ABSTRACT. We consider a bounded domain \( \Omega \) in \( \mathbb{C}^n \) and the Banach algebra \( A(\Omega) \) of all continuous functions on \( \overline{\Omega} \) which are holomorphic on \( \Omega \). Fix \( f_1, \ldots, f_k \in A(\Omega) \). We say they are a set of generators if \( A(\Omega) \) is the smallest closed subalgebra containing the \( f_i \). We restrict attention to the case when \( \Omega \) is strictly pseudoconvex and smoothly bounded and the \( f_i \) are smooth on \( \overline{\Omega} \). In this case, Theorem 1 below gives conditions assuring that a given set \( f_i \) is a set of generators.

1. Introduction. Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \). \( A(\Omega) \) denotes the algebra of all continuous complex-valued functions on \( \overline{\Omega} \) which are holomorphic on \( \Omega \). With \( \|f\| = \max_{\Omega} |f| \), \( A(\Omega) \) is a Banach algebra.

Fix \( f_1, \ldots, f_k \in A(\Omega) \). Denote by \( [f_1, \ldots, f_k]_\Omega \) the uniform closure on \( \overline{\Omega} \) of the algebra of all polynomials in \( f_1, \ldots, f_k \). We say that the \( f_i \) form a set of generators for \( A(\Omega) \) if \( [f_1, \ldots, f_k]_\Omega = A(\Omega) \).

Our problem is to decide when a given set \( f_1, \ldots, f_k \) is a set of generators for \( A(\Omega) \).

Two immediate necessary conditions are:

1. The \( f_i \) separate points on \( \overline{\Omega} \).
2. The matrix \( \left( \frac{\partial f_i}{\partial z_j} \right) \), \( 1 \leq i \leq k, 1 \leq j \leq n \), has rank \( n \) at each point \( z \in \Omega \).

Note that (1) implies that \( k \geq n \).

To be able to find sufficient conditions, we impose the following restrictions on \( \Omega \): \( \exists \) a function \( \rho \) of class \( C^4 \) and strictly plurisubharmonic in some neighborhood of \( \overline{\Omega} \) such that \( \Omega = \{z \mid \rho(z) < 0\} \), and \( \text{grad } \rho \neq 0 \) on \( \partial \Omega \).

In this case it is known (see Appendix (A.1)) that the spectrum of the Banach algebra \( A(\Omega) \) coincides with \( \overline{\Omega} \).

Fix \( f_1, \ldots, f_k \in A(\Omega) \). Put \( K = \{(f_1(z), \ldots, f_k(z)) \mid z \in \overline{\Omega}\} \). In order that the \( f_i \) be a set of generators it is now necessary that

3. \( K \) is polynomially convex in \( \mathbb{C}^k \).

We shall also assume that the \( f_i \) are smooth up to the boundary of \( \Omega \). For each multi-index \( I = (i_1, \ldots, i_n) \), put

\[
|I| = \sum_{r=1}^{n} i_r \quad \text{and} \quad D^I = \frac{\partial|I|}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}}.
\]

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Definition 1. Fix an integer $\sigma > 0$. $A^\sigma (\Omega)$ denotes the class of all functions $f$ on $\Omega$ such that, for each multi-index $I$ with $0 \leq |I| \leq \sigma$, $D^I f \in A(\Omega)$.

With the norm

$$
\|f\| = \sum_I \frac{1}{|I|!} \max_{\Omega} |D^I f|,
$$

where the sum is taken over all $I$ with $0 \leq |I| \leq \sigma$ and $I! = i_1! \cdot i_2! \cdots i_n!$ if $I = (i_1, \ldots, i_n)$, $A^\sigma (\Omega)$ is a Banach algebra.

For $f \in A^\sigma (\Omega)$, $(\partial f / \partial z_i)(z)$ is defined for each $z \in \partial \Omega$ by

$$
\frac{\partial f}{\partial z_i}(z) = \lim_{\zeta \to z} \frac{\partial f(\zeta)}{\partial z_i},
$$

where $\zeta \to z$ from inside $\Omega$, the limit existing by definition of $A^\sigma (\Omega)$. We can hence consider the matrix $((\partial f / \partial z_i))$ at all points of $\partial \Omega$.

Observation 1. Fix $\sigma > 0$. For $f_1, \ldots, f_k \in A^\sigma (\Omega)$ conditions (1), (2), (3) fail to be sufficient even when $n = 1$ and $\Omega$ is the unit disk: $|z| < 1$. We give an example in the Appendix (A.3).

We therefore strengthen condition (2) to

(2') The matrix $((\partial f / \partial z_i))$ has rank $n$ for all $z \in \partial \Omega$.

Theorem 1. Fix $\sigma \geq 4$. Let $\Omega$ be as above. Fix $f_1, \ldots, f_k \in A^\sigma (\Omega)$ such that (1), (2'), (3) are satisfied. Then $f_1, \ldots, f_k$ are a set of generators for $A(\Omega)$.

