

A FINITELY ADDITIVE GENERALIZATION OF THE  
FICHTENHOLZ-LICHTENSTEIN THEOREM<sup>(1)</sup>

BY

GEORGE EDWARD SINCLAIR

ABSTRACT. Let  $\mu$  and  $\nu$  be bounded, finitely additive measures on algebras over sets  $X$  and  $Y$ , respectively. Conditions are determined for a bounded function  $f: X \times Y \rightarrow \mathbb{R}$ , without assuming bimeasurability, so that the iterated integrals  $\int_X \int_Y f d\mu d\nu$  and  $\int_Y \int_X f d\nu d\mu$  exist and are equal. This result is then used to construct a product algebra and finitely additive product measure for  $\mu$  and  $\nu$ . Finally, a simple Fubini theorem with respect to this product algebra and product measure is established.

1. Introduction. Since the appearance in 1910 [7], [10] of the Fichtenholz-Lichtenstein theorem on the equality of repeated Riemann integrals, attempts have been made to extend this theorem to a more abstract setting without requiring the existence of the product integral. In recent years progress has been made by Luxemburg [12] and de Lucia [11]. However, the principal theorem of this paper [Theorem 4.4] provides the first general theorem for the existence and equality of iterated integrals with respect to two bounded, finitely additive measures. Further, Theorem 4.4 has a converse implication which shows that, in a certain sense, it is the best possible.

The principal theorems of classical measure theory applied to analysis are the Radon-Nikodym theorem, the Lebesgue decomposition theorem, and Fubini's theorem. A Radon-Nikodym theorem for finitely additive measures [6, p. 315] has long been known, while Darst [5] extended the Lebesgue decomposition to such measures. However, Theorem 4.4 supplies the first reasonable substitute for Fubini's theorem.

---

Received by the editors December 13, 1972.

AMS (MOS) subject classifications (1970). Primary 28A25, 28A35; Secondary 54C05, 54C20, 54D35.

Key words and phrases. A-continuous, Dunford-Schwartz integral, finitely additive product measure, Fubini theorem, Pták lemma, Pták theorem, separately continuous, Stone-Čech compactification, Stone space.

(<sup>1</sup>) This paper constitutes the main result of the author's dissertation which was completed at the University of Arizona under the direction of Dr. J. S. Lomont.

Copyright © 1974, American Mathematical Society

In §2 the terminology used in this article is established, while §3 is devoted to statements of preparatory theorems. The proof of Theorem 4.4 occupies §4. A new product measure and product algebra for finitely additive measures is defined in §5. This algebra contains the measurable rectangles. Certain properties of the new measure, including its relation to other product measures, are discussed in this section. The subjects of §6 are the proof of a "Fubini theorem" for the new product measure previously defined and the construction of related examples.

I am very grateful to my advisor, Dr. J. S. Lomont, for his help and advice during the research and preparation of this paper. I also wish to thank Dr. J. L. Denny, who originally suggested the use of Theorem 4.4 to construct a product measure.

2. **Terminology.** The object of interest in this paper is a triple  $(X, A, \mu)$  consisting of an algebra  $A$  over a set  $X$  and a bounded, finitely additive, real-valued set function  $\mu$  defined on  $A$ . The integral with respect to  $\mu$  is that defined by Dunford and Schwartz [6, p. 112]. It is a straightforward, although technically tedious, exercise to show that this integral and the Moore-Smith integral [9, p. 332] are the same for the class of functions considered in this paper. Hereafter, the term "measure" will be reserved for a finitely additive, bounded, real-valued set function defined on at least a semiring.

The following definition is the most important in this paper.

2.1. **Definition.** Let  $f: X \times Y \rightarrow \mathbb{R}$  be a real-valued function of two variables.  $f$  is a DLC function if, whenever  $\{x_i\}_{i=1}^{\infty} \subset X$  and  $\{y_j\}_{j=1}^{\infty} \subset Y$  are sequences such that the two iterated limits  $\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} f(x_i, y_j)$  and  $\lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} f(x_i, y_j)$  both exist, then they are equal.

3. **The theorem of Pták.** The primary tool required for the proof of Theorem 4.4 is Pták's Theorem 2 [14, p. 573]. Theorem 3.2 is a restatement of Pták's results in a form more useful to this paper. The following lemma is needed first. The proof of this lemma is transparent.

3.1. **Lemma.** Let  $f: X \times Y \rightarrow \mathbb{R}$  be a separately continuous, real-valued function on the topological spaces  $X$  and  $Y$ , respectively. Let  $\bar{X} \subset X$  and  $\bar{Y} \subset Y$  be dense subsets of  $X$  and  $Y$ , respectively. Let  $f^* = f|_{\bar{X} \times \bar{Y}}$ . Then  $f$  is bounded if and only if  $f^*$  is. Further, if  $f$  and  $f^*$  are bounded, they have the same supremum.

3.2. **Theorem.** Let  $X$  and  $Y$  be  $T_{3\frac{1}{2}}$  topological spaces. Let  $f: X \times Y \rightarrow \mathbb{R}$  be a real-valued, bounded, separately continuous function on  $X \times Y$ . Let  $\beta X$  and  $\beta Y$  be the Stone-Čech compactifications of  $X$  and  $Y$ , respectively. Let  $C^*(Z)$  be the topological dual of the space  $C(Z)$  of bounded, continuous, real-valued

functions on the topological space  $Z$ . Then the following are equivalent.

(A)  $f$  is a DLC function.

(B)  $f$  can be extended to a separately weak\* continuous, bilinear, real-valued function on  $C^*(X) \times C^*(Y)$ .

(C)  $f$  can be extended to a bounded, separately continuous, real-valued function on  $\beta X \times \beta Y$ .

(D) Let  $\mu$  and  $\nu$  be regular, signed, Borel measures on  $\beta X$  and  $\beta Y$ , respectively. Let  $f_x^*$  ( $f_y^*$ ) be the extension of  $f_x$  ( $f_y$ ) to a continuous function on  $\beta Y$  ( $\beta X$ ) for each  $x \in X$  ( $y \in Y$ ). Then:

(i) The function  $\psi(y) \equiv \int_{\beta X} f_y^* d\mu$  on  $Y$  has a continuous extension  $\psi^*$  on  $\beta Y$ .

(ii) The function  $\phi(x) \equiv \int_{\beta Y} f_x^* d\nu$  on  $X$  has a continuous extension  $\phi^*$  on  $\beta X$ .

(iii)  $\int_{\beta X} \phi^* d\mu = \int_{\beta Y} \psi^* d\nu$ .

**Proof.** Pták's Theorem 2 [14, p. 573] and Lemma 3.1 state that (A)  $\Rightarrow$  (B)  $\Rightarrow$  (C)  $\Rightarrow$  (A). Also, "infra" the proof of Theorem 1 [14, p. 567], Pták demonstrated that (A)  $\Rightarrow$  (D). (The Riesz representation theorem is needed here.) Hence, only the proof that (D)  $\Rightarrow$  (A) is required.

