finite x-interval. One then verifies easily that a.s. $A_t^{(n)}$ converges to $A_t^{(\infty)}$ for all t, uniformly for t in any compact set. Also $p_n^{\vee}(x)$ converges to $p_\infty^{\vee}(x)$ uniformly for x in any compact subset of $(p(-\infty), p(\infty))$. Therefore a.s. $Z_t^{(n)}$ converges to $Z_t^{(\infty)}$, uniformly for t in any compact interval. One also obtains (see Appendix (I.C)) that a.s.

$$\int_0^t b_n(Z_s^{(n)})\, ds = \int_{-\infty}^{\infty} L(A_t^{(n)}, p_n^{\vee}(x); X) b_n(x) m_n(dx)$$

$$\to \int_{-\infty}^{\infty} L(A_t, p^{\vee}(x); X) b(x) m(dx) = \int_0^t b(Z_s)\, ds,$$

the convergence being uniform for t in any compact interval.

Proposition 3. *Let* $P' \in \mathfrak{D}$, $P' \sim (p_b, m_b)$ *for some* $b \in L^2_{\text{loc}}$. *Then* $P' \prec P^0$ *and* (1.4) *holds.*

Proof. By Proposition 2, (1.6) holds. So under P', X_t differs from the Brownian motion $Y_t[b]$ by $\int_0^t b(X_u)du$. So $P' \prec P^0$ follows as soon as $\int_0^t b^2(X_u)du < \infty$ P'_x-a.s. is established (see Appendix III, Corollary). The fact that the integral is finite becomes obvious on writing

$$\int_0^t b^2(X_u)\, du = \int_{-\infty}^{\infty} L(t, x; X) b^2(x) m_b(dx)$$

(see Appendix (I.C)), remembering the nature of $L(t, x; X)$, $m_b(dx)$, and that $b \in L^2_{\text{loc}}$. So $P' \prec P^0$ is established. Finally (1.4) follows from applying Proposition 1; the fact that the b supplied by that proposition agrees with the one we started out with here is an immediate consequence of the uniqueness assertion contained in Remark 1.

Proposition 4. *Let* $P' \in \mathfrak{D}$, $P' \prec P^0$. *Then there exists a* $b \in L^2_{\text{loc}}$ *with* $P' \sim (p_b, m_b)$.

Proof. Proposition 1 applies and supplies a unique $b \in L^2_{\text{loc}}$. We wish to conclude that b satisfies the inaccessibility condition (1.10). Let $\tau_n = \inf\{t: |X_t| > n\}$ and let $b_n(x) = b(x)$ for $|x| \le n+1$, $b_n(x) = 0$ for $|x| > n+1$. Then $b_n \in L^2_{\text{loc}}$ and satisfies (1.10). So there exists $P^{(n)} \in \mathfrak{D}$, $P^{(n)} \sim (p_{b_n}, m_{b_n})$. Now $P^{(n)} \prec P^{(0)}$ and, using Proposition 1, we find

$$dP_x^{(n)}|_{\tau_n}/dP_x^0|_{\tau_n} = dP'_x|_{\tau_n}/dP_x^0|_{\tau_n} = L_{\tau_n}[b].$$

So $P' \sim (p, m)$ with p and m agreeing with p_n and m_n, respectively, on $[-n, n]$. Since n is arbitrary b must satisfy (1.10), for otherwise $P'_x[\sup_{s \le t}|X_s| = \infty] > 0$ for some finite t, contradicting $P' \prec P^0$. In fact then $P' \sim (p_b, m_b)$.

Theorem 1. *For $P' \in \mathfrak{D}$ the following conditions are equivalent.*

(a) $P' \prec P^0$.

(b) $P' \sim (p_b, m_b)$ *for some* $b \in L^2_{\text{loc}}$.

(c) $Y[b]$ *is Brownian motion under* P' *for some* $b \in L^2_{\text{loc}}$.

(d) $P' \sim (p, m)$, *where p has an absolutely continuous, strictly positive derivative p', m has a derivative m' satisfying* $\frac{1}{2}p'(x)m'(x) \equiv 1$, *and* $p'' \in L^2_{\text{loc}}$.

Proof. The equivalence of (a)–(c) follows from Propositions 1–4. Assuming (b), (d) follows at once; note that $b(x) = -\frac{1}{2}p''(x)/p'(x)$, and, since p' is strictly positive and continuous, the assumption $b \in L^2_{\text{loc}}$ gives $p'' \in L^2_{\text{loc}}$. Similarly one can go from (d) to (b).