Note. Our hypotheses on $\Omega$ are satisfied by all smoothly bounded strictly convex sets in $\mathbb{C}^n$, in particular by balls.

Theorem 1 admits the following generalisation:

Theorem 1 bis. Let $\Omega$ be as in Theorem 1. Let $\mathfrak{A}$ be a closed subalgebra of $A(\Omega)$ which contains a family $\mathfrak{F}$ of functions such that:

(i) $\mathfrak{F} \subset A^\sigma (\Omega)$.

(ii) $\mathfrak{F}$ separates points on $\Omega$.

(iii) For each $z \in \partial \Omega$, $\exists$ a finite subset $f_1(\zeta), \ldots, f_k(\zeta)$ of $\mathfrak{F}$ such that $((\partial f_j(\zeta) / \partial z_i))$ has rank $n$ at $z$.

(iv) The spectrum of $\mathfrak{A}$ is $\partial \Omega$.

Then $\mathfrak{A} = A(\Omega)$.

2. Modules over a Banach algebra. $A$ is a commutative semisimple Banach algebra with unit 1. Its spectrum is denoted $\sigma(A)$. Fix $k$.

$A^k$ denotes the $A$-module of all $k$-tuples $(a_1, \ldots, a_k)$ of elements $a_i \in A$.

Definition 2.1. Fix $l < k$, and fix elements $\xi^1, \ldots, \xi^l \in A^k$. We say the set $\xi^1, \ldots, \xi^l$ admits a completion if $\exists \xi^{l+1}, \ldots, \xi^{k} \in A^k$ such that $\xi^1, \xi^2, \ldots, \xi^l, \xi^{l+1}, \ldots, \xi^{k}$ is a module basis for $A^k$.

If $\xi = (\xi_1, \ldots, \xi_k) \in A^k$ and $M \in \Omega(k)$, we set $\xi(M) = (\xi_1(M), \ldots, \xi_k(M)) \in \mathbb{C}^k$, where, for $a \in A, a(M)$ denotes the value at $M$ of the Gel'fand transform.
of \( a \). It is clear that, given \( \xi^1, \ldots, \xi^l \in A^k \), a necessary condition in order that this set admits a completion is

(5) \( \xi^1(M), \ldots, \xi^l(M) \) are linearly independent in \( C^k \).

We ask: Under what restrictions is (5) a sufficient condition in order that \( \xi^1, \ldots, \xi^l \) admits a completion?

Our work is based on results of Forster [1], regarding finitely generated projective modules over a Banach algebra \( A \).

**Definition 2.2.** An \( A \)-module \( Q \) is a \( p \)-module (finitely generated projective module) if \( \exists n \) and a direct sum decomposition \( A^n = P \oplus Q \), where \( P \) is another module.

\( Q \) is free if it has a basis.

**Definition 2.3.** Let \( P \) be a \( p \)-module. \( P \) has rank \( k \) if for every \( M \in \mathfrak{M} \), the vector space \( P/MP \) has dimension \( k \) over the field \( A/M \cong C \).

**Lemma 2.1.** Assume \( \mathfrak{M} \) is connected. Then every \( p \)-module \( P \) over \( A \) has some rank \( k \).

**Proof.** \( \exists n \) and a decomposition

(6) \( A^n = P \oplus Q \).

Fix \( M \in \mathfrak{M} \). We claim:

\( P/MP \) is isomorphic to the vector space \( V_M = \{ \xi(M) \mid \xi \in P \} \).

For let \( \xi_1, \xi_2 \) be elements of \( P \) congruent mod \( MP \). Then

\[
\xi_1 - \xi_2 = \sum_{i=1}^{l} m_i p_i, \quad m_i \in M, p_i \in P.
\]

Hence \( \xi_1(M) = \xi_2(M) \). So the map \( [\xi] \rightarrow \xi(M) \) is well defined from the elements of \( P/MP \) to \( V_M \). It is evidently linear and surjective. Suppose, for some \( \xi \in P \), \( [\xi] \rightarrow 0 \), i.e., \( \xi(M) = 0 \). Let \( E_1, \ldots, E_n \) be the standard basis for \( A^n \). \( E_i = p_i + q_i, p_i \in P, q_i \in Q \), for each \( i \). We have

\[
\xi = (\xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} \xi_i E_i.
\]

For each \( i \), \( \xi_i(M) = 0 \), so \( \xi_i \in M \).

\[
\xi = \sum_{i=1}^{n} \xi_i(p_i + q_i) = \sum \xi_i p_i + \sum \xi_i q_i.
\]

Since (6) is a direct sum decomposition, \( \xi = \sum \xi_i p_i \in MP \), so \( [\xi] = 0 \).

Thus the map is injective, and the claim is proved. Put \( W_M = \{ \xi(M) \mid \xi \in Q \} \).

By (6), we have

(7) \( C^n = V_M \oplus W_M \).

