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GENERATORS FOR $A(\Omega)$

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ABSTRACT. We consider a bounded domain Ω in $\mathbf{C}^n$ and the Banach algebra $A(\Omega)$ of all continuous functions on $\overline{\Omega}$ which are analytic in Ω. Fix $f_1, \ldots, f_k$ in $A(\Omega)$. We say they are a set of generators if $A(\Omega)$ is the smallest closed subalgebra containing the f_i. We restrict attention to the case when Ω is strictly pseudoconvex and smoothly bounded and the f_i are smooth on $\overline{\Omega}$. In this case, Theorem 1 below gives conditions assuring that a given set f_i is a set of generators.

1. **Introduction.** Let Ω be a bounded domain in $\mathbf{C}^n$. $A(\Omega)$ denotes the algebra of all continuous complex-valued functions on $\overline{\Omega}$ which are holomorphic on Ω. With $\|f\| = \max_{\overline{\Omega}}|f|$, $A(\Omega)$ is a Banach algebra.

Fix $f_1, \ldots, f_k \in A(\Omega)$. Denote by $[f_1, \ldots, f_k \mid \overline{\Omega}]$ the uniform closure on $\overline{\Omega}$ of the algebra of all polynomials in $f_1, \ldots, f_k$. We say that the f_i form *a set of generators* for $A(\Omega)$ if $[f_1, \ldots, f_k \mid \overline{\Omega}] = A(\Omega)$.

Our problem is to decide when a given set $f_1, \ldots, f_k$ is a set of generators for $A(\Omega)$.

Two immediate necessary conditions are:

(1) *The f_i separate points on $\overline{\Omega}$.*

(2) *The matrix $((\partial f_i/\partial z_j))$, $1 \leq i \leq k$, $1 \leq j \leq n$, has rank n at each point $z \in \Omega$.*

Note that (1) implies that $k \geq n$.

To be able to find sufficient conditions, we impose the following restrictions on Ω: $\exists$ a function ρ of class C^4 and strictly plurisubharmonic in some neighborhood of $\overline{\Omega}$ such that $\Omega = \{z \mid \rho(z) < 0\}$, and grad $\rho \neq 0$ on $\partial\Omega$.

In this case it is known (see Appendix (A.1)) that the spectrum of the Banach algebra $A(\Omega)$ coincides with $\overline{\Omega}$.

Fix $f_1, \ldots, f_k \in A(\Omega)$. Put $K = \{(f_1(z), \ldots, f_k(z)) \mid z \in \overline{\Omega}\}$. In order that the f_i be a set of generators it is now necessary that

(3) *K is polynomially convex in $\mathbf{C}^k$.*

We shall also assume that the f_i are smooth up to the boundary of Ω. For each multi-index $I = (i_1, \ldots, i_n)$, put

$$|I| = \sum_{\nu=1}^{n} i_\nu \quad \text{and} \quad D^I = \frac{\partial^{|I|}}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}}.$$

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Definition 1. Fix an integer $\sigma > 0$. $A^\sigma(\Omega)$ denotes the class of all functions f on $\overline{\Omega}$ such that, for each multi-index I with $0 \leq |I| \leq \sigma$, $D^I f \in A(\Omega)$.

With the norm

$$\|f\| = \sum_I \frac{1}{I!} \max_{\Omega} |D^I f|, \tag{4}$$

where the sum is taken over all I with $0 \leq |I| \leq \sigma$ and $I! = i_1! \cdot i_2! \cdots i_n!$ if $I = (i_1, \ldots, i_n)$, $A^\sigma(\Omega)$ is a Banach algebra.

For $f \in A^\sigma(\Omega)$, $(\partial f/\partial z_i)(z)$ is defined for each $z \in \partial\Omega$ by

$$\frac{\partial f}{\partial z_i}(z) = \lim_{\zeta \to z} \frac{\partial f}{\partial z_i}(\zeta),$$

where $\zeta \to z$ from inside Ω, the limit existing by definition of $A^\sigma(\Omega)$. We can hence consider the matrix $((\partial f_i/\partial z_j))$ at all points of $\overline{\Omega}$.

Observation 1. Fix $\sigma > 0$. For $f_1, \ldots, f_k \in A^\sigma(\Omega)$ conditions (1), (2), (3) fail to be sufficient even when $n = 1$ and Ω is the unit disk: $|z| < 1$. We give an example in the Appendix (A.3).

We therefore strengthen condition (2) to

(2′) *The matrix* $((\partial f_i/\partial z_j))$ *has rank* n *for all* $z \in \overline{\Omega}$.

Theorem 1. *Fix* $\sigma \geq 4$. *Let* Ω *be as above. Fix* $f_1, \ldots, f_k \in A^\sigma(\Omega)$ *such that* (1), (2′), (3) *are satisfied. Then* $f_1, \ldots, f_k$ *are a set of generators for* $A(\Omega)$.

