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PAIRS OF DOMAINS WHERE ALL INTERMEDIATE DOMAINS ARE NOETHERIAN

BY

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ABSTRACT. For Noetherian integral domains R and T with $R \subseteq T$, (R, T) is called a *Noetherian pair* (NP) if every domain A, $R \subseteq A \subseteq T$, is Noetherian. When $\dim R = 1$ (Krull dimension) it is shown that the only NP's are those given by the Krull-Akizuki Theorem. For $\dim R \geq 2$, there is another type of NP besides the finite integral extension, namely $(R, \tilde{R})$ where $\tilde{R} = \bigcap\{R_P \mid \operatorname{rk} P \geq 2\}$. Further, for every NP (R, T) with $\dim R \geq 2$ there is an integral NP extension B of R with $T \subseteq \tilde{B}$. In all known examples B can be chosen to be a *finite* integral extension of R. For such NP's it is shown that the NP relation is transitive. T may itself be an infinite integral extension R, though, and an example of this is given. It is unknown exactly which infinite integral extensions are NP's.

In [5] Gilmer considered the property on an integral domain that all of its subdomains be Noetherian. He showed that there are no other such domains besides the few that have been known for many years. The concept of the Noetherian pair (NP)—two domains for which all intermediate domains are required to be Noetherian—was initially introduced in order to generalize Gilmer's work.

There are two basic, classical examples of Noetherian pairs (R, T):

(1) $\dim R = 1$ and T is a domain containing R and contained in a finite algebraic extension of the quotient field of R (the Krull-Akizuki Theorem);

(2) T is a finite integral extension of R.

(If R is a field, there are two more possibilities: T may be a domain contained in an algebraic function field in one variable over R, or a field algebraic over R.) The investigation of Noetherian pairs was begun with the particular hope of reaching a better understanding of the Krull-Akizuki Theorem, which stands out as one of the most remarkable and useful tools in the study of commutative rings of low dimensions. Indeed, a slight generalization of this theorem is obtained (see Theorem 8). But the strongest indication emerging from this study is that the

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range of Noetherian pairs extends little beyond the classical examples. Every NP is built up by a suitable admixture of 1-dimensionality (at least locally) and integral extensions.

The results presented here formed a part of my doctoral dissertation. I wish to express my thanks to my adviser, Professor Irving Kaplansky, for his encouragement in this undertaking and for his many helpful suggestions.

The notation and terminology herein will generally follow Kaplansky [6]. All rings are assumed to be commutative with 1. The dimension of a ring will always be its Krull dimension. x will always be an indeterminate. "Finitely generated" will often be abbreviated "f.g." When working with maximal ideals it will frequently be necessary to distinguish those of rank 1 from those with higher ranks; we will call the former "low maximals" and the latter "high maximals."

To simplify the statement of many theorems, we adopt the convention that R and T will always be integral domains, with $R \subseteq T$. After §1 it will be assumed further that $\dim R \geq 2$. Viewing T as an R-module, we will occasionally write T_P for the localization of T at $R - P$, where P is a prime ideal of R.

1. Quotient fields of Noetherian pairs. As an initial step toward understanding what pairs of domains (R, T) can be Noetherian pairs, we will determine how the quotient fields of R and T must be related. First, a pair of useful general results about NP's.

Lemma 1. (a) *If (R, T) is on NP and S is a multiplicatively closed subset of R then (R_S, T_S) is an NP.*

(b) *If R is quasisemilocal and (R_M, T_M) is an NP for each maximal ideal M of R, then (R, T) is an NP.*

Proof. (a) This follows immediately from the fact that if $R_S \subseteq A \subseteq T_S$, then $A = (A \cap T)_S$.

(b) Take a ring A, $R \subseteq A \subseteq T$, and an ideal I of A. For each maximal ideal M of R, I_M is a f.g. ideal of A_M, and the generators can be chosen to be in I. Take such a generating set of I_M for each M. The aggregate of elements thus chosen is a finite generating set for I. Thus, A is Noetherian and (R, T) is an NP.

It is an open question whether the semilocal condition in part (b) is necessary.

Theorem 2. *(R, T) is an NP iff R is Noetherian and for each proper ideal I of each ring A between R and T, A/I is a finitely generated R-module.*

Proof. Sufficiency is immediate, since the finiteness condition forces each intermediate ring A to be Noetherian.