That (7) is a direct sum decomposition is seen by the preceding argument. Thus \( n = \dim V_M + \dim W_M \). Fix \( M_0 \in \mathfrak{M} \) and put \( l = \dim V_{M_0} \). Thus \( \exists \) elements
\( \xi^1, \ldots, \xi^l \in P \) with \( \xi^i(M_0) \), \( \ldots, \xi^i(M) \) linearly independent. Hence for some choice of indices \( i_1, i_2, \ldots, i_l \), the determinant

\[
D(M) = \begin{vmatrix} 
\xi^1_i(M) & \cdots & \xi^l_i(M) \\
\vdots & & \vdots \\
\xi^1_i(M) & \cdots & \xi^l_i(M) 
\end{vmatrix} \neq 0
\]

when \( M = M_0 \). By continuity, \( D(M) \neq 0 \) for all \( M \) in some neighborhood of \( M_0 \) in \( M \). Hence \( \dim V_M \geq l \) for all \( M \) in some neighborhood of \( M_0 \). Similarly, \( \dim W_M \geq \dim W_{M_0} = n - l \) for all \( M \) in some neighborhood. But \( \dim V_M + \dim W_M = n \) for all \( M \). Hence \( \dim V_M = l \) for all \( M \) in some neighborhood of \( M_0 \).

Thus \( \dim V_M \) is locally constant on \( M \). Since \( M \) is connected, \( \exists k \) with \( \dim V_M = k \) for all \( M \in M \). Since \( P/MP \) is isomorphic to \( V_M \), we have \( \dim P/MP = k \) for all \( M \in M \). Q.E.D.

**Definition 2.4.** Let \( P \) be a \( \mathcal{P} \)-module \( \subseteq A^n \). By \( P \otimes C(\mathcal{R}) \), we denote the collection of all finite sums \( \sum_{i=1}^{l} f_i p_i, f_i \in C(\mathcal{R}), p_i \in P \), regarded as elements of \( (C(\mathcal{R}))^n \).

Note that \( A^n \otimes C(\mathcal{R}) \cong (C(\mathcal{R}))^n \).

We shall use the following two results from [1]:

**Proposition F.1.** Let \( Q \) be a \( \mathcal{P} \)-module. If \( Q \otimes C(\mathcal{R}) \) is free as a \( C(\mathcal{R}) \)-module, then \( Q \) is free as an \( A \)-module.

**Proof.** Satz 6 of [1].

**Proposition F.2.** Let the Banach algebra \( A \) have \( \mathcal{P} \) topological generators. Let \( P_1, P_2 \) be two \( \mathcal{P} \)-modules of rank \( k \) such that, for some \( l \),

\[
P_1 \oplus A^l \cong P_2 \oplus A^l.
\]

If \( k \geq [\rho/2] \), then \( P_1 \cong P_2 \).

**Proof.** Satz 10 of [1].

**Lemma 2.2.** \( A \) is a Banach algebra such that \( \mathcal{R} \) is a connected subset of \( C^n \). \( Q \) is a \( \mathcal{P} \)-module over \( A \) such that

\[
(\star) \quad A^k = P \oplus Q, \quad \text{where } P \cong A^l
\]

for some \( k, l \). If \( k - l \geq n \), then \( Q \) is free.

**Proof.** Tensoring (\( \star \)) with \( C = C(\mathcal{R}) \) gives

\[
C^k = C^l \oplus \{ Q \otimes C \}.
\]

The rank of \( Q \otimes C \) (as \( C \)-module) is \( k - l \). \( C(\mathcal{R}) \) possesses \( 2n \) topological generators, since \( \mathcal{R} \subset C^n \). \( k - l \geq n = [2n/2] \). Also
GENERATORS FOR $A(\omega)$

$$C^I \oplus C^{k-I} = C^I \oplus \{Q \otimes C\}.$$  

By Proposition F.2, it follows that

$$C^{k-I} \cong Q \otimes C.$$  

Thus $Q \otimes C$ is free as $C$-module. By Proposition F.1, this implies that $Q$ is free as $A$-module. Q.E.D.

**Theorem 2.1.** Let $A$ be a Banach algebra with spectrum $\mathbb{R}$ such that $\mathbb{R}$ is a connected subset of $C^n$. Fix $k, l$ satisfying

$$(8) \quad k \geq 2n, \quad l \leq n.$$  

Fix $\xi^1, \ldots, \xi^l \in A^k$ satisfying (5). Then this set admits a completion.

**Proof.** We hold $n$ fixed and proceed by induction on $l$.

Let $l = 1$; we are given $\xi^1 \in A^k, \xi^1(M) \neq 0$ for all $M \in \mathbb{R}$. $\xi^1 = (\xi^1, \ldots, \xi^1)$. Since the $\xi^1$ have no common zero on $\mathbb{R}$, $\exists a \in A$ with $\sum_{j=1}^l a_j \xi^1 = 1$.