Assume that  $f$ , as defined in the hypotheses of this theorem, is not a DLC function. So, there must exist sequences  $\{x_i\}_{i=1}^\infty \subset X$  and  $\{y_j\}_{j=1}^\infty \subset Y$  such that  $\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} f(x_i, y_j)$  and  $\lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} f(x_i, y_j)$  both exist, but  $\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} f(x_i, y_j) \neq \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} f(x_i, y_j)$ . Let  $f_x^*$  ( $f_y^*$ ) for any  $x \in X$  ( $y \in Y$ ) and the measures  $\mu$  and  $\nu$  be defined as in part (D) of this theorem. Due to continuity,  $\int_{\beta X} f_y^* d\mu$  ( $\int_{\beta Y} f_x^* d\nu$ ) exists for each  $y \in Y$  ( $x \in X$ ). Hence, the functions  $\psi(y) \equiv \int_{\beta X} f_y^* d\mu$  and  $\phi(x) \equiv \int_{\beta Y} f_x^* d\nu$  are well defined and obviously bounded on  $Y$  and  $X$  for any regular Borel measures  $\mu$  and  $\nu$  on  $\beta X$  and  $\beta Y$ , respectively. Assume that  $\psi$  and  $\phi$  are continuous and, therefore, have continuous extensions  $\psi^*$  and  $\phi^*$  on  $\beta Y$  and  $\beta X$ , respectively. If this assumption is false, then  $f$  does not satisfy condition (D) of Theorem 3.2.

Examine  $\{x_i\}_{i=1}^\infty \subset X$ . Either there exists  $x \in X$  such that  $x = x_i$  for an infinite number of  $i$ 's or  $\{x_i\}_{i=1}^\infty$  is an infinite set. In the second case there exists  $x \in \beta X$  such that  $x$  is an accumulation point of  $\{x_i\}_{i=1}^\infty$  in  $\beta X$ . In either case  $\lim_{i \rightarrow \infty} f(x_i, y_j) = f_{y_j}^*(x)$  for each  $j$ . The  $x$  obtained is not necessarily unique, but the value of  $f_{y_j}^*(x)$  is independent of the choice of  $x$ . Similarly, there exists  $y \in \beta Y$  such that  $\lim_{j \rightarrow \infty} f(x_i, y_j) = f_{x_i}^*(y)$  for each  $i$ .

Let  $\mu$  be the zero-one measure on the Borel sets of  $\beta X$  generated by  $x \in \beta X$ —that is

$$\mu(E) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $\nu$  be the zero-one measure on the Borel sets of  $\beta Y$  generated by  $y \in \beta Y$ . Clearly,  $\mu$  and  $\nu$  are regular measures. For each  $j$ ,

$$\psi(y_j) = \int_{\beta X} f_{y_j}^* d\mu = f_{y_j}^*(x) = \lim_{i \rightarrow \infty} f(x_i, y_j).$$

Also,

$$\phi(x_i) = \int_{\beta Y} f_{x_i}^* d\nu = f_{x_i}^*(y) = \lim_{j \rightarrow \infty} f(x_i, y_j)$$

for each  $i$ . As  $\psi^*$  is continuous on  $\beta Y$ ,  $\psi^*(y)$  is an accumulation point of the sequence  $\{\psi^*(y_j)\}_{j=1}^\infty$ . Since  $\lim_{j \rightarrow \infty} \psi(y_j) = \lim_{j \rightarrow \infty} \psi^*(y_j)$  exists by hypothesis,  $\lim_{j \rightarrow \infty} \psi^*(y_j) = \psi^*(y)$ . But,

$$\int_{\beta Y} \psi^* d\nu = \psi^*(y) = \lim_{j \rightarrow \infty} \psi(y_j) = \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} f(x_i, y_j).$$

Similarly,

$$\int_{\beta X} \phi^* d\mu = \phi^*(x) = \lim_{i \rightarrow \infty} \phi(x_i) = \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} f(x_i, y_j).$$

Therefore,  $\int_{\beta X} \phi^* d\mu \neq \int_{\beta Y} \psi^* d\nu$ . Hence,  $f$  cannot satisfy condition (D) in Theorem 3.2. One must conclude that (D)  $\Rightarrow$  (A). Q.E.D.

The following lemma, also due to Pták [14, p. 572], will be of use later.

**3.3. Lemma.** *Let  $X$  be a pseudocompact,  $T_2$  topological space, and let  $Y$  be a countably compact,  $T_{3\frac{1}{2}}$  topological space. Let  $f: X \times Y \rightarrow \mathbb{R}$  be a separately continuous, bounded, real-valued function on  $X \times Y$ . Then  $f$  is a DLC function.*

**4. The main theorem.** The following definition provides a description of the one-variable behavior of functions to which Theorem 4.4 will apply.

**4.1. Definition.** Let  $\mathbf{A}$  be an algebra on a set  $X$ .

(A)  $\mathbf{A}$  is said to "separate points" on  $X$  if, given  $x, y \in X$  such that  $x \neq y$ , there exists  $E, F \in \mathbf{A}$  such that  $x \in E, y \in F$ , and  $E \cap F = \emptyset$ . If  $\mathbf{A}$  "separates points" on  $X$ , then  $\mathbf{A}$  is an SP algebra on  $X$ .

(B) Let  $f: X \rightarrow \mathbb{R}$  be a real-valued function on  $X$ .  $f$  is  $\mathbf{A}$ -continuous if, given  $\epsilon > 0$ , there is a finite partition of  $X$  into  $\mathbf{A}$ -measurable sets  $\{E_i\}_{i=1}^N$  such that, given  $x, y \in E_i$  for any  $i$ ,  $|f(x) - f(y)| < \epsilon$ . The collection  $\{E_i\}_{i=1}^N$  will, in general, depend on  $\epsilon$ .

The next lemma, characterizing  $\mathbf{A}$ -continuous functions, is a collection of known facts.

**4.2. Lemma.** *Let  $f: X \rightarrow \mathbb{R}$  be a real-valued function on  $X$ . Let  $\mathbf{A}$  be an algebra on  $X$ . Then the following are equivalent.*

(A)  $f$  is  $\mathbf{A}$ -continuous.

(B) (Darst [4, p. 293]). *If  $\mu$  is a measure on  $\mathbf{A}$ , then  $f$  is  $\mu$ -integrable.*

(C) (Leader [9, p. 233]).  $f$  is a uniform limit of simple,  $\mathbf{A}$ -measurable functions. Let  $\mathbf{A}$  be an SP algebra on  $X$ .

(D) (Porcelli [13, p. 119]). Let  $X^*$  be the Stone space [16, p. 24] of  $\mathbf{A}$ . Then  $f$  can be extended to a continuous function on  $X^*$  if and only if one of (A), (B), or (C) above holds.

The following known facts concerning Stone spaces are collected here for easy reference. [See [1] and [16] for proofs.]

4.3. Let  $\mathbf{A}$  be an SP algebra on a set  $X$ .

(A) There exists a  $T_2$ , totally disconnected, compact space  $X^*$  (the Stone space of  $\mathbf{A}$ ) such that  $\mathbf{A}$  is isomorphic to the algebra of open-closed sets  $\mathcal{C}$  on  $X^*$ . Designate this isomorphism by  $I$  ( $I: \mathbf{A} \rightarrow \mathcal{C}$ ).

(B) There is a one-to-one correspondence  $J$  between the measures on  $\mathbf{A}$  and the regular, signed,  $\sigma$ -additive, Borel measures on  $X^*$ . Further, if  $\mu$  and  $\mu^*$  are measures on  $\mathbf{A}$  and  $\mathcal{C}$ , respectively, such that  $J(\mu) = \mu^*$ , and if  $E \in \mathbf{A}$ , then  $\mu(E) = \mu^*[I(E)]$ .