Note. Our hypotheses on Ω are satisfied by all smoothly bounded strictly convex sets in $\mathbf{C}^n$, in particular by balls.

Theorem 1 admits the following generalisation:

Theorem 1 bis. *Let* Ω *be as in Theorem* 1. *Let* $\mathfrak{A}$ *be a closed subalgebra of* $A(\Omega)$ *which contains a family* $\mathfrak{F}$ *of functions such that*:

(i) $\mathfrak{F} \subset A^\sigma(\Omega)$.

(ii) $\mathfrak{F}$ *separates points on* $\overline{\Omega}$.

(iii) *For each* $z \in \overline{\Omega}$, $\exists$ *a finite subset* $f_1^{(z)}, \ldots, f_s^{(z)}$ *of* $\mathfrak{F}$ *such that* $((\partial f_i^{(z)}/\partial z_j))$ *has rank* n *at* z.

(iv) *The spectrum of* $\mathfrak{A}$ *is* $\overline{\Omega}$.

Then $\mathfrak{A} = A(\Omega)$.

2. **Modules over a Banach algebra.** A is a commutative semisimple Banach algebra with unit 1. Its spectrum is denoted $\mathfrak{M}$. Fix k.

A^k denotes the A-module of all k-tuples $(a_1, \ldots, a_k)$ of elements $a_i \in A$.

Definition 2.1. Fix $l < k$, and fix elements $\xi^1, \ldots, \xi^l \in A^k$. We say the set $\xi^1, \ldots, \xi^l$ *admits a completion* if $\exists\, \xi^{l+1}, \ldots, \xi^k \in A^k$ such that $\xi^1, \xi^2, \ldots, \xi^l, \xi^{l+1}, \ldots, \xi^k$ is a module basis for A^k.

If $\xi = (\xi_1, \ldots, \xi_k) \in A^k$ and $M \in \mathfrak{M}$, we set $\xi(M) = (\xi_1(M), \ldots, \xi_k(M)) \in \mathbf{C}^k$, where, for $a \in A$, $a(M)$ denotes the value at M of the Gel′fand transform of a. It is clear that, given $\xi^1, \ldots, \xi^l \in A^k$, a necessary condition in order that this set admits a completion is

(5) $\xi^1(M), \ldots, \xi^l(M)$ *are linearly independent in* $\mathbf{C}^k$.

We ask: Under what restrictions is (5) a sufficient condition in order that $\xi^1, \ldots, \xi^l$ admits a completion?

Our work is based on results of Forster [1], regarding finitely generated projective modules over a Banach algebra A.

Definition 2.2. An A-module Q is a p-module (finitely generated projective module) if $\exists\, n$ and a direct sum decomposition $A^n = P \oplus Q$, where P is another module.

Q is *free* if it has a basis.

Definition 2.3. Let P be a p-module. P has *rank* k if for every $M \in \mathfrak{M}$, the vector space P/MP has dimension k over the field $A/M \cong \mathbf{C}$.

Lemma 2.1. *Assume $\mathfrak{M}$ is connected. Then every p-module P over A has some rank k.*

Proof. $\exists\, n$ and a decomposition

$$A^n = P \oplus Q. \tag{6}$$

Fix $M \in \mathfrak{M}$. We claim:

P/MP is isomorphic to the vector space $V_M = \{\xi(M) \mid \xi \in P\}$.

For let ξ_1, ξ_2 be elements of P congruent mod MP. Then

$$\xi_1 - \xi_2 = \sum_{i=1}^{s} m_i p_i, \qquad m_i \in M, p_i \in P.$$

Hence $\xi_1(M) = \xi_2(M)$. So the map $[\xi] \to \xi(M)$ is well defined from the elements of P/MP to V_M. It is evidently linear and surjective. Suppose, for some $\xi \in P$, $[\xi] \to 0$, i.e., $\xi(M) = 0$. Let $E_1, \ldots, E_n$ be the standard basis for A^n. $E_i = p_i + q_i, p_i \in P, q_i \in Q$, for each i. We have

$$\xi = (\xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} \xi_i E_i.$$

For each i, $\xi_i(M) = 0$, so $\xi_i \in M$.

$$\xi = \sum_{i=1}^{n} \xi_i(p_i + q_i) = \sum \xi_i p_i + \sum \xi_i q_i.$$

Since (6) is a direct sum decomposition, $\xi = \sum \xi_i p_i \in MP$, so $[\xi] = 0$.