Necessity. For a proper ideal I of an intermediate ring A, the ring $R+I$ is Noetherian, by hypothesis. Since $A \subseteq (a^{-1})(R+I)$ for any $0 \neq a \in I$, A must be a f.g. $(R+I)$-module. So, A/I is a f.g. $(R+I)/I$-module. But the latter is clearly a cyclic R-module. Thus, A/I is a f.g. R-module. Q.E.D.

Observe that the finiteness condition appearing in Theorem 2 is the central feature of most proofs of the Krull-Akizuki Theorem.

This theorem generalizes a result by Gilmer [5, Theorem 2]. Here is an easy corollary: $(R[x], T[x])$ is an NP iff T is a finite integral extension of R (R Noetherian). (Take $A = T[x]$ and $I = xA$.) Note that for (R, T) to be an NP it is not sufficient that the ideals of T satisfy the finiteness condition. (For example, if T is a field, the condition is vacuous.)

In Lemma 3 and Theorem 4, let K be the quotient field of R and L that of T.

Lemma 3. *If $[L: K]$ is infinite, then there is a domain A, $R \subseteq A \subseteq T$, which is a free but not f.g. R-module.*

Proof. If L is not algebraic over K, then there is a $t \in T$ transcendental over K, and we may take $A = R[t]$.

Assume now that L is algebraic over K. Take any t in T but not in K. Then there is an $r \in R$, $r \neq 0$, such that the minimal monic polynomial of rt over K has coefficients lying in R. (For example, express the coefficients of the minimal polynomial of t over K as quotients of elements of R. Then take r to be the product of the denominators.) Set $t_1 = rt$. Then $R_1 = R[t_1]$ is a free R-module with basis $B_1 = \{1, t_1, (t_1)^2, \cdots, (t_1)^{n_1-1}\}$, where $n_1 = [K[t_1]: K] \geq 2$. Now repeat this procedure with R_1 replacing R. We obtain a ring $R_2 = R_1[t_2]$ with basis $\{1, t_2, \cdots, (t_2)^{n_2-1}\}$ as a free R_1-module, so a free R-module with basis $B_2 = \{t_1^i t_2^j | 0 \leq i \leq n_1, 0 \leq j \leq n_2\}$. This process can be repeated without termination because $[L: K]$ is infinite.

We thus obtain an ascending chain of rings $R_1 \subseteq R_2 \subseteq, \cdots$, all within T. Each is a free R-module, and we have an ascending chain of bases $B_1 \subsetneqq B_2 \subsetneqq B_3 \subsetneqq \cdots$. Let $A = \bigcup_{i=1}^{\infty} R_i$, a domain between R and T. Because of the inclusion of the bases, it is evident that A is a free R-module with (infinite) base $B = \bigcup_{i=1}^{\infty} B_i$. Q.E.D.

Theorem 4. *Suppose (R, T) is an NP. Then,*

(i) *if R is not a field, then $[L: K] < \infty$;*

(ii) *if R is a field, then either L is algebraic over K (whence $T = L$) or L is an algebraic function field in one variable over K.*

Proof. (i) Assume R is not a field, and suppose $[L: K] = \infty$. By Lemma 3, there is a ring A, $R \subsetneq A \subsetneq T$, which is free but not f.g. as an R-module. Taking a proper ideal I of R, $IA \neq (0)$, and A/IA is not a f.g. R-module. This contradicts Theorem 2. Thus, $[L: K]$ must be finite.

(ii) Assume R is a field and L is not algebraic over R. Take an element $t \in T$ transcendental over R and apply part (i) to the NP $(R[t], T)$. L must be a finite algebraic extension of $K(t)$. Q.E.D.

Examples from the classical NP's demonstrate that each possible relation of quotient fields allowed by Theorem 4 can in fact occur. Further, the theorem shows that every NP (R, T) where $\dim R \leq 1$ is classical.

Corollary 5. *$(R, R[x])$ is an NP iff R is a field.*

Corollary 6 (Gilmer [5]). *Suppose T is a domain all of whose subrings are Noetherian. Then,*

(i) *if T has characteristic 0, then T is contained in a finite algebraic extension of the rationals;*

(ii) *if T has characteristic p, then either T is algebraic over Z_p, the field with p elements, or T is contained in a finitely generated field of transcendence degree 1 over Z_p.*

Both corollaries follow immediately from Theorem 4. In proving the second one, take R to be the prime ring of T.

2. **Higher-dimensional Noetherian pairs.** Having observed that every NP (R, T) with $\dim R \leq 1$ is classical, we now focus on higher-dimensional rings. In this context the only classical NP's (R, T) occur when T is a finite integral extension of R. We will construct another kind of NP, namely $(R, \tilde{R})$, as defined below. Then we will show that every higher-dimensional NP consists of an integral extension followed by an NP of this new type.