(C)  $X$  can be mapped one-to-one and onto a dense subset of  $X^*$ . Denote this map by  $b$ .

(D) Let  $f$  be an  $\mathbf{A}$ -continuous, real-valued function on  $X$ . Let  $f^*$  be the extension of  $f$  to a continuous function on  $X^*$ . If  $x \in X$ , then  $f(x) = f^*[b(x)]$ . Also  $\int_X f d\mu = \int_{X^*} f^* d\mu^*$ .

(E) There is a one-to-one correspondence,  $N$ , between the points in  $X^*$  and the zero-one measures on  $\mathbf{A}$ . Let  $\mu$  be a zero-one measure on  $\mathbf{A}$ , let  $N(\mu) = x \in X^*$ , and let  $J(\mu) = \mu^*$ . Then

$$\mu^*(E) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } E \in \mathcal{C}.$$

The next theorem is the main result of this article. [See [17].]

4.4. Theorem. Let  $\mathbf{A}$  and  $\mathbf{B}$  be SP algebras over the sets  $X$  and  $Y$ , respectively. Let  $\mu$  and  $\nu$  be bounded, finitely additive, real-valued, set functions on  $\mathbf{A}$  and  $\mathbf{B}$ , respectively. Let  $f: X \times Y \rightarrow \mathbf{R}$  be a real-valued, bounded function on  $X \times Y$ . Assume:

(A)  $f_y$  is  $\mathbf{A}$ -continuous for each  $y \in Y$ ;

(B)  $f_x$  is  $\mathbf{B}$ -continuous for each  $x \in X$ ;

(C)  $f$  is a DLC function.

Therefore:

(A') If  $\psi(y) \equiv \int_X f_y d\mu$  on  $Y$ , then  $\psi$  is  $\mathbf{B}$ -continuous;

(B') If  $\phi(x) \equiv \int_Y f_x d\nu$  on  $X$ , then  $\phi$  is  $\mathbf{A}$ -continuous;

(C')  $\int_Y \psi d\nu = \int_Y \int_X f d\mu d\nu = \int_X \int_Y f d\nu d\mu = \int_X \phi d\mu$ .

Further, if  $f$  is a real-valued, bounded function on  $X \times Y$  such that the integrals  $\int_X \int_Y f \, d\nu \, d\mu$  and  $\int_Y \int_X f \, d\mu \, d\nu$  exist and are equal for every pair of measures  $\mu$  and  $\nu$  on  $\mathbf{A}$  and  $\mathbf{B}$ , respectively, then  $f$  satisfies parts (A), (B), and (C) of this theorem.

**Proof. Notation.** Let  $X^*$  ( $Y^*$ ) be the Stone space of  $\mathbf{A}$  ( $\mathbf{B}$ ), and let  $\mathcal{U}$  ( $\mathcal{B}$ ) be the algebra of open-closed sets on that space. For a given measure  $\mu$  ( $\nu$ ) on  $\mathbf{A}$  ( $\mathbf{B}$ ), let  $\mu'$  ( $\nu'$ ) be the corresponding regular, signed,  $\sigma$ -additive measure on  $\mathcal{U}$  ( $\mathcal{B}$ ) (see 4.3(B)). Let  $\bar{X}$  ( $\bar{Y}$ ) be the embedded image of  $X$  ( $Y$ ) in  $X^*$  ( $Y^*$ ) which is dense there. Identify  $X$  with  $\bar{X}$  and  $Y$  with  $\bar{Y}$ . So,  $f$  can be considered to be a function on  $\bar{X} \times \bar{Y}$ . For each  $y \in \bar{Y}$  ( $x \in \bar{X}$ ),  $f_y$  ( $f_x$ ) is  $\mathbf{A}$ -continuous ( $\mathbf{B}$ -continuous) and can be extended to a continuous function  $f'_y$  ( $f'_x$ ) on  $X^*$  ( $Y^*$ ).

Now,  $\bar{X}$  and  $\bar{Y}$  under the relative topologies with respect to  $X^*$  and  $Y^*$ , respectively, are  $T_{3\frac{1}{2}}$  spaces. For the rest of this proof  $\bar{X}$  and  $\bar{Y}$  will be considered to be topological spaces with this topology. So,  $f_y$  ( $f_x$ ) is a continuous function on  $\bar{X}$  ( $\bar{Y}$ ) for each  $y \in \bar{Y}$  ( $x \in \bar{X}$ ). As  $f$  is bounded and a DLC function, Theorem 3.2 holds. So,  $f$  can be extended to a separately continuous, bounded function  $f^*$  on  $\beta\bar{X} \times \beta\bar{Y}$ —where  $\beta\bar{X}$  ( $\beta\bar{Y}$ ) is the Stone-Čech compactification of  $\bar{X}$  ( $\bar{Y}$ ).

**Embedding of  $X^*$  ( $Y^*$ ) in  $\beta\bar{X}$  ( $\beta\bar{Y}$ ).** The identity map  $i_x: \bar{X} \rightarrow \bar{X} \subset X^*$  is a homeomorphism of  $\bar{X}$  into  $X^*$ . Hence,  $i_x$  can be extended to a continuous map  $r: \beta\bar{X} \rightarrow X^*$  such that  $r(\beta\bar{X} - \bar{X}) \subset X^* - i_x(\bar{X}) = X^* - \bar{X}$  [8, p. 92]. Since  $\bar{X}$  is dense in  $X^*$  and  $\beta\bar{X}$  is compact,  $r$  is onto. Define the map  $\theta_x: X^* \rightarrow \beta\bar{X}$  by  $\theta_x(x) = z \in r^{-1}(\{x\})$ , where  $z$  is some fixed element (picked by the axiom of choice) in  $r^{-1}(\{x\})$ . Note that  $\theta$  is not unique.

If  $x \in \bar{X}$ , then  $r^{-1}(x) = i_x^{-1}(x) = x$ . So,  $\theta_x|_{\bar{X}} = i_x$ . Let  $\theta_x(X^*) = \mathfrak{X}$ . As  $\theta_x$  is one-to-one, it is a correspondence between  $X^*$  and  $\mathfrak{X}$ . Let  $i_y: \bar{Y} \rightarrow Y^*$  be the identity map on  $\bar{Y}$ . In a similar manner a one-to-one map  $\theta_y: Y^* \rightarrow \beta\bar{Y}$  can be defined so that  $\theta_y|_{\bar{Y}} = i_y$ . Let  $\theta_y(Y^*) = \mathfrak{Y}$ .

$\mathfrak{X}$  can have two topologies, the relative topology with respect to  $\beta\bar{X}$  (designated as the  $R$ -topology) and the topology obtained by identifying  $X^*$  with  $\mathfrak{X}$  (designated as the  $I$ -topology). Note that  $E'$  is open in  $\mathfrak{X}$  with respect to the  $I$ -topology if and only if there exists an open set  $E \subset X^*$  such that  $\theta_x(E) = E'$ . However,  $\theta_x(E) = r^{-1}(E) \cap \mathfrak{X}$ , which is open in  $\mathfrak{X}$  with respect to the  $R$ -topology. Hence, if  $E \subset \mathfrak{X}$  is open with respect to the  $I$ -topology, then it is open with respect to the  $R$ -topology.