Corollary. $P' \prec P^0$ *implies* $P^0 \prec P'$.

Proof. Note that the Radon-Nikodym derivative given in (1.4) is positive.

Remark 2. Also $P^0 \prec P'$ implies $P' \prec P^0$; this follows from the result in §2.

Here is an interesting consequence of Theorem 1. For $b \in L^2_{\text{loc}}$ and any x, $L_t[b]$ is always a supermartingale; it is a martingale if and only if $E_x L_t[b] = 1$ for all t (see Appendix III). We now have necessary and sufficient conditions for this.

Corollary 2. *Let $b \in L^2_{\text{loc}}$. If b satisfies* (1.10), $E_x^0[L_t[b]] = 1$ *for all x and t. Conversely, if for some x,* $E_x^0[L_t[b]] = 1$ *for all t, then b satisfies* (1.10).

Proof. If b satisfies (1.10), let $P' \sim (p_b, m_b)$ and use (1.5) of Proposition 1. Suppose now that, for some x, $E_x^0[L_t[b]] = 1$ for all t; a simple stopping time argument shows that this relation must then hold for all x, and we may use relation (1.4) to define the measures P'_x. It follows easily (see Appendix III, transformation theorem) that $P' = (P'_x) \in \mathfrak{D}$. By Propositions 4, 2, 1 and Remark 1, $P' \sim (p_b, m_b)$, which means that b must satisfy (1.10).

2. The general case. Let $X^i = (X_t, \mathcal{C}_t, t \geq 0, P^i)$, $i = 1, 2$, be the function space representation of two diffusions in $\mathfrak{D}_{(-\infty,\infty)}$, with $P^i = (P^i_x)$. Let $P^i \sim (p_i, m_i)$, $i = 1, 2$. Applying p_1 to the coordinate process we obtain two new processes. Say $X^i = (Y_t, \mathcal{C}^I_t, t \geq 0, P^i)$, $i = 3, 4$, where $Y_t = p_1(X_t)$, so that (Y_t) is the coordinate process on the space $\mathcal{C}^I$ of continuous functions with values in $I = (p_1(-\infty), p_1(\infty))$, $\mathcal{C}^I_t$ is the σ-field generated by $\{Y_s: 0 \leq s \leq t\}$, and P^3 and P^4 are the measures on $\mathcal{C}^I$ that are induced from P^1 and P^2, respectively, by the mapping p_1. Of course X^3 and X^4 are just X^1 and X^2 with the state space reparametrized, and the condition $P^1 \prec P^2$ ($P^2 \prec P^1$) is equivalent to $P^3 \prec P^4$ ($P^4 \prec P^3$).

Let q be the inverse function of p_1. Then $P^3 \sim (p_3, m_3) = (p_1 \circ q, m_3 \circ q)$, $P^4 \sim (p_4, m_4) = (p_2 \circ q, m_3 \circ q)$. Note X^3 has Lebesgue scale on its interval of definition I.

We now make a time change $\beta(t)$ such that under P^3 $Y_{\beta(t)}$ is a Brownian motion, defined up to first exit from I. Such a $\beta(t)$ is the inverse of the following additive functional on X^3 (see Appendix (I.D)),

$$\alpha(t) = \int_I L(t, x; X^3)\, 2dx, \tag{2.1}$$

since $2dx$ is the speed measure of Brownian motion.

Now we observe that if $P^4 \prec P^3$ then m_4 and m_3 must be equivalent, that is, have the same null sets. Indeed one sees easily that $P^i_x[\mu(t, B; X^i) > 0] = 0$ for every t if and only if $m_i(B) = 0$ (see Appendix I). So, if $P^4 \prec P^3$ and $m_3(B) = 0$, then also $m_4(B) = 0$. For the converse implication, suppose $m_3(B) > 0$. Then $P^3_x[\mu(t, B; X^3) > 0]$ is positive for some x and t. Also, for each x, $P^3_x[\mu(t, B; X^3) > 0$ for all $t]$ must equal 0 or 1 by the zero-one law. By considering $T = \inf\{t: \mu(t, B; X^3) > 0\}$ and using the strong Markov property we obtain the existence of some x with $P^3_x[\mu(t, B; X^3) > 0$ for all $t] = 1$. Then $P^4 \prec P^3$ implies also $m_4(B) > 0$.