Thus the map is injective, and the claim is proved. Put $W_M = \{\xi(M) \mid \xi \in Q\}$. By (6), we have

$$\mathbf{C}^n = V_M \oplus W_M. \tag{7}$$

That (7) is a direct sum decomposition is seen by the preceding argument. Thus $n = \dim V_M + \dim W_M$. Fix $M_0 \in \mathfrak{M}$ and put $l = \dim V_{M_0}$. Thus $\exists$ elements

$\xi^1, \ldots, \xi^l \in P$ with $\xi^1(M_0), \ldots, \xi^l(M_0)$ linearly independent. Hence for some choice of indices $i_1, i_2, \ldots, i_l$, the determinant

$$D(M) = \begin{vmatrix} \xi^1_{i_1}(M) & \cdots & \xi^1_{i_l}(M) \\ \vdots & & \vdots \\ \xi^l_{i_1}(M) & \cdots & \xi^l_{i_l}(M) \end{vmatrix} \neq 0$$

when $M = M_0$. By continuity, $D(M) \neq 0$ for all M in some neighborhood of M_0 in $\mathfrak{M}$. Hence dim $V_M \geq l$ for all M in some neighborhood of M_0. Similarly, dim $W_M \geq$ dim $W_{M_0} = n - l$ for all M in some neighborhood. But dim V_M + dim $W_M = n$ for all M. Hence dim $V_M = l$ for all M in some neighborhood of M_0.

Thus dim V_M is locally constant on $\mathfrak{M}$. Since $\mathfrak{M}$ is connected, $\exists k$ with dim $V_M = k$ for all $M \in \mathfrak{M}$. Since P/MP is isomorphic to V_M, we have dim $P/MP = k$ for all $M \in \mathfrak{M}$. Q.E.D.

Definition 2.4. Let P be a p-module $\subseteq A^n$. By $P \otimes C(\mathfrak{M})$, we denote the collection of all finite sums $\sum_{i=1}^s f_i p_i, f_i \in C(\mathfrak{M}), p_i \in P$, regarded as elements of $(C(\mathfrak{M}))^n$.

Note that $A^n \otimes C(\mathfrak{M}) \cong (C(\mathfrak{M}))^n$.

We shall use the following two results from [1]:

Proposition F.1. *Let Q be a p-module. If $Q \otimes C(\mathfrak{M})$ is free as a $C(\mathfrak{M})$-module, then Q is free as an A-module.*

Proof. Satz 6 of [1].

Proposition F.2. *Let the Banach algebra A have ρ topological generators. Let P_1, P_2 be two p-modules of rank k such that, for some l,*

$$P_1 \oplus A^l \cong P_2 \oplus A^l.$$

If $k \geq [\rho/2]$, then $P_1 \cong P_2$.

Proof. Satz 10 of [1].

Lemma 2.2. *A is a Banach algebra such that $\mathfrak{M}$ is a connected subset of $\mathbf{C}^n$. Q is a p-module over A such that*

$$(*) \qquad A^k = P \oplus Q, \quad \textit{where } P \cong A^l$$

for some k, l. If $k - l \geq n$, then Q is free.

Proof. Tensoring $(*)$ with $C = C(\mathfrak{M})$ gives

$$C^k = C^l \oplus \{Q \otimes C\}.$$

The rank of $Q \otimes C$ (as C-module) is $k - l$. $C(\mathfrak{M})$ possesses $2n$ topological generators, since $\mathfrak{M} \subset \mathbf{C}^n$. $k - l \geq n = [2n/2]$. Also

$$C^l \oplus C^{k-l} = C^l \oplus \{Q \otimes C\}.$$

By Proposition F.2, it follows that

$$C^{k-l} \cong Q \otimes C.$$

Thus $Q \otimes C$ is free as C-module. By Proposition F.1, this implies that Q is free as A-module. Q.E.D.

Theorem 2.1. *Let A be a Banach algebra with spectrum $\mathfrak{M}$ such that $\mathfrak{M}$ is a connected subset of $\mathbf{C}^n$. Fix k, l satisfying*

$$(8) \qquad k \geq 2n, \qquad l \leq n.$$

Fix $\xi^1, \dots, \xi^l \in A^k$ satisfying (5). *Then this set admits a completion.*

Proof. We hold n fixed and proceed by induction on l.

Let $l = 1$; we are given $\xi^1 \in A^k$, $\xi^1(M) \neq 0$ for all $M \in \mathfrak{M}$. $\xi^1 = (\xi_1^1, \dots, \xi_k^1)$. Since the ξ_j^1 have no common zero on $\mathfrak{M}$, $\exists\, a_j \in A$ with $\sum_{j=1}^k a_j \xi_j^1 = 1$. Define a map φ: $A^k \to A$ by

$$\varphi(x_1, \dots, x_k) = \sum_{j=1}^k a_j x_j.$$

Then φ is a homomorphism and $\varphi(\xi^1) = 1$. If $\xi \in A^k$, $\xi - \varphi(\xi) \cdot \xi^1 \in \ker \varphi$, so $\xi = \varphi(\xi) \cdot \xi^1 + r$, $r \in \ker \varphi$.