As a consequence of the previous statement, if  $g$  is a function on  $\mathfrak{X}$  which is continuous in the  $I$ -topology, then it is continuous in the  $R$ -topology. Also, if  $\{a_\lambda\}_{\lambda \in \Lambda}$  is a net in  $\mathfrak{X}$  which converges in the  $R$ -topology, then it converges in the  $I$ -topology. Similar statements can be made concerning the corresponding  $I$ - and  $R$ -topologies on  $\mathfrak{Y}$  with respect to  $Y^*$  and  $\beta\bar{Y}$ .

*Extension of  $f$  to  $X^* \times Y^*$ .* Let  $\ell = f^*|_{\mathfrak{X} \times \mathfrak{Y}}$ . Let  $f_y^\# = f_y' \circ \theta_x^{-1}$  for each  $y \in \bar{Y}$ . Then  $f_y^\#$  is continuous in both topologies on  $\mathfrak{X}$  (for each  $y \in \bar{Y}$ ). Also,  $f_y^\#|_{\bar{X}} = f_y = \ell_y|_{\bar{X}}$ . As  $f_y^\#$  and  $\ell_y$  are both continuous in the  $R$ -topology on  $\mathfrak{X}$  and agree on a dense subset of  $\mathfrak{X}$ , they agree on all of  $\mathfrak{X}$  (for each  $y \in \bar{Y}$ ). Similarly, if  $f_x^\# = f_x' \circ \theta_y^{-1}$ , then  $f_x^\# = \ell_x$  for each  $x \in \bar{X}$ .

Let  $\lambda \in \mathfrak{Y} - \bar{Y}$ . Since  $\bar{Y}$  is dense in  $\beta\bar{Y}$ , there exists a net  $\{y_\alpha\}_{\alpha \in A} \subset \bar{Y}$  which converges to  $\lambda$  in  $\beta\bar{Y}$  and, also, in both topologies on  $\mathfrak{Y}$ . Note that  $\{\ell_{y_\alpha}(x)\}_{\alpha \in A} = \{\ell_x(y_\alpha)\}_{\alpha \in A}$  converges to  $\ell_x(\lambda) = \ell_\lambda(x)$  for each  $x \in \mathfrak{X}$  as  $\ell_x$  is continuous in the  $R$ -topology on  $\mathfrak{Y}$  for each  $x \in \mathfrak{X}$ .

The net  $\{f_{y_\alpha}^*\}_{\alpha \in A}$  also converges pointwise to  $f_\lambda^*$  on  $\beta\bar{X}$ . Since  $f_{y_\alpha}^*$  for all  $\alpha \in A$  and  $f^*$  are continuous and bounded on  $\beta\bar{X}$  and since  $\beta\bar{X}$  is a compact,  $T_2$  space,  $\{f_{y_\alpha}^*\}_{\alpha \in A}$  converges quasiuniformly to  $f_\lambda^*$  [6, p. 268]. Therefore  $\{\ell_{y_\alpha}\}_{\alpha \in A}$  converges quasiuniformly to  $\ell_\lambda$ . As  $\ell_{y_\alpha}$ , for each  $\alpha \in A$ , is bounded and continuous on  $\mathfrak{X}$  in the  $I$ -topology, as  $\ell_\lambda$  is bounded, and as  $\mathfrak{X}$  is a compact,  $T_2$  space in this topology,  $\ell_\lambda$  is a continuous function on  $\mathfrak{X}$  in the  $I$ -topology.

Note that  $\lambda$  was arbitrary. Also,  $\mathfrak{X}$  and  $\mathfrak{Y}$  can be treated symmetrically in the previous proof. So,  $\ell$  is separately continuous in the  $I$ -topologies on  $\mathfrak{X} \times \mathfrak{Y}$ .

*Integration.* Let  $g = \ell \circ (\theta_x \times \theta_y)$ . Hence,  $g$  is separately continuous and bounded on  $X^* \times Y^*$ . Note that  $g|_{\bar{X} \times \bar{Y}} = f$ . By Lemma 3.3,  $g$  is a DLC function.

As  $X^*$  and  $Y^*$  are their own compactifications, Theorem 3.2, part (D), applies to  $g$ . So:

if  $\psi^*(y) \equiv \int_{X^*} g_y d\mu'$  for all  $y \in Y^*$ , then  $\psi^*$  is continuous on  $Y^*$ ;  
 if  $\phi^*(x) \equiv \int_{Y^*} g_x d\nu'$  for all  $x \in X^*$ , then  $\phi^*$  is continuous on  $X^*$ ; and  
 $\int_{Y^*} \psi^* d\nu' = \int_{X^*} \phi^* d\mu$ .

Let  $\psi = \psi^*|_{\bar{Y}}$ , and let  $\phi = \phi^*|_{\bar{X}}$ . Then by Lemma 4.2  $\psi$  is  $\mathbf{B}$ -continuous and  $\phi$  is  $\mathbf{A}$ -continuous. Also

$$\int_{Y^*} \psi^* d\nu' = \int_Y \psi d\nu = \int_Y \int_{X^*} g_y d\mu' d\nu \quad (\text{for } y \in \bar{Y}) = \int_Y \int_X f d\mu d\nu,$$

and

$$\int_{X^*} \phi^* d\mu' = \int_X \phi d\mu = \int_X \int_{Y^*} g_x d\nu' d\mu \quad (\text{for } x \in \bar{X}) = \int_X \int_Y f d\nu d\mu.$$

In conclusion,

$$\int_Y \psi d\nu = \int_Y \int_X f d\mu d\nu = \int_X \int_Y f d\nu d\mu = \int_X \phi d\mu.$$

*The converse implication.* Let  $f$  be a bounded, real-valued function on  $X \times Y$  for which the iterated integrals exist and are equal with respect to every pair of measures  $\mu$  and  $\nu$  on the SP algebras  $\mathbf{A}$  and  $\mathbf{B}$  over the sets  $X$  and  $Y$ , respectively. Let  $\psi_\mu$  be a real-valued function defined on  $Y$  by

$$\psi_\mu(y) \equiv \int_X f_y d\mu \quad \text{for a given measure } \mu \text{ on } \mathbf{A},$$

and let  $\phi_\nu$  be a real-valued function defined on  $X$  by

$$\phi_\nu(x) \equiv \int_Y f_x d\nu \text{ for a given measure } \nu \text{ on } B.$$

Since  $\psi_\mu$  is integrable with respect to any measure  $\nu$  on  $B$  (as the iterated integral exists),  $\psi_\mu$  is  $B$ -continuous by Lemma 4.2. Similarly,  $\phi_\nu$  is  $A$ -continuous. For the same reason,  $f_x$  is  $B$ -continuous for all  $x \in X$  and  $f_y$  is  $A$ -continuous for all  $y \in Y$ . It is left to show that  $f$  is a DLC function.

Recall the notation established in §4.2, part (E). Let  $(x, y) \in X^* \times Y^*$ . Let  $N^{-1}(x) = \mu_x$  (a zero-one measure on  $A$ ), and let  $N^{-1}(y) = \nu_y$  (a zero-one measure on  $B$ ). Let  $J(\mu_x) = \mu_x^*$  and  $J(\nu_y) = \nu_y^*$ . Define:

$$F(x, y) = \int_X \int_Y f d\nu_y d\mu_x \text{ on } X^* \times Y^*.$$

If  $(x, y) \in X \times Y$ , then  $\mu_x$  and  $\nu_y$  are one-point generated (by  $x$  and  $y$ ) measures on  $A$  and  $B$ , respectively. So,

$$F(x, y) = \int_X \int_Y f d\nu_y d\mu_x = \int_X f_y d\mu_x = f_y(x) = f(x, y).$$

Hence,  $F|X \times Y = f$ .