Define a map $\varphi: A^k \to A$ by

$$\varphi(x_1, \ldots, x_k) = \sum_{j=1}^k a_j x_j.$$  

Then $\varphi$ is a homomorphism and $\varphi(\xi^1) = 1$. If $\xi \in A^k, \xi - \varphi(\xi) \cdot \xi^1 \in \ker \varphi$, so

$$\xi = \varphi(\xi) \cdot \xi^1 + r, \quad r \in \ker \varphi.$$  

If $\xi = t \cdot \xi^1 + r, \quad t \in A, \quad r' \in \ker \varphi$, then $t = \varphi(t) = r$. Thus we have the direct sum decomposition $A^k = \{\xi^1\} \oplus \ker \varphi$ where $\{\xi^1\}$ is the cyclic module generated by $\xi^1$. $\{\xi^1\}$ is isomorphic to $A^k$, since $a\xi^1 = 0$ only if $a = 0$, in view of the fact that $\xi^1(M) \neq 0$ for all $M$. Lemma 2.2 applies since, by (8), $k - 1 \geq n$, and so $\ker \varphi$ is free.

Thus $\ker \varphi$ has a basis $\xi^2, \ldots, \xi^k$ and so the set $\xi^1, 1 \leq i \leq k$, is the required completion of $\xi^1$.

Now, assume the theorem is true when $l$ is replaced by $l - 1$ and consider a set $\xi^1, \ldots, \xi^l$ satisfying (5) and $l \leq n$. Then $\xi^1, \ldots, \xi^{l-1}$ satisfies (5). By the induction hypothesis, $\exists \eta^1, \eta^{l+1}, \ldots, \eta^k \in A^k$ such that $\xi^1, \ldots, \xi^{l-1}, \eta^1, \ldots, \eta^k$ forms a basis of $A^k$. Denote by $Q$ the module generated by $\eta^1, \ldots, \eta^k$. We have the direct sum decomposition

$$(9) \quad A^k = \{\xi^1\} \oplus \cdots \oplus \{\xi^{l-1}\} \oplus Q.$$  

In particular, $\exists c_r \in A, \quad q \in Q$ with

$$(10) \quad \xi^l = \sum_{r=1}^{l-1} c_r \xi^r + q.$$  

For each $M$ with $q(M) = 0$, we have

$$\xi^l(M) = \sum_{r=1}^{l-1} c_r(M) \xi^r(M),$$
contrary to (5). Hence $q(M) \neq 0$, for all $M$. Also $Q \cong A^{k-l+1}$. We can hence apply to $Q$ and $q$ the reasoning used above on $A^k$ and $\xi^1$ and obtain a homomorphism $\varphi': Q \to A$, with $Q = \{q\} \oplus \ker \varphi'$. The rank of $Q = k - l + 1$, so the rank of $\ker \varphi' = k - l$. But, by (8), $k - l \geq n$. So, by Lemma 2.2, $\ker \varphi'$ is free, and has rank $k - l$. Let $\xi^1, \ldots, \xi^k$ be a basis for $\ker \varphi'$.

We claim that $\{\xi^i | 1 \leq i \leq k\}$ is a basis for $A^k$. This follows directly from (9) and (10). The induction is complete, so the theorem is proved.

The application of Theorem 2.1 which we will require is the following.

**Theorem 2.2.** Let $\Omega$ be a domain in $C^n$ as in Theorem 1. Fix $\alpha > 0$. Let $(\xi^1, \ldots, \xi^n), \ldots, (\xi^1, \ldots, \xi^n)$ be $n$ $k$-tuples of elements of $A^n(\Omega)$, where $k \geq 2n$. Assume that the matrix

$$
\begin{pmatrix}
\xi^1 & \cdots & \xi^1 \\
\vdots & \ddots & \vdots \\
\xi^n & \cdots & \xi^n
\end{pmatrix}
$$

has rank $n$ at each point of $\Omega$. Then $\exists k - n$ $k$-tuples of elements of $A^n(\Omega)$, $(a_1, \ldots, a_k), \ldots, (a_1^{k-n}, \ldots, a_k^{k-n})$ such that the determinant

$$
\begin{vmatrix}
\xi^1 & \cdots & \xi^1 \\
\vdots & \ddots & \vdots \\
\xi^n & \cdots & \xi^n \\
a_1 & \cdots & a_k \\
\vdots & \cdots & \vdots \\
a_1^{k-n} & \cdots & a_k^{k-n}
\end{vmatrix}
$$

equals 1. We are done.

**Proof.** The spectrum of $A^\alpha$ coincides with $\overline{\Omega}$. (See Appendix (A.2).) Thus the spectrum is a connected subset of $C^n$. The assumption on the matrix $((\xi^i))$ tells us that the vectors $\xi^i(M) = (\xi^i_1(M), \ldots, \xi^i_n(M)), i = 1, \ldots, n$, are independent at each $M$ in the spectrum, i.e., satisfy (5).