We now adopt as a standing hypothesis that R will be a domain of dimension ≥ 2.

Definition. For a domain R, $\tilde{R} = \bigcap R_M$, where the intersection is taken over all the high maximal ideals M of R (i.e., those of rank ≥ 2).

Note that $\tilde{R}$ consists of all those elements of the quotient field of R whose conductor to R is contained in no high maximal.

Lemma 7. *Let R be Noetherian and T a domain algebraic over R with $\dim T \geq 2$. Then every prime ideal Q of T lying over a low maximal of R is itself a low maximal. Hence, $\tilde{R} \subseteq \tilde{T}$.*

Proof. Say $Q \cap R = P$, a low maximal. Then $R_P \subseteq T_Q$ and R_P is a 1-dimensional Noetherian domain. Hence, the Krull-Akizuki Theorem forces

$\dim T_Q = 1$. So rk $Q = 1$. The same argument applies to any prime $Q' \supseteq Q$, as P is maximal. Hence, Q must be maximal.

Now, let M be a high maximal of T. As we just proved, $N = M \cap R$ is *not* a low maximal of R. Therefore, $\tilde{R} \subseteq R_N \subseteq T_M$. Since this holds for each high maximal M of T, $\tilde{R} \subseteq \tilde{T}$. Q.E.D.

We note an immediate consequence of Lemma 7 and standard facts about localization: If R is Noetherian and $R \subseteq T \subseteq \tilde{R}$, then, after low maximals are excluded, the natural map from the spectrum of T to that of R is an order isomorphism. Furthermore, the corresponding primes generate the same local rings. Hence, $\tilde{T} = \tilde{R}$. In fact, if R is integrally closed and Noetherian, Davis [2] has shown that all the rings between R and $\tilde{R}$ are of the form $\tilde{R} \cap \bigcap_{P \in \mathcal{P}} R_P$, where $\mathcal{P}$ is a collection of low maximals of R.

Theorem 8. *If R is Noetherian, then $(R, \tilde{R})$ is an NP.*

Proof. Let A be an intermediate ring. Let $I \neq (0)$ be an ideal of A and $J = I \cap R$, a f.g. ideal of R. We will show that I/JA is a f.g. R-module. It then follows that I is a f.g. ideal of A, hence A is Noetherian.

Let M be a maximal ideal of R. There are three possible cases. First, $M \not\supseteq J$. Then $J_M = R_M$, hence $(JA)_M = I_M = A_M$ and $(I/JA)_M = (0)$. Second, M is a high maximal containing J. Then $R_M \subseteq A_M \subseteq (\tilde{R})_M \subseteq (R_M)_M = R_M$. Thus, $A_M = R_M$, $J_M = I_M \cap R_M = I_M$, and again $(I/JA)_M = (0)$. The third possibility is that M is a low maximal containing J. Then R_M is a 1-dimensional Noetherian domain, so that $(I/JA)_M$ is a f.g. R_M-module by the Krull-Akizuki Theorem. Now, there are only finitely many low maximals M under consideration, since each is minimal over J. For each such M pick a finite set of elements in I/JA mapping onto a generating set of $(I/JA)_M$. The finite collection of all elements thus chosen clearly generates I/JA as an R-module, completing the proof.

This theorem can be viewed as a globalization of the Krull-Akizuki Theorem. The method of proof was suggested by an argument of Davis [3].

Theorem 9. *Suppose (R, T) is an NP with R local and integrally closed in T. Then $T = R$.*

Proof. (Recall: we are assuming $\dim R \geq 2$!) Suppose not and pick $t \in T - R$. Let ϕ be the natural homomorphism of $R[x] \to R[t]$ mapping x to t. Assume for a moment that $\ker \phi \subseteq MR[x]$, where M is the maximal ideal of R. It follows then (cf. [6, Problem 31, pp. 43–44]) that $R[t]/MR[t] \cong R[x]/MR[x] \cong R/M[x]$. Thus, although $MR[t] \neq (0)$, $R[t]/MR[t]$ is not a f.g. R-module, contradicting Theorem 2. Thus, $\ker \phi \not\subseteq MR[x]$, i.e., t satisfies some polynomial over R with at least one coefficient a unit. It follows by the u, u^{-1} Lemma ([6, Theorem 67], observing that it suffices that R be integrally closed in $R[u]$) that $t \in R$ or $1/t \in R$. But $t \notin R$, by hypothesis.