Fix  $x \in X^*$ . Let  $\psi_{\mu_x}^*$  be the extension of  $\psi_{\mu_x}$  to a continuous function on  $Y^*$ . (This is possible as  $\psi_{\mu_x}$  is  $B$ -continuous.) Then

$$\begin{aligned} F_x(y) &= \int_X \int_Y f d\nu_y d\mu_x = \int_Y \int_X f d\mu_x d\nu_y = \int_Y \psi_{\mu_x} d\nu_y \\ &= \int_{Y^*} \psi_{\mu_x}^* d\nu_y^* = \psi_{\mu_x}^*(y), \end{aligned}$$

as  $\nu_y^*$  is one-point generated by  $y \in Y^*$ . Since  $y \in Y^*$  was arbitrary,  $F_x = \psi_{\mu_x}^*$ . Therefore,  $F_x$  is continuous on  $Y^*$ . Similarly,  $F_y$  is continuous on  $X^*$  for an arbitrary  $y \in Y^*$ . So,  $F$  is separately continuous on  $X^* \times Y^*$ .

Note that  $f$  is bounded and that  $X \times Y$  is dense in  $X^* \times Y^*$ . By Lemma 3.1,  $F$  is also bounded. Since  $X^*$  and  $Y^*$  are compact,  $T_2$  spaces, Lemma 3.3 applies. Therefore  $F$  is a DLC function. So is  $f$  as  $F|X \times Y = f$ . Q.E.D.

**4.5. Corollary.** *Let  $A$  and  $B$  be SP algebras over the sets  $X$  and  $Y$ , respectively. Let  $f$  be a real-valued, bounded function on  $X \times Y$ . Let  $X^*$  and  $Y^*$  be the Stone spaces of  $A$  and  $B$ , respectively.*

*Then  $f$  can be extended to a separately continuous, real-valued function on  $X^* \times Y^*$  if and only if:*

- (A)  $f_y$  is  $A$ -continuous for all  $y \in Y$ ;
- (B)  $f_x$  is  $B$ -continuous for all  $x \in X$ ; and
- (C)  $f$  is a DLC function.

**Proof.** If  $f$  satisfies hypotheses (A), (B), and (C), then it was proved "intra" Theorem 4.4 that  $f$  could be extended to a separately continuous function on  $X^* \times Y^*$ .

Suppose  $f$  has a separately continuous extension  $F$  on  $X^* \times Y^*$ . As  $f$  is bounded, Lemma 3.1 implies that  $F$  is also bounded. By Lemma 4.2, hypotheses (A) and (B) must hold. By Lemma 3.3,  $F$  is a DLC function. Hence, as  $f = F|X \times Y$ ,  $f$  is also a DLC function. Q.E.D.

5. Product measures. The following definition is made in reference to Theorem 4.4 and Corollary 4.5.

5.1. Definition. Let  $A$  and  $B$  be SP algebras over the sets  $X$  and  $Y$ , respectively. Let  $f: X \times Y \rightarrow \mathbb{R}$  be a real-valued, bounded function on  $X \times Y$ .  $f$  is a Stone space function (or  $S$ -function)—with respect to  $A$  and  $B$ —if:

- (A)  $f_y$  is  $A$ -continuous for all  $y \in Y$ ;
- (B)  $f_x$  is  $B$ -continuous for all  $x \in X$ ; and
- (C)  $f$  is a DLC function.

5.2. Lemma. Let  $A$  and  $B$  be SP algebras over the sets  $X$  and  $Y$ , respectively. Let  $f$  and  $g$  be  $S$ -functions on  $X$  and  $Y$ . Then:

- (A)  $f + g$ ,  $f \vee g$ , and  $f \wedge g$  are  $S$ -functions.
- (B)  $f^+$ ,  $f^-$ ,  $|f|$ , and  $\alpha f$  for all  $\alpha \in \mathbb{R}$  are  $S$ -functions.
- (C) If  $\{f_n\}_{n=1}^\infty$  is a sequence of  $S$ -functions which converges uniformly to a function  $f$  on  $X \times Y$ , then  $f$  is an  $S$ -function.

Proof. Let  $f$  and  $g$  be  $S$ -functions, and let  $\alpha \in \mathbb{R}$ . Then  $f$  and  $g$  can be extended to functions  $F$  and  $G$ , respectively, which are bounded and separately continuous on  $X^*$  and  $Y^*$ , the Stone spaces with respect to  $A$  and  $B$ , respectively. However,  $F + G$ ,  $F \vee G$ ,  $F \wedge G$ , and  $\alpha F$  are bounded, separately continuous functions on  $X^* \times Y^*$ . Since  $F + G|X \times Y = f + g$ ,  $F \vee G|X \times Y = f \vee g$ ,  $F \wedge G|X \times Y = f \wedge g$ , and  $\alpha F|X \times Y = \alpha f$ ,  $f + g$ ,  $f \wedge g$ ,  $f \vee g$ , and  $\alpha f$  are necessarily  $S$ -functions. (See Corollary 4.5.)

The zero function is clearly an  $S$ -function. By the above statement—as  $f^+ \equiv f \vee 0$ ,  $f^- \equiv f \wedge 0$ , and  $|f| \equiv f^+ + f^- - f^+$ ,  $f^-$ , and  $|f|$  are also  $S$ -functions.

Let  $\{f_n\}_{n=1}^\infty$  be a sequence of  $S$ -functions which converge uniformly to a function  $f$ . Since the sequence is uniformly bounded as each  $f_n$  is bounded,  $f$  must also be bounded (say, by  $m \in \mathbb{R}$ ). Let  $\mu$  and  $\nu$  be any pair of measures on  $A$  and  $B$ , respectively. As uniform convergence implies convergence in measure and as  $f$  and all but a finite number of  $f_n$ 's are dominated by the  $\mu$ - and  $\nu$ -integrable constant function  $m + 1$ , the Dunford-Schwartz dominated convergence theorem [6, p. 124] holds. So,  $\int_X f_y d\mu$  ( $\int_Y f_x d\nu$ ) exists and

$$\lim_{n \rightarrow \infty} \int_X f_{n,y} d\mu = \int_X f_y d\mu \left( \lim_{n \rightarrow \infty} \int_Y f_{n,x} d\nu = \int_Y f_x d\nu \right) \text{ for all } y \in Y \text{ (all } x \in X).$$

Let  $\phi_n(x) \equiv \int_Y f_{n,x} d\nu$ , and let  $\phi(x) = \int_Y f_x d\nu$ . Let  $\epsilon > 0$  be given. Then  $N$  can be found so that, for  $n \geq N$ ,  $|\phi(x) - \phi_n(x)| < \epsilon$  for all  $(x, y) \in X \times Y$ .