Theorem 2.1 thus applies, and yields $\xi^{n+1}, \ldots, \xi^k \in (A^n(\Omega))^k$ which together with $\xi^1, \ldots, \xi^n$ form a basis of $(A^n(\Omega))^k$. Elementary algebra now gives that the determinant

$$
\begin{vmatrix}
\xi^1 \\
\vdots \\
\xi^k
\end{vmatrix}
$$

is a unit in the ring $A^n(\Omega)$. Without loss of generality that determinant then equals 1. We are done.

3. **Proof of Theorems 1 and 1 bis.** Let now $\Omega$ and $f_1, \ldots, f_k$ satisfy the hypothesis of Theorem 1. If $k < 2n$, define $f_{k+1} = \cdots = f_{2n} = f_k$. Then the set
$f_1, \ldots, f_{2n}$ again satisfies (1), (2'), (3). Also

$$[f_1, \ldots, f_k | \Omega] = [f_1, \ldots, f_{2n} | \Omega].$$

If we can show that $[f_1, \ldots, f_{2n} | \Omega] = A(\Omega)$, then of course $[f_1, \ldots, f_k | \Omega] = A(\Omega)$. Thus it is no loss of generality to suppose $k \geq 2n$, and we do so from now on.

For $i = 1, \ldots, n$, put $p_i = (\partial f_i/\partial z_1, \ldots, \partial f_k/\partial z_i)$. Each $\partial f_i/\partial z_i \in A^{e-1}(\Omega)$. Put $A = A^{e-1}(\Omega)$. Then $p_i \in A^k$ for $i = 1, 2, \ldots, n$. Hypothesis (2') gives exactly that, for each $M \in \Omega$, the vectors $p_1(M), \ldots, p_n(M)$ are independent in $C^k$.

By Theorem 2.2, we can select vectors such that the determinant

$$\begin{vmatrix}
\partial f_1/\partial z_1 & \cdots & \partial f_k/\partial z_1 \\
\vdots & \ddots & \vdots \\
\partial f_1/\partial z_n & \cdots & \partial f_k/\partial z_n \\
\partial f_1/\partial z_{k-n} & \cdots & \partial f_k/\partial z_{k-n}
\end{vmatrix} = 1$$

identically on $\Omega$. Each $a_j' \in A = A^{e-1}(\Omega)$. We choose an open neighborhood $U$ of $\Omega$ in $C^*$ such that each of the functions $f_i$, $1 \leq i \leq k$, admits an extension to $U$ lying in $C^e(U)$ and each $a_j'$ admits an extension to an element of $C^{e-1}(U)$.

We form the product domain in $C^k$: $U \times C^{k-n} = \{(z, w) \mid z \in U, w \in C^{k-n}\}$. We define a map $\chi$ of $U \times C^{k-n}$ into $C^k$ as follows: If $(z, w) \in U \times C^{k-n},$

$$\chi(z, w) = \left( f_1(z) + \sum_{j=1}^{k-n} a_j(z)w_j, \ldots, f_k(z) + \sum_{j=1}^{k-n} a_k(z)w_j \right).$$

Note that $\chi$ is of class $\sigma - 1$ in $U \times C^{k-n}$.

**Lemma 3.1.** An open set $V$ in $C^k$ containing $\Omega \times \{0\} = \{(z, 0) \in C^k \mid z \in \Omega\}$, such that $\chi$ is a diffeomorphism of class $\sigma - 1$ on $V$.

**Proof of Lemma 3.1.** Denote by $D_{\chi}$ the Jacobian determinant of $\chi$ when we regard $\chi$ as a map from a domain in $R^{2k}$ into $R^{2k}$. We have, for $z \in \Omega$,

$$D_{\chi}(z, 0) = \begin{vmatrix}
\partial f_1/\partial z_1 & \cdots & \partial f_k/\partial z_1 \\
\vdots & \ddots & \vdots \\
\partial f_1/\partial z_n & \cdots & \partial f_k/\partial z_n \\
a_1 & \cdots & a_k \\
a_1^2 & \cdots & a_k^2 \\
a_1^{k-n} & \cdots & a_k^{k-n}
\end{vmatrix}$$

in modulus squared, hence $=1$ by choice of the $a_j'$. By continuity this relation remains true for $z \in \Omega$. 

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Since $D\chi(z) \neq 0$ for each $z \in \Omega \times \{0\}$, $\chi$ is a local homeomorphism in $C^k$ at each such point. Also $\chi$ is one-to-one on $\Omega \times \{0\}$ by (1). It follows that $\exists$ an open set $V$ in $C^k$ with $\Omega \times \{0\} \subset V$ such that $\chi$ is one-to-one in $V$ and with $D\chi \neq 0$ at each point of $V$. Hence $\chi$ is a diffeomorphism on $V$, and we are done.