If $\xi = t \cdot \xi^1 + r'$, $t \in A$, $r' \in \ker \varphi$, then $t = \varphi(\xi)$ and $r' = r$. Thus we have the direct sum decomposition. $A^k = \{\xi^1\} \oplus \ker \varphi$ where $\{\xi^1\}$ is the cyclic module generated by ξ^1. $\{\xi^1\}$ is isomorphic to A^1, since $a\xi^1 = 0$ only if $a = 0$, in view of the fact that $\xi^1(M) \neq 0$ for all M. Lemma 2.2 applies since, by (8), $k - 1 \geq n$, and so $\ker \varphi$ is free.

Thus $\ker \varphi$ has a basis $\xi^2, \dots, \xi^k$ and so the set ξ^i, $1 \leq i \leq k$, is the required completion of ξ^1.

Now, assume the theorem is true when l is replaced by $l - 1$ and consider a set $\xi^1, \dots, \xi^l$ satisfying (5) and $l \leq n$. Then $\xi^1, \dots, \xi^{l-1}$ satisfies (5). By the induction hypothesis, $\exists\, \eta^l, \eta^{l+1}, \dots, \eta^k \in A^k$ such that $\xi^1, \dots, \xi^{l-1}, \eta^l, \dots, \eta^k$ forms a basis of A^k. Denote by Q the module generated by $\eta^l, \dots, \eta^k$. We have the direct sum decomposition

$$(9) \qquad A^k = \{\xi^1\} \oplus \cdots \oplus \{\xi^{l-1}\} \oplus Q.$$

In particular, $\exists\, c_\nu \in A$, $q \in Q$ with

$$(10) \qquad \xi^l = \sum_{\nu=1}^{l-1} c_\nu \xi^\nu + q.$$

For each M with $q(M) = 0$, we have

$$\xi^l(M) = \sum_{\nu=1}^{l-1} c_\nu(M) \xi^\nu(M),$$

contrary to (5). Hence $q(M) \neq 0$, for all M. Also $Q \cong A^{k-l+1}$. We can hence apply to Q and q the reasoning used above on A^k and ξ^1 and obtain a homomorphism φ': $Q \to A$, with $Q = \{q\} \oplus \ker \varphi'$. The rank of $Q = k - l + 1$, so the rank of $\ker \varphi' = k - l$. But, by (8), $k - l \geq n$. So, by Lemma 2.2, $\ker \varphi'$ is free, and has rank $k - l$. Let $\xi^{l+1}, \ldots, \xi^k$ be a basis for $\ker \varphi'$.

We claim that $\{\xi^i \mid 1 \leq i \leq k\}$ is a basis for A^k. This follows directly from (9) and (10). The induction is complete, so the theorem is proved.

The application of Theorem 2.1 which we will require is the following.

Theorem 2.2. *Let Ω be a domain in $\mathbf{C}^n$ as in Theorem* 1. *Fix $\alpha > 0$. Let $(\xi_1^1, \ldots, \xi_k^1), \ldots, (\xi_1^n, \ldots, \xi_k^n)$ be n k-tuples of elements of $A^\alpha(\Omega)$, where $k \geq 2n$. Assume that the matrix*

$$\begin{pmatrix} \xi_1^1 & \cdots & \xi_k^1 \\ \vdots & & \vdots \\ \xi_1^n & \cdots & \xi_k^n \end{pmatrix}$$

has rank n at each point of $\overline{\Omega}$. Then $\exists\, k - n$ k-tuples of elements of $A^\alpha(\Omega)$, $(a_1^1, \ldots, a_k^1), \ldots, (a_1^{k-n}, \ldots, a_k^{k-n})$ such that the determinant

$$\begin{vmatrix} \xi_1^1 & \cdots & \xi_k^1 \\ \vdots & & \vdots \\ \xi_1^n & \cdots & \xi_k^n \\ a_1^1 & \cdots & a_k^1 \\ \vdots & & \vdots \\ a_1^{k-n} & \cdots & a_k^{k-n} \end{vmatrix} = 1$$

at each point $z \in \overline{\Omega}$.

Proof. The spectrum of A^α coincides with $\overline{\Omega}$. (See Appendix (A.2).) Thus the spectrum is a connected subset of $\mathbf{C}^n$. The assumption on the matrix $((\xi_j^i))$ tells us that the vectors $\xi^i(M) = (\xi_1^i(M), \ldots, \xi_k^i(M))$, $i = 1, \ldots, n$, are independent at each M in the spectrum, i.e., satisfy (5).

Theorem 2.1 thus applies, and yields $\xi^{n+1}, \ldots, \xi^k \in (A^\alpha(\Omega))^k$ which together with $\xi^1, \ldots, \xi^n$ form a basis of $(A^\alpha(\Omega))^k$. Elementary algebra now gives that the determinant

$$\begin{vmatrix} \xi^1 \\ \vdots \\ \xi^k \end{vmatrix}$$

is a unit in the ring $A^\alpha(\Omega)$. Without loss of generality that determinant then equals 1. We are done.