Let  $x \in X$  be fixed. Let  $\nu(\lambda)$  be the variation measure [6, p. 97] with respect to a given measure  $\lambda$ . Then,

$$|\phi(x) - \phi_n(x)| \leq \int_Y |f_x(y) - f_{n,x}(y)| d\nu(y) \\ \leq \max_{y \in Y} |f(x, y) - f_n(x, y)| \nu(Y) < \epsilon \nu(Y) \text{ for } n \geq N.$$

Hence,  $\{\phi_n\}_{n=1}^\infty$  converges uniformly to  $\phi$  as  $\epsilon$  does not depend on  $x$ . Let  $\psi_n(y) \equiv \int_X f_y d\mu$ . It can be shown in a similar manner that  $\{\psi_n\}_{n=1}^\infty$  converges uniformly to  $\psi$ .

By use of arguments similar to those above, one can show that  $\int_X \phi d\mu$  and  $\int_Y \psi d\nu$  both exist, that  $\lim_{n \rightarrow \infty} \int_X \phi_n d\mu = \int_X \phi d\mu$  and that  $\lim_{n \rightarrow \infty} \int_Y \psi_n d\nu = \int_Y \psi d\nu$ . Hence,

$$\int_Y \psi d\nu = \lim_{n \rightarrow \infty} \int_Y \psi_n d\nu = \lim_{n \rightarrow \infty} \int_Y \int_X f_n d\mu d\nu = \lim_{n \rightarrow \infty} \int_X \int_Y f_n d\nu d\mu \\ = \lim_{n \rightarrow \infty} \int_X \phi_n d\mu = \int_X \phi d\mu.$$

Since  $\mu$  and  $\nu$  were arbitrary, Theorem 4.4 applies. Therefore,  $f$  is an  $S$ -function. Q.E.D.

5.3. Definition. Let  $A$  and  $B$  be SP algebras over the sets  $X$  and  $Y$ , and let  $\mu$  and  $\nu$  be measures on  $A$  and  $B$ , respectively.

(A) The limit product algebra of  $A$  and  $B$ , denoted by  $A * B$ , is defined by:

$$A * B = \{E \subset X \times Y : \chi_E \text{ is an } S\text{-function with respect to } A \text{ and } B\}.$$

(B) The finitely additive product set function  $\mu$  and  $\nu$  on  $A * B$ , denoted by  $\mu * \nu$ , is defined by:

$$\mu * \nu(E) = \int_X \int_Y \chi_E d\nu d\mu \text{ for all } E \in A * B.$$

5.4. Theorem. Let  $A$  and  $B$  be SP algebras on the sets  $X$  and  $Y$ , and let  $\mu$  and  $\nu$  be measures on  $A$  and  $B$ , respectively. Then:

(A)  $A * B$  contains the measurable rectangles—that is, sets of the form  $E \times F$  for  $E \in A$  and  $F \in B$ .

(B) If  $E \times F$  is a measurable rectangle, then  $\mu * \nu(E \times F) = \mu(E) \nu(F)$ .

(C)  $\mu * \nu(E) = \int_Y \int_X \chi_E d\mu d\nu$  for all  $E \in A * B$ .

(D)  $A * B$  is an algebra.

(E)  $\mu * \nu$  is a measure on  $A * B$ .

(F) If  $\mu$  and  $\nu$  are both nonnegative or zero-one measures, then so is  $\mu * \nu$ .

(G) If  $\mu$  and  $\nu$  are both  $\sigma$ -additive, then so is  $\mu * \nu$ .

**Proof.**  $A * B$  is an algebra containing the measurable rectangles.

Let  $E \times F$  be a measurable rectangle. Let  $\pi$  and  $\lambda$  be any pair of measures on  $A$  and  $B$ , respectively. Then

$$\int_X \int_Y \chi_{E \times F} d\lambda d\pi = \int_X \lambda(F) \chi_E d\pi = \lambda(F) \pi(E),$$

and

$$\int_Y \int_X \chi_{E \times F} d\pi d\lambda = \int_Y \pi(E) \chi_F d\lambda = \pi(E) \lambda(F).$$

By Theorem 4.4,  $\chi_{E \times F}$  is an  $S$ -function. So,  $E \times F \in A * B$ . Part (B) has also been established.

In particular, the rectangles  $X \times Y$  and  $\phi = \phi \times \phi$  are in  $A * B$ . Let  $E, F \in A * B$ . Then  $\chi_E$  and  $\chi_F$  are  $S$ -functions. By Lemma 5.2,  $\chi_E \vee \chi_F = \chi_{E \cup F}$  and  $\chi_E \wedge \chi_F = \chi_{E \cap F}$  are also  $S$ -functions. So,  $E \cup F$  and  $E \cap F$  are in  $A * B$ . As  $\chi_E - \chi_{E \cap F} = \chi_{E - F}$  is also an  $S$ -function,  $E - F \in A * B$ . Hence,  $A * B$  is an algebra.

$\mu * \nu$  is a measure. Parts (C) and (F) are clearly true. Also,  $\mu * \nu(\phi) = \int_X \int_Y 0 d\nu d\mu = 0$ . Further,  $\mu * \nu$  is well defined on  $A * B$ . Let  $E, F \in A * B$  such that  $E \cap F = \emptyset$ . Then  $\chi_{E \cup F} = \chi_E + \chi_F$ . So,

$$\begin{aligned} \mu * \nu(E \cup F) &= \int_X \int_Y \chi_{E \cup F} d\nu d\mu = \int_X \int_Y (\chi_E + \chi_F) d\nu d\mu \\ &= \int_X \int_Y \chi_E d\nu d\mu + \int_X \int_Y \chi_F d\nu d\mu = \mu * \nu(E) + \mu * \nu(F). \end{aligned}$$

Hence,  $\mu * \nu$  is finitely additive on  $A * B$ .

Let  $\nu(\lambda)$  be the variation measure with respect to a given measure  $\lambda$ . Then, given  $E \in A * B$ ,

$$|\mu * \nu(E)| = \left| \int_X \int_Y \chi_E d\nu d\mu \right| \leq \int_X \int_Y 1 d\nu(\nu) d\nu(\mu) = \nu(\nu)(Y) \nu(\mu)(X).$$

Therefore,  $\mu * \nu$  is bounded and is a measure.

If  $\mu$  and  $\nu$  are  $\sigma$ -additive, so is  $\mu * \nu$ . Let  $\mu$  and  $\nu$  be  $\sigma$ -additive. Let  $\{E_n\}_{n=1}^\infty \subset A * B$  be a pairwise disjoint family of sets such that  $\bigcup_{n=1}^\infty E_n \equiv E \in A * B$ . Then

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \chi_{E_n} = \sum_{n=1}^\infty \chi_{E_n} = \chi_E.$$

Fix  $x \in X$ . Then  $\int_Y \sum_{n=1}^N \chi_{E_n}(x, ) d\nu = \sum_{n=1}^N \int_Y \chi_{E_n}(x, ) d\nu$  exists for all  $N \geq 1$ , and  $\int_Y \chi_E(x, ) d\nu$  also exists. By use of the Lebesgue dominated convergence theorem for algebras, one can conclude that

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \int_Y \chi_{E_n}(x, ) d\nu = \int_Y \chi_E(x, ) d\nu.$$

Let  $\phi_N(x) \equiv \sum_{n=1}^N \int_Y \chi_{E_n}(x, ) d\nu$ , and let  $\phi(x) \equiv \int_Y \chi_E(x, ) d\nu$ . Note that  $\lim_{N \rightarrow \infty} \phi_N = \phi$ . Also  $\int_X \phi_N d\mu = \int_X \sum_{n=1}^N \int_Y \chi_{E_n} d\nu d\mu = \sum_{n=1}^N \int_X \int_Y \chi_{E_n} d\nu d\mu$  for all  $N \geq 1$  and  $\int_X \phi d\mu = \int_X \int_Y \chi_E d\nu d\mu$  exist. By use of the Lebesgue dominated convergence theorem for algebras a second time, it is seen that

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \int_X \int_Y \chi_{E_n} d\nu d\mu = \int_X \int_Y \chi_E d\nu d\mu.$$

Hence,

$$\mu * \nu(E) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \mu * \nu(E_n) = \sum_{n=1}^{\infty} \mu * \nu(E_n).$$

So,  $\mu * \nu$  is  $\sigma$ -additive. Q.E.D.