**Notations.** Let $F$ be a holomorphic map from an open set in $C^N$ into $C^N$, $F = (F_1, \ldots, F_N)$. We denote by $J_F$ the Jacobian matrix

$$
\begin{bmatrix}
\frac{\partial F_1}{\partial \zeta_1} & \cdots & \frac{\partial F_1}{\partial \zeta_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_N}{\partial \zeta_1} & \cdots & \frac{\partial F_N}{\partial \zeta_N}
\end{bmatrix}
$$

Let $\rho$ be the defining function of $\Omega$. For each $t > 0$, put $\varphi_t(z, w) = \rho(z) + t|w|^2$. For large $t$, the set $\{(z, w) \mid \varphi_t(z, w) \leq 0\} \subset V$, where $V$ is as in Lemma 3.1. Fix such a $t$, put $\varphi = \varphi_t$, and put

$$
\mathcal{E} = \{(z, w) \in U \times C^{k-n} \mid \varphi(z, w) < 0\}.
$$

Thus $\mathcal{E} \subset V$. Also $\chi$ is holomorphic in $\mathcal{E}$. Put $\varphi^* = \varphi \circ \chi^{-1}$.

$\varphi^*$ is a smooth function defined in $\chi(V)$.

**Lemma 3.2**. $\varphi^*$ is strictly plurisubharmonic in a neighborhood of $\chi(\mathcal{E})$.

**Proof.** Fix $p = (z, w) \in \mathcal{E}$. The entries of the matrix $J_\chi(p)$ involve the numbers $\left(\frac{\partial f_i}{\partial z_j}(z)\right)$ and $\alpha_\beta(z)$, which are bounded independently of $p$, since if $p \in \mathcal{E}$ then $z \in \Omega$ and the $f$ and $\alpha_\beta \in A^{\infty}(\Omega)$. Also $|w|$ is bounded for $(z, w) \in \mathcal{E}$. Thus $\exists$ constant $M$ such that $|J_\chi(p)\eta| \leq M|\eta|$ for all $\eta \in C^k$ and all $p \in \mathcal{E}$. Also $(J_\chi(p))^{-1} = J_{\chi^{-1}}(\chi(p))$, whence, for all $\xi \in C^k$,

$$
|\xi| = |J_\chi(p)(J_{\chi^{-1}}(\chi(p))\xi)| \leq M|J_{\chi^{-1}}(\chi(p))\xi|
$$

and so

$$
|J_{\chi^{-1}}(\chi(p))\xi| \geq \frac{|\xi|}{M}.
$$

Thus, for all $p^* \in \chi(\mathcal{E})$ and all $\xi \in C^k$, we have

$$
|J_{\chi^{-1}}(p^*)\xi| \geq \frac{|\xi|}{M}.
$$

Also, since $\varphi$ is strictly plurisubharmonic on a neighborhood of $\mathcal{E}$, $\exists$ constant $c > 0$ with

$$
\sum_j \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(p)\eta_i\bar{\eta}_j \geq c|\eta|^2
$$

for all $p \in \mathcal{E}$, $\eta \in C^k$.

Denote $\chi^{-1} = (\psi_1, \ldots, \psi_k)$. Each $\psi_j$ is a smooth function on $\chi(V)$ and is holomorphic on $\chi(\mathcal{E})$. Fix $\xi = (\xi_1, \ldots, \xi_k) \in C^k$ and fix $p \in \mathcal{E}$. Put $p^* = \chi(p)$,
\[ \eta = J_{\mathcal{A}^{-1}}(p^*)\xi. \] Thus, for each \( j \),

\[ \eta_j = \sum_a \frac{\partial \psi_j}{\partial \xi_a}(p^*)\xi_a. \]

Direct calculation gives

\[ \sum_{a,\beta} \frac{\partial^2 \psi^*}{\partial \xi_a \partial \xi_\beta}(p^*)\xi_a \xi_\beta = \sum_{i,j} \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_j}(p)\eta_i \eta_j. \]

By (12), the right-hand side

\[ \geq c|\eta|^2 = c|J_{\mathcal{A}^{-1}}(p^*)\xi|^2 \]

\[ \geq c \cdot \frac{|\xi|^2}{M^2}, \quad \text{by (11)}. \]

This inequality holds for all \( p \in \mathbb{C} \), hence by continuity for all \( p \in \overline{\mathbb{C}} \). The assertion of Lemma 3.2 follows.

**Proof of Theorem 1.** Note first that

\[ \chi(\mathbb{C}) = \{ \zeta \in \chi(V) | \psi^*(\zeta) < 0 \}. \]

It is easy to see that \( \text{grad} \ \psi^* \neq 0 \) on the boundary of \( \chi(\mathbb{C}) \). Hence the domain \( \chi(\mathbb{C}) \) satisfies the hypothesis of the approximation theorem of Henkin [3], and hence if \( F \) is any function continuous in \( \chi(\mathbb{C}) \), holomorphic in \( \chi(\mathbb{C}) \), then \( \exists \) a sequence \( \{F_n\} \) of functions with the following properties:

- For each \( n \), \( F_n \) is holomorphic in some neighborhood \( W_n \) of \( \chi(\mathbb{C}) \), and \( F_n \to F \) uniformly on \( \chi(\mathbb{C}) \).