Now, since R is Noetherian and $\dim R \geq 2$, there are nonzero primes of R not containing $1/t$. Hence, there is a nonzero maximal ideal N of $R[t]$, a localization of R. By Theorem 2, the field $R[t]/N$ is integral over $R/(N \cap R)$. Therefore, $N \cap R$ must be the maximal ideal of R. But $1/t$ lies in M but not in N, a contradiction. Hence, $T = R$.

Theorem 10. *Suppose (R, T) is an NP. Let A be the integral closure of R in T. Then $T \subseteq \tilde{A}$ and* $\dim T = \dim R$.

Proof. Of course (A, T) is an NP. Let M be a high maximal of A. Then A_M is integrally closed in T_M, so that, by Lemma 1 and Theorem 9, $T_M = A_M$. Thus, $T \subseteq T_M = A_M$ for each high maximal M of A. Hence, $T \subseteq \tilde{A}$.

As to the dimensions, observe that $\dim R = \dim A = \dim T$. The first equality holds because A is integral over R, and the second follows immediately from the remarks after Lemma 7.

Corollary 11. *If (R, T) is an NP and the integral closure of R has no low maximals then T is integral over R.*

Proof. Let A be the integral closure of R in T. By Theorem 10, $T \subseteq \tilde{A}$. We will show that A has no low maximals. It then follows that $\tilde{A} = A$, so $T = A$ which is integral over R.

Let A' be the integral closure of A and R' that of R. Suppose A has a low maximal P. Since A' is integral over A there is a prime P' of A' lying over P, and P' must also be a low maximal. By the going-down theorem [8, 10.13], $P' \cap R'$ is a low maximal of R', contradicting the hypothesis. Thus, A can have no low maximal, completing the proof.

Note that the hypothesis of Corollary 11 cannot be replaced by the weaker condition that R itself have no low maximals. This is illustrated by the example in Proposition 18 below. It is an open question whether the conclusion of this theorem can be strengthened to assert that T is a *finite* integral extension of R.

The following corollary provides a particularly easy way to construct nonclassical NP's of any desired dimension. (For example, let U be a Noetherian unique factorization domain containing a prime Q of rank $n \geq 2$ and a principal prime (p) not in Q. Take $R = U_X$, where $X = U - (Q \cup (p))$ and take $S = \{p, p^2, p^3, \cdots\}$. In this case, there are no intermediate domains whatever between R and R_S.)

Corollary 12. *Let C be the set of all elements of a Noetherian domain R*

which are contained in no high maximal ideal. Then, for any multiplicatively closed subset S of R, (R, R_S) is an NP iff $S \subseteq C$.

Proof. Suppose $S \subseteq C$. Then, for each $s \in S$, the conductor of $1/s$ to R is sR, an ideal contained in no high maximal of R. Thus, $R_S \subseteq \tilde{R}$ and (R, R_S) is an NP by Theorem 8.

For the converse, suppose $(R, R[1/s])$ is an NP, where $s \in R$. It follows from Theorem 2 that every maximal ideal of $R[1/s]$ must contract to a maximal ideal of R. That is, all primes of R maximal with respect to exclusion of s are actually maximal ideals of R. With R Noetherian this can occur only if every prime containing s is a low maximal, i.e., $s \in C$. Thus, if (R, R_S) is an NP, then $S \subseteq C$. Q.E.D.

3. **Transitivity of the NP property.** Theorem 10 provides a decomposition of Noetherian pairs (R, T) into an integral NP extension (R, A) followed by a $\sim$-extension (A, T), with $A \subseteq T \subseteq \tilde{A}$. Now taking the opposite approach, we ask: Given a pair of domains (R, T) with such a decomposition into NP components, must (R, T) itself be an NP? More generally, if (R, A) and (A, T) are NP's, is (R, T) an NP?

A quick examination of the classical NP's shows that, for $\dim R \leq 1$, the transitivity property will hold except when Theorem 4 would be violated. It suffices to compare the quotient fields. That settled, we will for the rest of this section consider exclusively rings of dimension ≥ 2.

Theorem 13. *If A is a finite integral extension of R and (A, T) is an NP, then (R, T) is an NP.*

Proof. Say $A = R[u_1, \cdots, u_n]$, with $u_1, \cdots, u_n$ integral over R. Suppose B is a ring between R and T. Then $B[u_1, \cdots, u_n]$ is Noetherian, as (A, T) is an NP. By Eakin's Theorem [4], it follows that B is Noetherian. Q.E.D.