5.5. Remark. If  $\mu * \nu$  is nonnegative, then  $\mu * \nu$  induces a finitely additive [18] outer measure on  $X \times Y$ . If  $\mu * \nu$  is  $\sigma$ -additive, then it induces the usual outer measure on  $X \times Y$ . So, the algebra  $A * B$  can be enlarged in these cases and can always be completed (enlarged to include all subsets of sets of variation measure zero). However, due to the converse implication of Theorem 4.4,  $A * B$  is the largest product algebra upon which arbitrary product measures can be compared.

5.6. Example. This example will show that  $A * B$  is not, in general, a  $\sigma$ -algebra. Let  $X = Y = \mathcal{N}$ , the natural numbers. Let  $A = B = \{E \subset \mathcal{N}: E \text{ or } \mathcal{N} - E \text{ is finite}\}$ . Consider the measures  $\mu = \nu$  defined by:

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is finite} \\ 1, & \text{if } \mathcal{N} - E \text{ is finite} \end{cases} \quad \text{for } E \in A = B.$$

Let  $E = \bigcup_{n=1}^{\infty} \{n\} \times \bigcup_{i=n}^{\infty} i$ . Let  $\sigma(A \times B)$  be the  $\sigma$ -algebra generated by the measurable rectangles.

Then,  $E \in \sigma(A \times B)$ . But,

$$\int_X \int_Y \chi_E d\nu d\mu = \int_X \sum_{n=1}^{\infty} \chi_n d\mu = \int_X \chi_X d\mu = 1$$

and

$$\int_Y \int_X \chi_E d\mu d\nu = \int_Y 0 d\nu = 0.$$

Therefore,  $\chi_E$  is not an  $S$ -function, and  $E \notin A * B$ .

5.7. Example. The following example will show that  $A * B$  can contain sets not measurable with respect to the classical product algebra. Let  $X = Y = [0, 1]$ ;  $A = B =$  the Lebesgue measurable sets in  $[0, 1]$ ;  $\mu = \nu =$  Lebesgue measure.

Sierpiński [15] has constructed a set  $E \subset [0, 1] \times [0, 1]$  which intersects any vertical or horizontal line in at most two points. Further, this set is not Lebesgue measurable with respect to Lebesgue measure on the plane.

A little reflection will show that  $\chi_E$  is a DLC function. Also,  $\chi_E$  is  $C$ -continuous (for fixed  $x$  or fixed  $y$ ) with respect to any algebra  $C$  which contains the singleton sets in  $[0, 1]$ . Hence,  $\chi_E$  is an  $S$ -function and  $E \in A * B$ .

Note that  $E$  is measurable with respect to the Bledsoe-Morse product outer measure [3]. The relationship between the outer measure induced by  $\mu * \nu$  on  $A * B$  and the Bledsoe-Morse outer measure is not known.

6. A "Fubini" theorem. The following theorem, although seemingly trivial, is not true for all extensions of product measures on the measurable rectangles. An example where this theorem is false is given after the theorem.

6.1. Theorem. Let  $A$  and  $B$  be SP algebras on the sets  $X$  and  $Y$ , and let  $\mu$  and  $\nu$  be measures on  $A$  and  $B$ , respectively. Let  $f: X \times Y \rightarrow \mathbb{R}$  be a real-valued, bounded function on  $X \times Y$ .

If  $f$  is a uniform limit of simple,  $A * B$ -measurable functions, then

$$\int_X \int_Y f d\nu d\mu = \int_Y \int_X f d\mu d\nu = \int_{X \times Y} f d(\mu * \nu).$$

Proof. Let  $f$  be a uniform limit of a sequence  $\{f_n\}_{n=1}^\infty$  of simple,  $A * B$ -measurable functions. Then each of the  $f_n$ 's is an  $S$ -function and, by Lemma 5.2, so is  $f$ . Therefore, Theorem 4.4 guarantees the existence and equality of the iterated integrals

$$\int_X \int_Y f d\nu d\mu = \int_Y \int_X f d\mu d\nu \quad \text{and} \quad \int_X \int_Y f_n d\nu d\mu = \int_Y \int_X f_n d\mu d\nu$$

for all  $n$ .

Due to the uniform convergence of the  $f_n$ 's and the Dunford-Schwartz dominated convergence theorem, the existence of the integral  $\int_{X \times Y} f d(\mu * \nu)$  and the limit  $\int_{X \times Y} f d(\mu * \nu) = \lim_{n \rightarrow \infty} \int_{X \times Y} f_n d(\mu * \nu)$  is also guaranteed.

Let  $f$  be a simple,  $A * B$ -measurable function ( $f = \sum_{n=1}^N a_n \chi_{E_n}$  for  $E_n \in A * B$ ). Hence,

$$\begin{aligned} \int_{X \times Y} f d(\mu * \nu) &= \sum_{n=1}^N a_n \int_{X \times Y} \chi_{E_n} d(\mu * \nu) = \sum_{n=1}^N a_n \mu * \nu(E_n) \\ &= \sum_{n=1}^N a_n \int_X \int_Y \chi_{E_n} d\nu d\mu = \int_X \int_Y \sum_{n=1}^N a_n \chi_{E_n} d\nu d\mu \\ &= \int_X \int_Y f d\nu d\mu. \end{aligned}$$

So, the theorem is true in this case.

Consider the more general sense. Note that  $\int_{X \times Y} f_n d(\mu * \nu) = \int_X \int_Y f_n d\nu d\mu$  for all  $n$ . Let  $v(\lambda)$  be the variation measure with respect to a give measure  $\lambda$ . Then,

$$\left| \int_X \int_Y f d\nu d\mu - \int_X \int_Y f_n d\nu d\mu \right| \leq \max_{(x,y) \in X \times Y} |(f - f_n)(x,y)| v(\nu)(Y) v(\mu)(X).$$

Since  $\{f_n\}_{n=1}^\infty$  converges uniformly to  $f$ ,  $\lim_{n \rightarrow \infty} \int_X \int_Y f_n d\nu d\mu = \int_X \int_Y f d\nu d\mu$ .

Let  $\epsilon > 0$  be given. Then, for  $N$  large enough and for all  $n \geq N$ ,

$$\begin{aligned} \left| \int_{X \times Y} f d(\mu * \nu) - \int_X \int_Y f dv d\mu \right| &\leq \left| \int_{X \times Y} f d(\mu * \nu) - \int_{X \times Y} f_n d(\mu * \nu) \right| \\ &\quad + \left| \int_{X \times Y} f_n d(\mu * \nu) - \int_X \int_Y f_n dv d\mu \right| \\ &\quad + \left| \int_X \int_Y f_n dv d\mu - \int_X \int_Y f dv d\mu \right| \\ &< \epsilon. \end{aligned}$$

Hence,  $\int_{X \times Y} f d(\mu * \nu) = \int_X \int_Y f dv d\mu$ . Q.E.D.