Fix \( f \in A(\Omega) \). Let \( (z, w) \in \mathbb{C} \). Then \( z \in \Omega \). For \( (z, w) \in \mathbb{C} \), put \( \tilde{f}(z, w) = f(z) \). \( \tilde{f} \) is then continuous in \( \mathbb{C} \), holomorphic in \( \mathbb{C} \).

\( \tilde{f}(\chi^{-1}) \) is hence continuous in \( \chi(\mathbb{C}) \), holomorphic in \( \chi(\mathbb{C}) \). By the preceding, we can choose neighborhoods \( W_n \) of \( \chi(\mathbb{C}) \) and \( F_n \) holomorphic in \( W_n \) with \( F_n \to \tilde{f}(\chi^{-1}) \) uniformly on \( \chi(\mathbb{C}) \). Put

\[ K = \{ x(z, 0) | z \in \overline{\Omega} \}. \]

Then \( K \subset \overline{\chi(\mathbb{C})} \). By (3), \( K \) is polynomially convex. The Oka-Weil theorem allows us to approximate \( F_n \) uniformly by polynomials on \( K \), and so we obtain a sequence of polynomials \( \mathcal{P}_n \) in the coordinates with \( \mathcal{P}_n \to \tilde{f}(\chi^{-1}) \) uniformly on \( K \). Hence \( \mathcal{P}_n(x(z, 0)) \to f(z) \) uniformly on \( \overline{\Omega} \). But \( \chi(z, 0) = (f_1(z), \ldots, f_k(z)) \). So \( f \in [f_1, \ldots, f_k] \). Q.E.D.

**Proof of Theorem 1 bis.** Condition (iii) gives that each \( z \in \overline{\Omega} \) has a neighborhood \( U_z \) such that \( \exists \) a finite subset of \( \mathbb{C} \) separating points in \( U_z \). This, plus the compactness of \( \overline{\Omega} \), implies that \( \exists \) a finite set of elements \( f_1, \ldots, f_t \in \mathbb{C} \) such that each point of \( \overline{\Omega} \) has a neighborhood where \( f_1, \ldots, f_t \) separate points. Using this fact and (ii), a standard argument shows \( \exists \) a finite set \( f_{t+1}, \ldots, f_k \in \mathbb{C} \) which
together separate points on \( \overline{\Omega} \). The set \( \{ f_i \mid 1 \leq i \leq k \} \) then satisfies our conditions (1) and (2'). Using the functions \( f_1, \ldots, f_k \) we proceed as in the proof of Theorem 1 to obtain a map \( \chi \) and a set \( \mathcal{E} \) as earlier. The situation differs from the preceding one in that

\[
K = \{ \chi(z, 0) \mid z \in \overline{\Omega} \} = \{ (f_1(z), \ldots, f_k(z)) \mid z \in \overline{\Omega} \}
\]

is not necessarily polynomially convex. However, we can do the following: Fix \( f \in A(\Omega) \). Define \( \tilde{f}(z, w) = f(z) \), as earlier, for \( (z, w) \in \mathcal{E} \). By the proof of Theorem 1, \( \exists f_n \) holomorphic in a neighborhood of \( K \) with

\[
f_n \to \tilde{f}(\chi^{-1})
\]

uniformly on \( K \). Since the spectrum of \( \mathfrak{A} \) coincides with \( \overline{\Omega} \) by (iv), the operational calculus applied to \( \mathfrak{A} \) yields that, if \( g_n = F_n(f_1, \ldots, f_k) \), \( g_n \) lies in \( \mathfrak{A} \), for each \( n \). By (13), \( g_n(z) \to f(z) \) uniformly on \( \overline{\Omega} \). Hence \( f \in \mathfrak{A} \). Q.E.D.

Appendix. (A.1) Let \( \Omega \) be a domain in \( \mathbb{C}^n \), \( \Omega = \{ z \mid p(z) < 0 \} \), where \( p \) is strictly plurisubharmonic in a neighborhood of \( \overline{\Omega} \) and \( \text{grad } p \neq 0 \) on \( \partial \Omega \). Then the spectrum of \( A(\Omega) \) is \( \overline{\Omega} \).