The transformation taking Y_t into $Y_{\beta(t)}$ transformed X^3 into Brownian motion defined up to leaving I. What process is obtained by applying the same transformation to X^4? To see this, recall the definition of local time to write

$$\alpha(t) = \int_I \frac{d\mu(t, \cdot; X^3)}{dm_3}(x) \cdot 2dx.$$

Interpret the indicated derivative as the limit superior of the difference quotients ordinarily defining a derivative. Because of the continuity of local time we know that $P^3_{x'}$-a.s., for any x', this limit superior will actually be a limit for all t and

x. Assume now that m_3 and m_4 are equivalent. To consider α as a functional on X^4 write

$$\alpha(t) = \int_I \frac{d\mu(t, \cdot; X^4)}{dm_3}(x) \cdot 2dx = \int_I \frac{d\mu(t, \cdot; X^4)}{dm_4}(x) \frac{dm_4}{dm_3}(x) \cdot 2dx.$$

(Note that both X^3 and X^4 are coordinate processes, so $\mu(t, B; X^3)$ and $\mu(t, B; X^4)$ are two names for the same quantity.) If $t > 0$, the derivative in the first integrand will exist as a limit of difference quotients and be positive P_x^3-a.s.; the same applies to the first derivative in the second integrand P_x^4-a.s. So, unless P_x^3 and P_x^4 are singular, the second derivative in the second integrand, which is not random, must also exist as a limit of difference quotients. This will be true always if $P^3 \prec P^4$ or $P^4 \prec P^3$. Then we may write

$$\alpha(t) = \int_I L(t, x; X^4) \frac{dm_4}{dm_3}(x) \cdot 2dx.$$

Now let

$$X^5 = (Y_{\beta(t)}, \mathcal{C}^I_{\beta(t)}, 0 \leq t < \alpha(\infty); P^3) \quad \text{and} \quad X^6 = (Y_{\beta(t)}, \mathcal{C}^I_{\beta(t)}, 0 \leq t < \alpha(\infty), P^4).$$

We know already that X^5 is a diffusion on I, defined up to the first exit time from I, with scale x and speed $2dx$. From the final form of $\alpha(t)$ we learn that (see Appendix (I.D)) X^6 corresponds to a diffusion with scale p_6 and speed m_6 given by

$$p_6(x) = p_4(x), \qquad m_6(dx) = 2\frac{dm_4}{dm_3}(x)\,dx$$

defined on I. Since X^5 and X^6 have life times $\alpha(\infty)$ which need not be infinite they do not necessarily belong to $\mathfrak{D}_I$, strictly speaking. However, X^5 and X^6 induce measures $P^5 = (P_x^5)$ and $P^6 = (P_x^6)$ on the space of all continuous functions from $[0, \infty)$ into I, defined up to the first time that the function approaches a boundary point of I. The measures P^5, P^6 come from the original measures P^3, P^4 via the map taking ω into $\beta[\omega]$, where $\beta[\omega](t) = \omega(\beta(t, \omega))$. If $\eta = \beta[\omega]$, $\omega(t) = \eta(\alpha(t, \omega))$, since α is the inverse of β. However, α can be considered as a function of η, because

$$\int L(t, x; X^5)m^3(dx) = \int L(\beta(t), x; X^3)m^3(dx) = \beta(t)$$

(see Appendix (I.C)), so that on a set having P_x^3-measure one for all x, the map ω into $\beta[\omega]$ is invertible. Observe now that for P^i and $P^j \in \mathfrak{D}_I$, $P^i \prec P^j$ if and only if $P_x^i|_\tau \ll P_x^j|_\tau$ for every $x \in I$, and every τ which is the first exit time from a compact subinterval of I, where $|_\tau$ denotes restriction to $\mathcal{C}_\tau$. Similarly we can define $P^6 \prec P^5$ $(P^5 \prec P^6)$ to hold if the measures are absolutely continuous when restricted up to the first exit time from any compact subinterval of I. From what we have said it follows that $P^4 \prec P^3$ if and only if $P^6 \prec P^5$, and both $P^4 \prec P^3$ and $P^3 \prec P^4$ if and only if both $P^6 \prec P^5$ and $P^5 \prec P^6$. Since P^5 is Brownian motion, defined up to the first exit time from I, we can use the work of §1.

