SOME MAPPING THEOREMS

BY

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ABSTRACT. Various mapping theorems are proved, culminating in the following result for mappings $f$ from a closed $(2k + 1)$-manifold $M$ to another, $N$: If "almost all" point-inverses of $f$ are strongly acyclic in dimensions less than $k$ and if "almost all" point-inverses of $f$ have Euler characteristic equal to one, then all but finitely many point-inverses are totally acyclic. (Here "almost all" means "except on a zero-dimensional set in $N".") More can be said when $k = 1$: If $f$ is a monotone map between closed 3-manifolds and if the Euler characteristic of almost-all point-inverses is one, then all but finitely many point-inverses of $f$ are cellular in $M$; consequently $M$ is the connected sum of $N$ and some other closed 3-manifold and $f$ is homotopic to a spine map. Other results include an acyclicity criterion using the idea of "nonalternating" mapping and the following result for PL maps $\phi$ between finite polyhedra $X$ and $Y$: If the Euler characteristic of each point-inverse of $\phi$ is the integer $c$ then $x(X) = cx(Y)$.

We begin with a clarification of the term "mapping theorem": this is to mean a theorem in which "local" assumptions are made on a map $f$ (i.e., assumptions are made on point-inverses of $f$) and "global" conclusions are drawn. The global conclusions may be topological (e.g., deducing that the domain and range of $f$ are homeomorphic) or algebraic (e.g., concluding $f$ has degree $\pm 1$). The classical Vietoris mapping theorem [3] is the best known example of the sort of result we have in mind.

Global conclusions can sometimes be obtained using "finiteness" theorems. As an illustration, consider a map $f: M^n \to N^n$ between closed topological manifolds. If the set $C_f$ of points $y \in N$ for which $f^{-1}(y)$ is not cellular in $M$ can be shown to be finite, $n \neq 4$, then results of S. Armentrout [2] and L. C. Siebenmann [10] imply that $M$ is homeomorphic to the connected sum of $N$ and another manifold. (See [8] for a more complete discussion of finiteness theorems.)

For mappings between closed even-dimensional manifolds there is a finite-
ness theorem as follows (cf. [7]): If each $f^{-1}(y)$ has $(k-1)$-connected shape, $n = 2k$, then $C_f$ is finite ($k \neq 2$). In §2, after reviewing the even-dimensional case in more detail in §1, we give an analogous result for odd dimensions: If each $f^{-1}(y)$ has $(k-1)$-connected shape and Euler characteristic one, $n = 2k + 1$, then $C_f$ is finite (no restriction on $k$). For $n = 3$, this generalizes (and uses) a result of A. Wright [17] and thus seems to explain Wright’s theorem as a phenomenon about odd-dimensional manifolds.

For functions of a real variable, the concepts of monotone map and nonalternating map coincide. In §3, we show how this equivalence is a special case of a general fact for mappings between odd-dimensional manifolds.

A result of E. G. Skljarenko [11] states that a map between compact ANR’s which is locally acyclic almost everywhere (i.e., except on a set of dimension $\leq 0$) is in fact locally acyclic except on a finite set. After slightly modifying Skljarenko’s result in §4, we prove in §5 a finiteness theorem for maps $f$ between closed $K$-orientable manifolds ($K$ a field): If $f$ is locally $(k-1)$-acyclic almost everywhere, and $2k \geq n$, then $f$ is locally acyclic except on a finite set. A similar result holds with $2k + 1 = n$ and a local Euler characteristic hypothesis. As applications some of the finiteness theorems of §§1, 2 are improved and a further generalization of Wright’s theorem is given.

The paper concludes with §6, where we consider PL maps $f: X \to Y$ between finite polyhedra. A proof of the following elementary formula is sketched: Suppose $Y_0$ is a subpolyhedron of $Y$ such that $\chi(f^{-1}(y)) = c_0$ for all $y \in Y_0$ and $\chi(f^{-1}(y)) = c$ for all $y \in Y \setminus Y_0$; then $\chi(X) = c\chi(Y) + (c_0 - c)\chi(Y_0)$.