Fix \( \varepsilon > 0 \) and put \( \Omega_{\varepsilon} = \{ z \mid p(z) < \varepsilon \} \). For small \( \varepsilon \), we have that \( \Omega_{\varepsilon} \) is a Stein manifold and \( \overline{\Omega} \) is convex with respect to the algebra \( H(\Omega_{\varepsilon}) \) of all functions holomorphic in \( \Omega_{\varepsilon} \). Let \( B \) denote the uniform closure on \( \overline{\Omega} \) of the restrictions to \( \overline{\Omega} \) of functions in \( H(\Omega_{\varepsilon}) \). Since \( \overline{\Omega} \) is convex with respect to \( H(\Omega_{\varepsilon}) \), \( B \) contains the restriction to \( \overline{\Omega} \) of each function \( F \) holomorphic in some neighborhood of \( \overline{\Omega} \). Hence, by the approximation theorem in [3], \( B = A(\Omega) \). On the other hand, by Theorem 7.2.10 of [4], the spectrum of \( B \) coincides with \( \overline{\Omega} \). We are done.

(A.2) Let \( \Omega \) be as in (A.1) and fix \( \sigma \). The spectrum of \( A^\sigma(\Omega) \) coincides with \( \overline{\Omega} \).

For fix a homomorphism \( m: A^\sigma(\Omega) \to \mathbb{C} \). Let \( f \in A^\sigma(\Omega) \).

Put \( \lambda = m(f) \). If \( |\lambda| > \max_{z} |f| \), \( f - \lambda \) is invertible in \( A^\sigma(\Omega) \), contradicting that \( m(f - \lambda) = 0 \). Hence \( |\lambda| \leq \max_{z} |f| \). It follows that \( m \) extends to an element of the spectrum of the closure of \( A^\sigma(\Omega) \) in \( A(\Omega) \). But that closure is \( A(\Omega) \), by the approximation theorem in [3]. Hence, by (A.1), \( m \) coincides with a point of \( \overline{\Omega} \), and we are done.

(A.3) Consider the open disk \( E: |z| < 1 \) in the plane. Put

\[
\varphi(z) = \exp(-(1 + z)/(1 - z)).
\]

Put \( f = (z - 1)^3 \cdot \varphi \), \( g = (z - 1)^4 \cdot \varphi \).

Assertion. \( f, g \in A^1(E) \) and satisfy (1), (2), (3) on \( E \). Yet \( f, g \) fail to generate \( A(E) \).

The fact that \( f, g \in A^1(E) \) as well as conditions (1) and (2) are checked by direct calculation. The argument to prove (3) is longer, and we do not give it here.

The fact that \( [f, g \mid E] \neq A(E) \) is seen as follows: Fix \( h \in [f, g \mid E] \) with \( h(1) = 0 \). Choose a sequence \( g_n \to h \) uniformly on \( E \) with each \( g_n \) a polynomial...
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in $f$ and $g$. We may assume $g_n(1) = 0$, all $n$.

$$g_n = \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(n)} \cdot g^\alpha \cdot f^\beta, \quad c_{\alpha, \beta}^{(n)} = g_n(1) = 0.$$ 

Hence $g_n = h_n \cdot \varphi$, where $h_n \in A(E)$. Since $|\varphi| = 1$ on $\partial E$, $g_n \varphi = h_n$, so $h_n \to h \varphi$ in the uniform norm on $|z| = 1$. Hence $h \varphi \in A(E)$ or $h = H \cdot \varphi$, $H \in A(E)$. Hence, every $h \in [f, g | E]$ with $h(1) = 0$ has $\varphi$ as a factor, and so $[f, g | E] \neq A(E)$, as claimed.

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Note 1. Recall that condition (3) required the set $K = \{(f_1(z), \ldots, f_k(z)) \mid z \in \Omega\}$ to be polynomially convex.

When $\Omega$ is the unit disk in $\mathbb{C}$, condition (3) is superfluous, i.e., is implied by (1) and (2') assuming the $f_j \in A'(\Omega)$. This was proved by J.-E. Björk in [11]. For the case when the $f_j$ are holomorphic on the closed disk, it was proved by Wermer in [8].

For the case when $\Omega$ is the disk in the plane, our Theorem 1 is known and is due to R. Blumenthal (to appear). This proof uses the measures orthogonal to the algebra. He needs only one derivative for the $f_j$.

In the case when $\Omega$ is the ball in $\mathbb{C}^2$, condition (3) is no longer a consequence of (1) and (2'). The example in Wermer [9] shows this.

Questions related to the work of the present paper are studied by Gamelin [2, Theorem 7], and by Sakai in [6] and [7].

Note 2. Every smoothly bounded strictly pseudoconvex domain $\Omega$ in $\mathbb{C}^n$ satisfies our condition; i.e., one can find a function $\rho$ smooth and strictly plurisubharmonic in a neighborhood of $\Omega$ such that $\Omega = \{z \mid \rho(z) < 0\}$ and $\text{grad } \rho \neq 0$ on $\partial \Omega$.

Added in proof. We have recently noted the paper by V. Iu. Lin, Holomorphic fiber bundles and multi-valued functions of an element of a Banach algebra, Functional Anal. Appl. 7 (1973), 43–51 (Russian). This paper overlaps with §2 of this article.

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