Theorem 2. *$P^2 \prec P^1$ implies $P^1 \prec P^2$. Necessary and sufficient conditions for $P^2 \prec P^1$ are as follows:*

(i) *the derivative $dp_2(x)/dp_1$ exists everywhere and defines a positive function absolutely continuous with respect to p_1;*

(ii) *$dm_2(x)/dm_1$ exists everywhere and satisfies $dm_2(x)/dm_1 \cdot dp_2(x)/dp_1 = 1$;*

(iii) *the second derivative $d^2p_2(x)/dp_1^2$, defined dp_1-a.e. belongs to $L^2_{\mathrm{loc}}(dp_1)$.*

3. Appendix. We organize some known results, occasionally with trivial variations, for easy reference.

I. Diffusion local time. All the basic facts we need are in [7]. As a reference for our purposes here the more leisurely [4] suffices and might be found more convenient.

(A) *Brownian local time*. Let $X = (X_t, \mathcal{C}_t, 0 \le t < \infty, (P_x))$ be coordinate representation of Brownian motion on function space C. The associated scale and speed are x and $2dx$.

Trotter's Theorem [14]. *For each $t \geq 0$ and $x \in (-\infty, \infty)$ there exists a random variable $L(t, x; X)$ such that for all ω in C outside some fixed set Λ with $P_x[\Lambda] = 0$ for all x, the following two conditions hold: $(t, x) \to L(t, x; X)(\omega)$ is continuous, and $\mu(t, B; X) = \int_B L(t, x; X)2dx$, B a Borel set of R^1, $t \geq 0$. $L(t, x; X)$ is called Brownian local time.*

(B) *Ito-McKean representation.* Let $P' = (P'_x)$, $x \in I$, be a diffusion in $\mathfrak{D}_I$, where I is an open interval. Say $P' \sim (p, m)$. A diffusion Z corresponding to P' is constructed from Brownian motion X in two steps. Let $P^* \sim (p^*, m^*) = (p \circ q, m \circ q)$, where q is the inverse function of p. $P^* \in \mathfrak{D}_{p(I)}$. Then $Z^* = (Z^*_t)$ is obtained as $Z^*_t = X_{\beta(t)}$, $\beta(t)$ $(= \beta(t, \omega))$ being the inverse of $\alpha(t)$ $(= \alpha(t, \omega))$ defined by

$$\alpha(t) = \int_{p(I)} L(t, y; X) m^*(dy), \qquad t < \tau = \inf\{s: X_s \notin p(I)\}.$$

Finally $Z_t = q(Z^*_t)$. Note that as t increases to τ, $\alpha(t)$ approaches infinity. Both $\alpha(t)$ and $\beta(t)$ are continuous, strictly increasing.

(C) *Diffusion local time.* Keeping the notations of (A) and (B), $L(t, x; Z) = L(\beta(t), q(x); X)$ defines the *local time of* Z. Then $\mu(t, B; Z) = \int_B L(t, x; Z)m(dx)$, B a Borel set of I, $0 \le t < \infty$, and $(t, x) \to L(t, x; Z)$ is continuous, both assertions again holding outside the exceptional null set Λ. The last formula allows an obvious extension:

$$\int_0^t f(Z_u)du = \int_I L(t, x; Z)f(x)m(dx), \qquad f \text{ Borel measurable, } \int |f|\, dm < \infty.$$

So in particular, for the case of Brownian motion, $g \in L^2_{\text{loc}}$ is necessary and sufficient for $\int_0^t g^2(X_u)du < \infty$, P_x-a.s. for all $t < \infty$, $x \in (-\infty, \infty)$.

(D) *Change of time scale.* We continue with the notations introduced. Let n be a positive measure on I, finite on compact sets, assigning strictly positive weight to every open interval. Let $\gamma(t) = \int_I L(t, y; Z)n(dy)$ and let $\delta(t)$ be the inverse function of $\gamma(t)$. The situation is similar to (B) above, but as t tends to infinity $\gamma(t)$ tends to a limit $\gamma(\infty)$ which need not be infinite. So $\delta(t)$ is defined only for $0 \le t < \gamma(\infty)$. The same considerations as in (B) show that $(Z_{\delta(t)}, 0 \le t < \gamma(\infty))$ is a diffusion on I, defined up to the first exit time from I, and governed in the interior of I by the scale p and speed n.