1. Definitions and a review of the even-dimensional case. For the following definitions suppose that $f: M \to N$ is a map with compact point-inverses. Recall three “singular sets”

$$A_i(f; R) = \{y \in N | f^{-1}(y) \text{ does not have property } i - uv(R)\},$$

$$A^h_i(f; R) = \{y \in N | f^h(f^{-1}(y); R) \neq 0\},$$

$$C_f = \{y \in N | f^{-1}(y) \text{ is not cellular in } M\}.$$

(A compact set $X \subset M$ has property $i - uv(R)$ if, for any open set $U$ containing $X$, there is an open set $V$ with $X \subset V \subset U$ such that $f^h_i(V; R) \to f^h_i(U; R)$ is zero. An equivalent condition is that the reduced Skljarenko homology of $X$ vanish in dimension $i$; see [12].) The map $f$ is called strongly acyclic in dimension $i$ (over $R$) if $A_i(f; R) = \emptyset$ and strongly acyclic (over $R$) if $A(f; R) = \emptyset$, where

$$A_i(f; R) = \bigcup_{i \geq 0} A_i^h(f; R) = \bigcup_{i \geq 0} A^h_i(f; R).$$
The first equality is a definition. The second follows from (3.3) of [7]. The above terminology is slightly different from that used in [9], where the following is proved:

**Theorem 1.1.** Suppose \( f: M^n \to N^n \) is a proper map between \( R \)-orientable \( n \)-manifolds and that \( f \) is strongly acyclic in dimensions less than \( k \).

1. If \( 2k > n \) then \( f \) is strongly acyclic.
2. If \( 2k = n \) then \( A(f; R) \) is a locally finite subset of \( N^n \).

Below we give analogues of (1) for \( 2k = n \) and \( 2k + 1 = n \) (§3) and (2) for \( 2k + 1 = n \) (§2). Another even-dimensional result we shall analogize is the following, proved in [7].

**Theorem 1.2.** Suppose \( f: M^n \to N^n \) is a proper map between \( n \)-manifolds and that \( f^{-1}(y) \) has property \( UV^{k-1} \) for each \( y \in N^n \).

1. If \( 2k > n \neq 4 \) then \( f \) is cellular (when \( n = 3 \), we need to assume each \( f^{-1}(y) \) has a neighborhood containing no fake cubes).
2. If \( 2k = n \neq 4 \) then \( C_f \) is a locally finite set in \( N^n \).

See [1] or [7] for explanations of the terminology used in (1.2).

**Conventions.** A manifold is understood to be a connected, locally Euclidean metric space (without boundary points). \( R \) always means a principal ideal domain. A double arrow \( M \to N \) indicates a surjective map. Otherwise, our notation is that of [14].

2. Finiteness theorems in odd dimensions. We will use local Euler characteristic assumptions in this section. When \( G \) is a module over our PID \( R \), we define rank \( G \) to be the minimum number of generators of \( \text{Hom}_R(G, R) \), i.e., the rank of the free part of \( G \). If and only if \( X \) is a compactum with rank \( \tilde{H}^i(X; R) \) finite for all \( i \) and zero for all but finitely many \( i \), we write

\[
\chi(X; R) = \sum (-1)^i \text{rank } \tilde{H}^i(X; R).
\]

The question of dependence on \( R \) will be ignored.

**Theorem 2.1.** Suppose that \( f: M^{2k+1} \to N^{2k+1} \) is a proper map between \( R \)-orientable manifolds. If \( f \) is strongly acyclic in dimensions less than \( k \) and if \( \chi(f^{-1}(y); R) = 1 \) for each \( y \in N^{2k+1} \), then \( A(f; R) \) is a locally finite set in \( N^{2k+1} \).

**Proof.** The set \( A^{k+1}(f; R) \) is locally finite by Theorem 2.3 of [9] and
\( A^q(f; R) = \varnothing \) for \( q \neq k, k + 1 \) by (1.3) of [9]. We claim that \( A(f; R) = A^{k+1}(f; R) \). To see this, let \( y \in N \setminus A^{k+1}(f; R) \). Then

\[
\chi(f^{-1}(y); R) = 1 + (-1)^k \text{rank } \tilde{H}^k(f^{-1}(y); R)
\]

and hence rank \( \tilde{H}^k(f^{-1}(y); R) = 0 \). Since \( f^{-1}(y) \) has property \((k - 1) - \nu f(R)\), it follows from the homology/cohomology universal coefficient theorem that \( \tilde{H}^k(f^{-1}(y); R) = 0 \). Therefore \( y \in N \setminus A(f; R) \).