3. **Proof of Theorems 1 and 1 bis.** Let now Ω and $f_1, \ldots, f_k$ satisfy the hypothesis of Theorem 1. If $k < 2n$, define $f_{k+1} = \cdots = f_{2n} = f_k$. Then the set

$f_1, \ldots, f_{2n}$ again satisfies (1), (2′), (3). Also

$$[f_1, \ldots, f_k \mid \overline{\Omega}] = [f_1, \ldots, f_{2n} \mid \overline{\Omega}].$$

If we can show that $[f_1, \ldots, f_{2n} \mid \overline{\Omega}] = A(\Omega)$, then of course $[f_1, \ldots, f_k \mid \overline{\Omega}] = A(\Omega)$. Thus it is no loss of generality to suppose $k \geq 2n$, and we do so from now on.

For $i = 1, \ldots, n$, put $\mathbf{p}_i = (\partial f_1/\partial z_i, \ldots, \partial f_k/\partial z_i)$. Each $\partial f_j/\partial z_i \in A^{\sigma-1}(\Omega)$. Put $A = A^{\sigma-1}(\Omega)$. Then $\mathbf{p}_i \in A^k$ for $i = 1, 2, \ldots, n$. Hypothesis (2′) gives exactly that, for each $M \in \overline{\Omega}$, the vectors $\mathbf{p}_1(M), \ldots, \mathbf{p}_n(M)$ are independent in $\mathbf{C}^k$.

By Theorem 2.2, we can select vectors such that the determinant

$$\begin{vmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_n \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{k-n} \end{vmatrix} = \begin{vmatrix} \partial f_1/\partial z_1 & \cdots & \partial f_k/\partial z_1 \\ \vdots & & \vdots \\ \partial f_1/\partial z_n & \cdots & \partial f_k/\partial z_n \\ a_1^1 & \cdots & a_k^1 \\ \vdots & & \vdots \\ a_1^{k-n} & \cdots & a_k^{k-n} \end{vmatrix} = 1$$

identically on $\overline{\Omega}$. Each $a_j^i \in A = A^{\sigma-1}(\Omega)$. We choose an open neighborhood U of $\overline{\Omega}$ in $\mathbf{C}^n$ such that each of the functions f_i, $1 \leq i \leq k$, admits an extension to U lying in $C^\sigma(U)$ and each a_j^i admits an extension to an element of $C^{\sigma-1}(U)$.

We form the product domain in $\mathbf{C}^k$: $U \times \mathbf{C}^{k-n} = \{(z, w) \mid z \in U, w \in \mathbf{C}^{k-n}\}$. We define a map χ of $U \times \mathbf{C}^{k-n}$ into $\mathbf{C}^k$ as follows: If $(z, w) \in U \times \mathbf{C}^{k-n}$,

$$\chi(z, w) = \left(f_1(z) + \sum_{j=1}^{k-n} a_1^j(z) w_j, \ldots, f_k(z) + \sum_{j=1}^{k-n} a_k^j(z) w_j\right).$$

Note that χ is of class $\sigma - 1$ in $U \times \mathbf{C}^{k-n}$.

Lemma 3.1. $\exists$ *an open set V in $\mathbf{C}^k$ containing $\overline{\Omega} \times \{0\} = \{(z, 0) \in \mathbf{C}^k \mid z \in \overline{\Omega}\}$, such that χ is a diffeomorphism of class $\sigma - 1$ on V.*

Proof of Lemma 3.1. Denote by D_χ the Jacobian determinant of χ when we regard χ as a map from a domain in $\mathbf{R}^{2k}$ into $\mathbf{R}^{2k}$. We have, for $z \in \Omega$,

$$D_\chi(z, 0) = \begin{vmatrix} \partial f_1/\partial z_1 & \cdots & \partial f_k/\partial z_1 \\ \vdots & & \vdots \\ \partial f_1/\partial z_n & \cdots & \partial f_k/\partial z_n \\ a_1^1 & \cdots & a_k^1 \\ a_1^2 & \cdots & a_k^2 \\ \vdots & & \vdots \\ a_1^{k-n} & \cdots & a_k^{k-n} \end{vmatrix}$$

in modulus squared, hence $=1$ by choice of the a_i^j. By continuity this relation remains true for $z \in \overline{\Omega}$.

Since $D_\chi(z) \neq 0$ for each $z \in \overline{\Omega} \times \{0\}$, χ is a local homeomorphism in $\mathbf{C}^k$ at each such point. Also χ is one-to-one on $\overline{\Omega} \times \{0\}$ by (1). It follows that $\exists$ an open set V in $\mathbf{C}^k$ with $\overline{\Omega} \times \{0\} \subset V$ such that χ is one-to-one in V and with $D_\chi \neq 0$ at each point of V. Hence χ is a diffeomorphism on V, and we are done.