II. **Additive functionals of Brownian motion.** Again $X = (X_t, \mathcal{C}_t, 0 \le t < \infty, (P_x))$ is coordinate representation of Brownian motion, (θ_t) are the associated shift operators. A stochastic process (A_t) is called an *additive functional* of Brownian motion if A_t is $\mathcal{F}_t$-measurable, A_t assumes values in $(-\infty, \infty]$ and, for each pair of nonnegative numbers s, t, $A_{t+s} = A_s + A_t \circ \theta_s$, P_x-a.s., $-\infty < x < \infty$. The following result is also given in Ventcel [15].

Tanaka's representation [13]. If (A_t) is a finite-valued, continuous additive functional of Brownian motion, then there exists a continuous function g, and a function $k \in L^2_{\text{loc}}$ such that $A_t = g(X_t) - g(X_0) + \int_0^t k(X_s)dX_s$. We will require the following lemma.

Lemma. *Let (A_t) be an additive functional of Brownian motion with values in $(-\infty, \infty]$, continuous in the topology of the extended real line, with $A_0 = 0$ P_x-a.s. for all x. Then (A_t) is finite valued.*

Proof. For every x and every positive δ there exists a positive ϵ and a positive finite M such that, with τ the first exit time of Brownian motion from $[x - \epsilon, x + \epsilon]$, $P_x[\sup_{0 \le t \le \tau} |A_t| > M] < \delta$. Now one can repeat, word for word, the argument of Tanaka [13, Theorem 1], to conclude that there exist positive constants c and ρ, with $\rho < 1$, such that

$$P_x\left[\sup_{0 \le t \le \tau} |A_t| > \lambda\right] \le c\rho^\lambda, \qquad \lambda \ge 0.$$

In particular

$$P_x\left[\sup_{0\le t\le T} |A_t| < \infty\right] = 1.$$

An easy covering argument concludes the proof.

III. **Absolute continuity.** Let $(\Omega, \mathfrak{F})$ be a measurable space, $(\mathfrak{F}_t)$ an increasing family of sub-σ-fields of $\mathfrak{F}$. A stochastic process $X = (X_t, \mathfrak{F}_t, t \geq 0)$ is adapted if each X_t is $\mathfrak{F}_t$-measurable. Given a probability measure P on $(\Omega, \mathfrak{F})$, a continuous *adapted* process $X = (X_t, \mathfrak{F}_t, t \geq 0)$ is said to be a *Brownian motion under* P (the last phrase can be omitted if it is understood that a fixed P is used) if X is a martingale under P-measure with finite dimensional distributions as given by Wiener measure.

If P is a probability measure on $\mathfrak{F}$, $P|_t$ is the restriction of P to $\mathfrak{F}_t$. If P' is another such measure, with $P'|_t \ll P|_t$ for each t, there exists a Radon-Nikodym derivative L_t such that $P'(\Lambda) = \int_\Lambda L_t dP$, $\Lambda \in \mathfrak{F}_t$, and $(L_t, \mathfrak{F}_t, t \geq 0)$ must be a nonnegative martingale with respect to P. We can choose a right continuous version. If $T_0 = \inf\{t: L_t = 0\} \leq \infty$, then $L_t = 0$ for $t \geq T_0$. One verifies at once that if S and T are two bounded stopping times with $S \leq T$ and H is an $\mathfrak{F}_T$-measurable P'-integrable random variable then

$$E'[H|\mathfrak{F}_S] = E[H(L_T/L_S)|\mathfrak{F}_S] \qquad P'\text{-a.s.} \tag{3.1}$$

where the possible vanishing of L_S causes no problem since $P'[L_S = 0] = 0$.

Let $M = (M_t, \mathfrak{F}_t, t \geq 0, P)$ be a continuous local martingale. Let $A_t = \langle M, M\rangle_t$ be the associated increasing process, where we use the bracket notation of Meyer [12]. One defines a new process $X = \operatorname{Exp}[M]$ by $X_t = \exp[M_t - \tfrac{1}{2}A_t]$. By Ito's formula this is again a continuous increasing process with $dX_t = X_t dM_t$. Since X is in fact a positive continuous local martingale an easy limiting argument using Fatou's lemma shows it is a supermartingale. Evidently X will be a martingale if and only if $EX_t \equiv 1$.