Corollary 14. *Suppose L is an extension field of the quotient field of R, such that the integral closure R' of R in L is a finite R-module. Then for a subring T of L, (R, T) is an NP iff $R \subseteq T \subseteq \tilde{R}'$.*

Proof. $(R, \tilde{R}')$ is an NP by Theorems 8 and 13. Sufficiency follows at once. Conversely, assume $T \subseteq L$ and (R, T) is an NP. By Theorem 10, $T \subseteq \tilde{A}$, where A is the integral closure of R in T. Since $A \subseteq R'$, Lemma 7 shows that $\tilde{A} \subseteq \tilde{R}'$. Thus, $R \subseteq T \subseteq \tilde{R}'$, as desired.

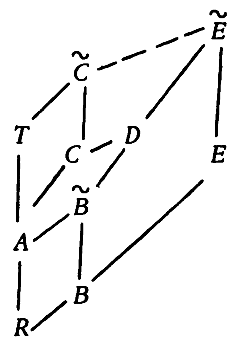
Lemma 15. *If B is Noetherian and D is a finite integral extension of $\tilde{B}$, then there is a finite integral extension E of B, such that $D \subseteq \tilde{E}$.*

Proof. Say $D = \tilde{B}[u_1, \cdots, u_n]$. For each i, let $J_i = \{r \in B \mid ru_i$ is integral over $B\}$, an ideal of B (so, finitely generated). Note that, for any high maximal M of B, since $\tilde{B} \subseteq B_M$, u_i is integral over B_M. Hence, there is an $s \in B - M$ such that su_i is integral over B. That is, $J_i \nsubseteq M$.

Let $E = B[J_1u_1, J_2u_2, \cdots, J_nu_n]$, a finite integral extension of B. We will show that $D \subseteq \tilde{E}$. Observe first that, by Lemma 7, $\tilde{B} \subseteq \tilde{E}$. Now, let I_i be the conductor of u_i to E. Clearly, $J_i \subseteq I_i \cap B$. Take a maximal ideal N of E with $u_i \notin E_N$. Then $N \supseteq I_i$, and $N \cap B$ is a maximal ideal of B containing J_i. As noted above, it follows that $N \cap B$ is a low maximal of B. Hence, N is a low maximal of E, since E is integral over B. Therefore, for each high maximal M of E, $u_i \in E_M$. That is, $u_i \in \tilde{E}$. Thus, $D = \tilde{B}[u_1, \cdots, u_n] \subseteq \tilde{E}$. Q.E.D.

Theorem 16. *Suppose (R, A) and (A, T) are NP's. If there is a finite integral extension B of R such that $A \subseteq \tilde{B}$ and a finite integral extension C of A such that $T \subseteq \tilde{C}$, then (R, T) is an NP.*

Proof. Let $C = A[u_1, \cdots, u_n]$. Set $D = \tilde{B}[u_1, \cdots, u_n]$, a finite integral extension of $\tilde{B}$. By Lemma 15, there is a finite integral extension E of B, such that $D \subseteq \tilde{E}$. Since $C \subseteq D \subseteq \tilde{E}$, we have $\tilde{C} \subseteq \tilde{\tilde{E}} = \tilde{E}$, by Lemma 7 and the remarks following that lemma. But, since E is a finite integral extension of R, $(R, \tilde{E})$ is an NP by Theorems 8 and 13. Thus, (R, T) is an NP.



Let us call an NP (R, A) "normal" if there is a finite integral extension B of R, such that $A \subseteq \tilde{B}$. Theorem 16 shows that transitivity holds for normal NP's (and that the large NP (R, T) is again normal). Any further results on transitivity await resolution of the more fundamental question: Are there NP's which are not normal? This question is completely open. Even the infinite integral NP's constructed in the next section are normal.

4. Infinite integral Noetherian pairs. In this section we give a method of constructing examples of Noetherian pairs (R, T) of any desired dimension ≥ 2 in which T is an infinite integral extension of R. The basic construction yields semilocal rings. Adding further hypotheses, we then obtain the same result with local rings. The construction is modelled on Nagata's counterexample to the saturated chain condition (see [7, pp. 23–27] or [9, pp. 327–329]).