Notations. Let F be a holomorphic map from an open set in $\mathbf{C}^N$ into $\mathbf{C}^N$, $F = (F_1, \ldots, F_N)$. We denote by J_F the Jacobian matrix

$$\begin{bmatrix} \partial F_1/\partial \zeta_1 & \cdots & \partial F_1/\partial \zeta_N \\ \vdots & & \vdots \\ \partial F_N/\partial \zeta_1 & \cdots & \partial F_N/\partial \zeta_N \end{bmatrix}.$$

Let ρ be the defining function of Ω. For each $t > 0$, put $\varphi_t(z,w) = \rho(z) + t|w|^2$. For large t, the set $\{(z,w) \mid \varphi_t(z,w) \leq 0\} \subset V$, where V is as in Lemma 3.1. Fix such a t, put $\varphi = \varphi_t$ and put

$$\mathfrak{G} = \{(z,w) \in U \times \mathbf{C}^{k-n} \mid \varphi(z,w) < 0\}.$$

Thus $\overline{\mathfrak{G}} \subset V$. Also χ is holomorphic in $\mathfrak{G}$. Put $\varphi^* = \varphi \circ \chi^{-1}$.
φ^* is a smooth function defined in $\chi(V)$.

Lemma 3.2. *φ^* is strictly plurisubharmonic in a neighborhood of* $\overline{\chi(\mathfrak{G})}$.

Proof. Fix $p = (z,w) \in \mathfrak{G}$. The entries of the matrix $J_\chi(p)$ involve the numbers $(\partial f_i/\partial z_j)(z)$ and $a^\alpha_\beta(z)$, which are bounded independently of p, since if $p \in \mathfrak{G}$ then $z \in \Omega$ and the f_i and $a^\alpha_\beta \in A^{\sigma-1}(\overline{\Omega})$. Also $|w|$ is bounded for $(z,w) \in \mathfrak{G}$. Thus $\exists$ constant M such that $|J_\chi(p)\eta| \leq M|\eta|$ for all $\eta \in \mathbf{C}^k$ and all $p \in \mathfrak{G}$. Also $(J_\chi(p))^{-1} = J_{\chi^{-1}}(\chi(p))$, whence, for all $\zeta \in \mathbf{C}^k$,

$$|\zeta| = |J_\chi(p)\{J_{\chi^{-1}}(\chi(p))\zeta\}| \leq M|J_{\chi^{-1}}(\chi(p))\zeta|$$

and so

$$|J_{\chi^{-1}}(\chi(p))\zeta| \geq \frac{|\zeta|}{M}.$$

Thus, for all $p^* \in \chi(\mathfrak{G})$ and all $\xi \in \mathbf{C}^k$, we have

$$|J_{\chi^{-1}}(p^*)\xi| \geq \frac{|\xi|}{M}. \tag{11}$$

Also, since φ is strictly plurisubharmonic on a neighborhood of $\overline{\mathfrak{G}}$, $\exists$ constant $c > 0$ with

$$\sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(p)\eta_i\bar{\eta}_j \geq c|\eta|^2 \tag{12}$$

for all $p \in \mathfrak{G}$, $\eta \in \mathbf{C}^k$.

Denote $\chi^{-1} = (\psi_1, \ldots, \psi_k)$. Each ψ_j is a smooth function on $\chi(V)$ and is holomorphic on $\chi(\mathfrak{G})$. Fix $\xi = (\xi_1, \ldots, \xi_k) \in \mathbf{C}^k$ and fix $p \in \mathfrak{G}$. Put $p^* = \chi(p)$,

$\eta = J_{\chi^{-1}}(p^*)\xi$. Thus, for each j,

$$\eta_j = \sum_{\alpha} \frac{\partial \psi_j}{\partial \zeta_\alpha}(p^*)\xi_\alpha.$$

Direct calculation gives

$$\sum_{\alpha,\beta} \frac{\partial^2 \varphi^*}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}(p^*)\xi_\alpha \bar{\xi}_\beta = \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(p)\eta_i \bar{\eta}_j.$$

By (12), the right-hand side

$$\geq c|\eta|^2 = c|J_{\chi^{-1}}(p^*)\xi|^2$$

$$\geq c \cdot \frac{|\xi|^2}{M^2}, \quad \text{by (11)}.$$

This inequality holds for all $p \in \mathfrak{E}$, hence by continuity for all $p \in \overline{\mathfrak{E}}$. The assertion of Lemma 3.2 follows.