Remarks. 1. The assumption \( \chi(f^{-1}(y); R) = 1 \) in (2.1) can be weakened to the inequality \( \chi(f^{-1}(y); R) \geq 1 \) \((k \text{ odd})\) or \( \chi(f^{-1}(y); R) \leq 1 \) \((k \text{ even})\) without altering the conclusion. In the context of (2.1), where \( \chi(f^{-1}(y); R) = 1 + (-1)^k \beta_k + (-1)^{k+1} \beta_{k+1} \), this means assuming \( \beta_k \leq \beta_{k+1} \) instead of \( \beta_k = \beta_{k+1} \). (Note that \( \beta_{k+1} \) is finite; see the remark following the proof of (2.2) in [9].) Under this weaker assumption, one can use the conclusion of the theorem to show that \( \chi(f^{-1}(y); R) = 1 \) for all \( y \) so that the weaker hypothesis never arises in practice.

2. Under the hypothesis of (2.1), take \( R = \mathbb{Z} \) or \( \mathbb{Z}_2 \). Then \( \deg f = \pm 1 \), so \( f_* \) is an isomorphism for \( i \neq k, k + 1 \) and an epimorphism for all \( i \). Moreover,

\[
\ker f_* \cong \bigoplus_{y \in N} \tilde{H}^{k+1}(f^{-1}(y); R) \quad \text{and} \quad \ker f_{*+1} \cong \bigoplus_{y \in N} \tilde{H}^{k}(f^{-1}(y); R).
\]

See the analysis in §7 of [9].

3. When \( R = \mathbb{Z} \) or \( \mathbb{Z}_2 \), the orientability hypotheses in (2.1) may be dropped. See §4 of [9].

Theorem 2.2. Suppose \( f: M^{2k+1} \to N^{2k+1} \) is a proper map between manifolds and that \( f^{-1}(y) \) has property \( UV^{k-1} \) for each \( y \in N^{2k+1} \). If \( \chi(f^{-1}(y); \mathbb{Z}_2) = 1 \) for all \( y \in N^{2k+1} \), then \( C_f \) is a locally finite set in \( N^{2k+1} \).

Proof. Suppose first that \( k \geq 2 \). We have \( A(f; \mathbb{Z}_2) \) locally finite by (2.1) and \( A^{k+1}(f; \mathbb{Z}) \) locally finite by Theorem 2.3 of [9]. Let \( F = A(f; \mathbb{Z}_2) \cup A^{k+1}(f; \mathbb{Z}) \). We claim that \( F = A(f; \mathbb{Z}) \). To prove this, let \( y \in N \setminus F \). Then

\[
\tilde{H}^i(f^{-1}(y); \mathbb{Z}) = 0 \quad \text{for } i \neq k \quad \text{and} \quad \tilde{H}^i(f^{-1}(y); \mathbb{Z}_2) = 0 \quad \text{for all } i.
\]

It follows from the universal coefficient theorem for Čech cohomology that \( \tilde{H}^k(f^{-1}(y); \mathbb{Z}) = 0 \). This proves the claim and the local finiteness of \( A(f; \mathbb{Z}) \).

Now we are finished since \( f^{-1}(y) \) has \( UV^{\infty} \) for each \( y \in N \setminus A(f; \mathbb{Z}) \). (See §4 of [7].)

Now assume \( k = 1 \). Then \( A(f; \mathbb{Z}_2) \) is locally finite, hence zero-dimensional, so a result of [17] applies.

Remarks. 1. If \( f \) is as in (2.2) with \( M \) and \( N \) closed manifolds, then we can find a closed, \((k - 1)\)-connected manifold \( K \) such that \( M \) is homeomorphic to the connected sum \( N \# K \). Conversely, if \( M = N \# K \), where \( K \) is \((k - 1)\)-co-
nected, we can construct a map \( f: M \rightarrow N \) which satisfies (2.2). See the discussion in §7 of [7].

2. An interesting aspect of (2.2) is that its statement makes no dimensional restrictions on the manifolds and thus seems to explain the "Wright phenomenon" (cf. [17]) as a statement about manifolds in general. The dichotomy between \( k = 1 \) and \( k > 1 \) occurs in the proof for two reasons: first, of course, the Poincaré conjecture, and second, the fact that \( UV^{k-1} \) implies \( 1 - UV \) in the latter situations while \( UV^0 \) is merely the statement that \( f \) is monotone.