Proposition 17. *Let field L be a finite (inseparable) extension of field K, and V a discrete valuation ring with quotient field K whose integral closure V' in L is not a f.g. V-module. For any $n \geq 2$, let W be an integrally closed n-dimensional local domain not contained in V, whose integral closure W' in L is a f.g. W-module. Let $R = V \cap W$ and $T = V' \cap W'$. Then* $\dim R = n$, *T is an infinite integral extension of R, and (R, T) is an NP.*

Proof. It is known [7, p. 23] that R has exactly two maximal ideals, P of rank 1 with $R_P = V$, and Q of rank n with $R_Q = W$. Since V and W are integrally closed, T is the integral closure of R in L. So, T_P is the integral closure of R_P in L, i.e., $T_P = V'$, and likewise $T_Q = W'$. T cannot be a finite integral extension of R, since T_P is not a finite integral extension of R_P. (R_P, T_P) is an NP by the Krull-Akizuki Theorem, and (R_Q, T_Q) is an NP because T_Q is a finite integral extension of R_Q. It follows from Lemma 1 that (R, T) is an NP. Q.E.D.

Note that $\tilde{R} = W$ in this setup. Thus (after verifying that R is Noetherian), Theorem 16 shows that (R, W') is a "normal" NP.

Proposition 18. *In addition to the hypotheses of Proposition* 17, *assume that R contains a subfield k, such that V'/M and W'/N are finitely algebraic over k, for each maximal ideal M of V' and N of W'. Let J be the Jacobson radical of R and J' that of T. Set $R_0 = k + J$ and $T_0 = k + J'$. Then R_0 and T_0 are n-dimensional local domains, T_0 is an infinite integral extension of R_0, and (R_0, T_0) is an NP.*

Proof. Once we show that R (resp. T) is a finite integral extension of R_0 (resp. T_0) the conclusion follows easily. For, since T is an infinite integral extension of R, T_0 must be an infinite integral extension of R_0. $\dim R_0 = \dim R = n$ and $\dim T_0 = \dim T = n$. Because (R, T) is an NP, it follows from Theorem 13 that (R_0, T_0) is an NP. Since every maximal ideal in T contracts to J' in T_0, J' is the unique maximal ideal of T_0. Likewise, J is the unique maximal ideal of R_0.

To see that T is a finite integral extension of T_0 it suffices to prove that T/J' is a finite integral extension of $T_0/J' \cong k$. But, let $M_1, \cdots, M_m$ be the maximal ideals of T (finitely many, as T is integral over R which is semilocal).

Then, $T/J' \cong T/M_1 \oplus \cdots \oplus T/M_m$ by the Chinese Remainder Theorem, and each summand is a finite algebraic extension field of k. For, if $M_i \cap R = P$, then M_{iP} is a maximal ideal of $T_P = V'$, and $T/M_i \cong V'/M_{iP}$ which was assumed to be finitely algebraic over k. Likewise, if $M_i \cap R = Q$, then $T/M_i \cong W'/M_{iQ}$ and we have the same result.

Since the condition on residue fields of V' and W' is inherited by V and W, the same argument shows that R is a finite integral extension of R_0, and the proof is complete.

We conclude with one specific example to illustrate Proposition 18. Let k be any field of characteristic $p \neq 0$. With $m = n - 1$, pick $a_1, a_2, \cdots, a_m$ in the power series ring $A = k[[x]]$ such that $x, a_1, a_2, \cdots, a_m$ are algebraically independent over k. Set $K = k(x, a_1{}^p, \cdots, a_m{}^p)$ and $L = k(x, a_1, \cdots, a_m)$ a finite purely inseparable extension of K. Let $V = A \cap K$. Then $V' = A \cap L$. Note that both V and V' have residue class field k. Furthermore, since $x \in V$, the ramification index from V to V' is 1. Therefore, V' is an infinite integral extension of V, by standard valuation theory [1, p. 147]. Let W be the localization at a maximal ideal of an affine k-algebra, not in V, with quotient field K. It is well known that W is n-dimensional, W' is a finite integral extension of W, and the residue fields of W' modulo a maximal ideal are finite algebraic extensions of k. Thus, V and W satisfy all the hypotheses of Proposition 18.

It is an open question exactly which infinite integral extensions are NP's.

Added in proof. J. R. Matijevic of the University of Kentucky has informed me of the following theorems he has recently proved, which provide interesting new examples of Noetherian pairs. First, if R is a local domain, and T is the ideal transform of its maximal ideal, then (R, T) is an NP. Second, if R is a Noetherian domain and $A = \bigcap_{P \in \mathcal{P}(R)} R_P$, where $\mathcal{P}(R) = \{P \mid P$ is an associated prime of a principal ideal, P not maximal$\}$, then (R, A) is an NP.

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