Proof of Theorem 1. Note first that

$$\chi(\mathfrak{E}) = \{\zeta \in \chi(V) | \varphi^*(\zeta) < 0\}.$$

It is easy to see that grad $\varphi^* \neq 0$ on the boundary of $\chi(\mathfrak{E})$. Hence the domain $\chi(\mathfrak{E})$ satisfies the hypothesis of the approximation theorem of Henkin [3], and hence if F is any function continuous in $\overline{\chi(\mathfrak{E})}$, holomorphic in $\chi(\mathfrak{E})$, then $\exists$ a sequence $\{F_n\}$ of functions with the following properties:

For each n, F_n is holomorphic in some neighborhood W_n of $\overline{\chi(\mathfrak{E})}$, and $F_n \to F$ uniformly on $\overline{\chi(\mathfrak{E})}$.

Fix $f \in A(\Omega)$. Let $(z,w) \in \mathfrak{E}$. Then $z \in \Omega$. For $(z,w) \in \overline{\mathfrak{E}}$, put $\tilde{f}(z,w) = f(z)$. $\tilde{f}$ is then continuous in $\overline{\mathfrak{E}}$, holomorphic in $\mathfrak{E}$.

$\tilde{f}(\chi^{-1})$ is hence continuous in $\overline{\chi(\mathfrak{E})}$, holomorphic in $\chi(\mathfrak{E})$. By the preceding, we can choose neighborhoods W_n of $\overline{\chi(\mathfrak{E})}$ and F_n holomorphic in W_n with $F_n \to \tilde{f}(\chi^{-1})$ uniformly on $\overline{\chi(\mathfrak{E})}$. Put

$$K = \{\chi(z,0) \mid z \in \overline{\Omega}\}.$$

Then $K \subset \overline{\chi(\mathfrak{E})}$. By (3), K is polynomially convex. The Oka-Weil theorem allows us to approximate F_n uniformly by polynomials on K, and so we obtain a sequence of polynomials P_n in the coordinates with $P_n \to \tilde{f}(\chi^{-1})$ uniformly on K. Hence $P_n(\chi(z,0)) \to f(z)$ uniformly on $\overline{\Omega}$. But $\chi(z,0) = (f_1(z), \ldots, f_k(z))$. So $f \in [f_1, \ldots, f_k \mid \overline{\Omega}]$. Q.E.D.

Proof of Theorem 1 bis. Condition (iii) gives that each $z \in \overline{\Omega}$ has a neighborhood U_z such that $\exists$ a finite subset of $\mathfrak{F}$ separating points in U_z. This, plus the compactness of $\overline{\Omega}$, implies that $\exists$ a finite set of elements $f_1, \ldots, f_l \in \mathfrak{F}$ such that each point of $\overline{\Omega}$ has a neighborhood where $f_1, \ldots, f_l$ separate points. Using this fact and (ii), a standard argument shows $\exists$ a finite set $f_{l+1}, \ldots, f_k \in \mathfrak{F}$ which together separate points on $\overline{\Omega}$. The set $\{f_i \mid 1 \le i \le k\}$ then satisfies our conditions (1) and (2′). Using the functions $f_1, \ldots, f_k$ we proceed as in the proof of Theorem 1 to obtain a map χ and a set $\mathfrak{E}$ as earlier. The situation differs from the preceding one in that

$$K = \{\chi(z,0) \mid z \in \overline{\Omega}\} = \{(f_1(z), \ldots, f_k(z)) \mid z \in \overline{\Omega}\}$$

is not necessarily polynomially convex. However, we can do the following: Fix $f \in A(\Omega)$. Define $\tilde{f}(z,w) = f(z)$, as earlier, for $(z,w) \in \overline{\mathfrak{E}}$. By the proof of Theorem 1, $\exists F_n$ holomorphic in a neighborhood of K with

$$F_n \to \tilde{f}(\chi^{-1}) \tag{13}$$

uniformly on K. Since the spectrum of $\mathfrak{A}$ coincides with $\overline{\Omega}$ by (iv), the operational calculus applied to $\mathfrak{A}$ yields that, if $g_n = F_n(f_1, \ldots, f_k)$, g_n lies in $\mathfrak{A}$, for each n. By (13), $g_n(z) \to f(z)$ uniformly on $\overline{\Omega}$. Hence $f \in \mathfrak{A}$. Q.E.D.

Appendix. (A.1) Let Ω be a domain in $\mathbf{C}^N$, $\Omega = \{z \mid \rho(z) < 0\}$, where ρ is strictly plurisubharmonic in a neighborhood of $\overline{\Omega}$ and grad $\rho \neq 0$ on $\partial\Omega$. Then the spectrum of $A(\Omega)$ is $\overline{\Omega}$.