Conversely if (Z_t) is a continuous adapted process such that $X_t = \exp[Z_t]$ is a continuous local martingale one sees, by applying Ito's formula to $\log X_t$, that $Z_t = M_t - \tfrac{1}{2}\langle M, M\rangle_t$ for some continuous local martingale M. In particular

(3.2) $(M_t, \mathfrak{F}_t, t \geq 0)$ *is Brownian motion if and only if* $(\exp(M_t - \tfrac{1}{2}t), \mathfrak{F}_t, t \geq 0)$ *is a continuous local martingale.*

It is also immediate that for two continuous local martingales M and N

$$\operatorname{Exp}[M + N] = \operatorname{Exp}[M] \cdot \operatorname{Exp}[N] \cdot \exp(-\langle M, N\rangle). \tag{3.3}$$

Girsanov theorem [5]. *Let* $W = (W_t, \mathfrak{F}_t, t \geq 0)$ *be Brownian motion under* P. *Let* $H = (H_t, \mathfrak{F}_t, t \geq 0)$ *be a previsible process with* $\int_0^t H_u^2 du < \infty$ P*-a.s. Let* $V = W - \int_0 H_u du$ (*i.e.* $V_t = W_t - \int_0^t H_u du$, $t \geq 0$) *and set* $L = \mathrm{Exp}[\int_0 H_u dW_u]$.

If $L = (L_t, \mathfrak{F}_t, t \geq 0)$ *is a martingale and* P' *is determined by* $P'(\Lambda) = \int_\Lambda L_t dP$, $\Lambda \in \mathfrak{F}_t$, *then with respect to* P', $(V_t, \mathfrak{F}_t)$ *is Brownian motion.*

Proof. By (3.2) it must be proved that $(\exp(V_t - \frac{1}{2}t), \mathfrak{F}_t, t \geq 0)$ is a local martingale with respect to P'. Writing out what this means, using (3.1) and (3.3) this follows at once.

Corollary (Kailath-Zakai [10]; with different proof Kadota-Shepp [8]). *Let* W, H, V *be as in the statement of Girsanov's theorem.* (*No hypothesis on* L *is made now.*) *Let* P^0 *and* P' *be the measures induced in function space* $(C, \mathcal{C})$ *by* W *and* V *respectively. Then* $P'_t \ll P^0_t$ *for all* t.

Proof. If L, defined as in Girsanov's theorem, is a martingale, the conclusion follows from Girsanov's theorem. In the general case there exist stopping times $T_n \uparrow \infty$ such that $L^{(n)}$, with $L_t^{(n)} = L_{t \wedge T_n}$, is a martingale for each n. Let $H_t^{(n)} = H_t \cdot \chi_{t \leq T_n}$. Then $L^{(n)} = \mathrm{Exp}\, H^{(n)}$ and, setting $V^{(n)} = W - \int_0 H_u^{(n)} du$, we find that the measures $P^{(n)}$ induced in function space by $V^{(n)}$ satisfy $P_t^{(n)} \ll P_t^0$. Since for every $K \in \mathcal{C}_t$, $P^{(n)}(K)$ converges to $P'(K)$ as n goes to infinity $P'_t \ll P^0_t$ follows.

The following is a variation of Dynkin [2, Theorem 10.4]. The notation $L_t[b]$ is defined in (1.2).

Transformation theorem. *Let* (P_x) *be a diffusion in* $\mathfrak{D}_{(-\infty,\infty)}$, $b \in L^1_{\mathrm{loc}}$, *and* $E_x[L_t[b]] \equiv 1$, $-\infty < x < \infty$. *Determine* P'_x *on* $(C, \mathcal{C})$ *by* $P'_x(\Lambda) = \int_\Lambda L_t dP$, $\Lambda \in \mathcal{C}_t$. *Then* $(P'_x) \in \mathfrak{D}_{(-\infty,\infty)}$.

Proof. The existence of the P'_x is evident. In order to prove that (P'_x) is a strong Markov process consider a bounded Markov time T (the unbounded case is handled by a limiting argument). Let Y be a bounded, $\mathcal{C}_s$-measurable random variable. Using (3.1), the fact that (L_t) is a multiplicative functional, and the strong Markov property of (P_x)

$$E'_x[Y \circ \theta_T | \mathfrak{F}_T] = E_x[Y \circ \theta_T \cdot (L_{T+s}/ L_T) | \mathfrak{F}_T]$$
$$= E_x[Y \circ \theta_T \cdot L_s \circ \theta_T | \mathfrak{F}_T] = E_{X_T}[L_s Y] = E'_{X_T}[Y].$$

A monotone class argument extends this to all bounded $\mathcal{C}$-measurable Y.

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