In the following two examples both iterated integrals and the product integral will exist. In the first example the two iterated integrals will be equal to each other, but not to the product integral. In the second example no two integrals will be equal.

6.2. Example. Let  $X, Y, A, B$ , and  $\mu$  be defined as in Example 5.6. Let  $\nu$  be defined by

$$\nu(E) = \begin{cases} \sum_{n \in E} 1/n^2, & \text{if } E \text{ is finite} \\ 3 - \sum_{n \in \mathbb{N} - E} 1/n^2, & \text{otherwise} \end{cases} \quad \text{for } E \in \mathcal{B}.$$

Let  $A_1$  be the smallest algebra containing the  $A$  and  $B$  measurable rectangles. Let  $\mu \times \nu$  be the unique product measure on  $A_1$  such that  $\mu \times \nu(E \times F) = \mu(E)\nu(F)$  for each measurable rectangle  $E \times F$ .

Let  $G_1 = \bigcup_{n=1}^{\infty} \{n\} \times \{n\}$ . Extend  $A_1$  to  $A_2$ , an algebra containing  $A_1$  and  $G_1$ , via the Tarski extension method [2, p. 185]—that is,

$$A_2 = \{ (G_1 \cap F_1) \cup (G_1' \cap F_2) : F_1, F_2 \in A_1 \}.$$

Through this method  $\mu \times \nu$  can be extended to a measure  $\pi$  on  $A_2$  by

$$\pi(F) = \pi[(G_1 \cap F_1) \cup (G_1' \cap F_2)] = \inf_{E \in A_1; G \cap F_1 \subset E} \mu \times \nu(E) + \sup_{E \in A_1; E \subset G' \cap F_2} \mu \times \nu(E)$$

for all  $E \in A_2$ . Note that  $\pi(G_1) = 3 - \pi^2/6$ .

Now,

$$\int_X \int_Y \chi_{G_1} d\pi(y) d\pi(x) = 0 = \int_Y \int_X \chi_{G_1} d\pi(x) d\pi(y),$$

but

$$\int_{X \times Y} \chi_{G_1} d\pi = 3 - \pi^2/6.$$

6.3. Example. Use the same notation as Example 6.2. Let  $G_2 = \bigcup_{n=2}^{\infty} (\{n\} \times \bigcup_{i=1}^{n-1} i)$ . Note that  $\int_X \int_Y \chi_{G_2} dv d\mu = \pi^2/6$  and  $\int_Y \int_X \chi_{G_2} d\mu dv = 3$ . Let  $A_3$  and  $\lambda$  be defined by

$$A_3 = \{(G_2 \cap F_1) \cup (G'_2 \cap F_2) : F_1, F_2 \in A_2\}$$

and

$$\lambda(G) = \lambda[(G_2 \cap F_1) \cup (G'_2 \cap F_2)] = \inf_{E \in A_2; G_2 \cap F_1 \subseteq E} \pi(E) + \sup_{E \in A_2; E \subseteq G'_2 \cap F_2} \pi(E)$$

for all  $E \in A_3$ . Note that  $\lambda(G_2) = \pi^2/6$ .

Let  $H = \chi_{G_2} + 2\chi_{G_1}$ . Then:  $\int_X \int_Y H d\lambda(y) d\lambda(x) = \pi^2/6$ ;  $\int_Y \int_X H d\lambda(x) d\lambda(y) = 3$ ; and  $\int_{X \times Y} H d\lambda = \lambda(G_2) + 2\lambda(G_1) = 6 - \pi^2/6$ .

**Note added in proof.** The direct implication of Theorem 4.4 was previously announced by N. J. Young, using a different proof [Proc. Edinburgh Math. Soc. 193 (1973), 199. (Zbl. 237, #46009)]. However, Dr. Young's hypotheses "measurable and bounded" should be replaced by "A-continuous" for the non-countably additive case. (See [4] for a discussion of this point.)

#### REFERENCES

1. M. Alić and B. Kronfeld, *A remark on finitely additive measures*, Glasnik Mat. Ser. III 4 (24) (1969), 197–200. MR 41 #415.
2. G. Birkhoff, *Lattice theory*, 2nd rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R. I., 1948. MR 10, 673.
3. W. W. Bledsoe and A. P. Morse, *Product measures*, Trans. Amer. Math. Soc. 79 (1955), 173–215. MR 16, 1008.
4. R. B. Darst, *A note on abstract integration*, Trans. Amer. Math. Soc. 99 (1961), 292–297. MR 22 #11092.
5. ———, *A decomposition of complete normed Abelian groups with applications to spaces of additive set functions*, Trans. Amer. Math. Soc. 103 (1962), 549–558. MR 25 #1256.
6. N. Dunford and J. T. Schwartz, *Linear operators. I: General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.
7. G. Fichtenholz, *Un théorème sur l'intégration sous le signe intégrale*, Rend. Circ. Mat. Palermo 36 (1913), 111–114.
8. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR 22 #6994.
9. S. Leader, *On universally integrable functions*, Proc. Amer. Math. Soc. 6 (1955), 232–234. MR 16, 805.
10. L. Lichtenstein, *Ueber die Integration eines bestimmten Integrals in Bezug auf einen Parameter*, Nach. Akad. Wiss. Göttingen Math.-Phys. Kl. (1910), 468–475.
11. P. de Lucia, *Sull'inversione dell'ordine d'integrazione*, Rend. Acad. Sci. Fis. Mat. Napoli (4) 33 (1966), 473–487. MR 37 #366.
12. W. A. J. Luxemburg, *The abstract Riemann integral and a theorem of G. Fichtenholz on equality of repeated Riemann integrals IA and IB*, Nederl. Akad. Wetensch. Proc. Ser. A 64 = Indag. Math. 23 (1961), 516–545. MR 25 #166a, b.
13. P. Porcelli, *Adjoint spaces of abstract  $L_p$  spaces*, Portugal. Math. 25 (1966), 105–122. MR 35 #720.

14. V. Pták, *An extension theorem for separately continuous functions and its application to functional analysis*, Czechoslovak Math. J. 14 (89) (1964), 562–581. (Russian) MR 30 #2334.

15. W. Sierpiński, *Sur un problème concernant les ensembles mesurables superficiellement*, Fund. Math. 1 (1920), 112–115.

16. R. Sikorski, *Boolean algebras*, 2nd ed., Ergebnisse der Math und ihrer Grenzgebiete, Band 25, Academic Press, New York; Springer-Verlag, Berlin, 1964. MR 31 #2178.

17. G. E. Sinclair, *A finitely additive extension of the Fichtenholz-Lichtenstein theorem for Riemann integrals*, Ph.D. dissertation under Dr. J. S. Lomont at the University of Arizona, Tucson, Arizona, 1972.

18. B. C. Strydom, *Abstract Riemann integration*, Getal en figuur, 10, Van Gorcum, Assen, Netherlands, 1959. MR 21 #4216.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARIZONA, TUCSON, ARIZONA 85719

*Current address:* 1212 Via Pavion, Santa Maria, California 93454