Fix $\varepsilon > 0$ and put $\Omega_\varepsilon = \{z \mid \rho(z) < \varepsilon\}$. For small ε, we have that Ω_ε is a Stein manifold and $\overline{\Omega}$ is convex with respect to the algebra $H(\Omega_\varepsilon)$ of all functions holomorphic in Ω_ε. Let B denote the uniform closure on $\overline{\Omega}$ of the restrictions to $\overline{\Omega}$ of functions in $H(\Omega_\varepsilon)$. Since $\overline{\Omega}$ is convex with respect to $H(\Omega_\varepsilon)$, B contains the restriction to $\overline{\Omega}$ of each function F holomorphic in some neighborhood of $\overline{\Omega}$. Hence, by the approximation theorem in [3], $B = A(\Omega)$. On the other hand, by Theorem 7.2.10 of [4], the spectrum of B coincides with $\overline{\Omega}$. We are done.

(A.2) Let Ω be as in (A.1) and fix σ. The spectrum of $A^\sigma(\Omega)$ coincides with $\overline{\Omega}$.

For fix a homomorphism m: $A^\sigma(\Omega) \to \mathbf{C}$. Let $f \in A^\sigma(\Omega)$.

Put $\lambda = m(f)$. If $|\lambda| > \max_{\overline{\Omega}}|f|$, $f - \lambda$ is invertible in $A^\sigma(\Omega)$, contradicting that $m(f-\lambda) = 0$. Hence $|\lambda| \le \max_{\overline{\Omega}}|f|$. It follows that m extends to an element of the spectrum of the closure of $A^\sigma(\Omega)$ in $A(\Omega)$. But that closure is $A(\Omega)$, by the approximation theorem in [3]. Hence, by (A.1), m coincides with a point of $\overline{\Omega}$, and we are done.

(A.3) Consider the open disk E: $|z| < 1$ in the plane. Put

$$\varphi(z) = \exp(-(1+z)/(1-z)).$$

Put $f = (z-1)^3 \cdot \varphi$, $g = (z-1)^4 \cdot \varphi$.

Assertion. $f, g \in A^1(E)$ and satisfy (1), (2), (3) on E. Yet f, g fail to generate $A(E)$.

The fact that $f, g \in A^1(E)$ as well as conditions (1) and (2) are checked by direct calculation. The argument to prove (3) is longer, and we do not give it here.

The fact that $[f,g \mid \overline{E}] \neq A(E)$ is seen as follows: Fix $h \in [f,g \mid \overline{E}]$ with $h(1) = 0$. Choose a sequence $g_n \to h$ uniformly on $\overline{E}$ with each g_n a polynomial in f and g. We may assume $g_n(1) = 0$, all n.

$$g_n = \sum_{\alpha,\beta\geq 0} c^{(n)}_{\alpha,\beta} f^\alpha \cdot g^\beta, \qquad c^{(n)}_{00} = g_n(1) = 0.$$

Hence $g_n = h_n \cdot \varphi$, where $h_n \in A(E)$. Since $|\varphi| = 1$ on ∂E, $g_n \overline{\varphi} = h_n$, so $h_n \to h\overline{\varphi}$ in the uniform norm on $|z| = 1$. Hence $h\overline{\varphi} \in A(E)$ or $h = H \cdot \varphi$, $H \in A(E)$. Hence, every $h \in [f, g \mid \overline{E}]$ with $h(1) = 0$ has φ as a factor, and so $[f, g \mid \overline{E}] \neq A(E)$, as claimed.

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Note 1. Recall that condition (3) required the set $K = \{(f_1(z), \ldots, f_k(z)) \mid z \in \overline{\Omega}\}$ to be polynomially convex.

When Ω is the unit disk in **C**, condition (3) is superfluous, i.e., is implied by (1) and (2′) assuming the $f_i \in A^1(\Omega)$. This was proved by J.-E. Björk in [11]. For the case when the f_i are holomorphic on the closed disk, it was proved by Wermer in [8].

For the case when Ω is the disk in the plane, our Theorem 1 is known and is due to R. Blumenthal (to appear). This proof uses the measures orthogonal to the algebra. He needs only one derivative for the f_i.

In the case when Ω is the ball in $\mathbf{C}^2$, condition (3) is no longer a consequence of (1) and (2′). The example in Wermer [9] shows this.

Questions related to the work of the present paper are studied by Gamelin [2, Theorem 7], and by Sakai in [6] and [7].

Note 2. Every smoothly bounded strictly pseudoconvex domain Ω in $\mathbf{C}^n$ satisfies our condition; i.e., one can find a function ρ smooth and strictly plurisubharmonic in a neighborhood of $\overline{\Omega}$ such that $\Omega = \{z \mid \rho(z) < 0\}$ and grad $\rho \neq 0$ on $\partial\Omega$.

Added in proof. We have recently noted the paper by V. Iu. Lin, *Holomorphic fiber bundles and multi-valued functions of an element of a Banach algebra*, Functional Anal. Appl. **7** (1973), 43–51 (Russian). This paper overlaps with §2 of this article.

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