3. Both (1.1) and (2.1) can be generalized in the case where \( R \) is a field by requiring only that \( \dim A_i(f; R) \leq 0 \) for \( i < k \) (and that \( \dim \{ y \in N | \chi(f^{-1}(y)) \neq 1 \} \leq 0 \) in (2.1)). See §5 below.

4. Some kind of hypothesis on \( f^{-1}(y) \) in dimension \( k \) is necessary in both (2.1) and (2.2). See §6 of [7].

3. Nonalternating mappings. Suppose that \( f: M \rightarrow N \) is a mapping. We shall say that \( f \) is nonalternating in dimension \( k \) provided that, for each pair, \( y, z \in N \) with \( y \neq z \), there exists a neighborhood \( V \) of \( f^{-1}(y) \) in \( M \) such that \( H_k(V) \rightarrow H_k(M - f^{-1}(z)) \) is the zero homomorphism. (Throughout §3, \( R \) is assumed to be a fixed principal ideal domain, and all homology/cohomology is taken with coefficients in \( R \). Otherwise, our notation follows that of previous sections.) Notice that if \( M \) and \( N \) are locally compact ANR's and each \( f^{-1}(y) \) is compact, then "nonalternating in dimension zero" agrees with the classical notion of nonalternating. See [16].

Similarly, we shall say \( f \) is weakly acyclic in dimension \( k \) if each \( f^{-1}(y) \) has a neighborhood \( V \) in \( M \) such that \( \tilde{H}_k(V) \rightarrow \tilde{H}_k(M) \) is zero.

The following result is a corollary to R. Soloway's version of the Vietoris mapping theorem for singular homology. (See [13, Theorem 5].)

**Theorem S.** Suppose that \( M \) and \( N \) are locally compact ANR's and that \( f: M \rightarrow N \) is a proper map which is strongly acyclic in dimensions less than \( k \). Then \( f_*: H_i(M) \rightarrow H_i(N) \) is an isomorphism for \( i \leq k - 1 \) and an epimorphism for \( i = k \). If, in addition, \( f \) is weakly acyclic in dimension \( k \), then \( f_*^{w_k} \) is an isomorphism.

We will be applying Theorem S to certain types of maps between manifolds.

**Theorem 3.1.** Suppose \( f: M^n \rightarrow N^n \) is a proper map between \( R \)-orientable \( n \)-manifolds, \( k < n \). If \( f \) is strongly acyclic in dimensions less than \( k \) and weakly acyclic in dimension \( k \) then \( H^i(f^{-1}(y)) = 0 \) for \( i \geq n - k \) and all \( y \in N^n \).

**Proof.** Let \( y \in N \), and consider
a commutative diagram. By Theorem 5, $f_*$ is an isomorphism when $i \leq k$ and $f|_{\ast}$ is an isomorphism when $i < k$ and an epimorphism when $i = k$. Since the lower horizontal map is an isomorphism for $i < n$, we can conclude that the upper horizontal map is an isomorphism when $i < k$ and an epimorphism when $i = k$. It follows from the homology sequence of $(M, M - f^{-1}(y))$ that $H_i(M, M - f^{-1}(y)) = 0$ for $i \leq k$; the result follows from duality.

Corollary 3.2. Suppose $f: M^{2k} \to N^{2k}$ is a proper map between R-orientable manifolds which is strongly acyclic in dimensions less than $k$. If $f$ is weakly acyclic in dimension $k$, then $f$ is strongly acyclic.

Corollary 3.2 is not surprising in view of the local finiteness of $A(f)$ noted above. We should point out, however, that one cannot conclude, in Theorem 3.1, that $f$ is strongly acyclic in dimension $k$: there is a map $f: S^{2k+1} \to S^{2k+1}$ which is strongly acyclic in dimensions less than $k$ (and, obviously, weakly acyclic in dimension $k$) which is not strongly acyclic in dimension $k$. (See §6 of [7].)

Changing from weakly acyclic to nonalternating, we can obtain an acyclicity criterion in odd dimensions as follows.

Theorem 3.3. Suppose $f: M^{2k+1} \to N^{2k+1}$ is a proper map between R-orientable manifolds. If $f$ is strongly acyclic in dimensions less than $k$ and nonalternating in dimension $k$, then $f$ is strongly acyclic.

Proof. Let $y \in N$. By (3.1), we have $\tilde{H}_i(f^{-1}(y)) = 0$ for $i > k$; and by §3 of [7] $\tilde{H}_i(f^{-1}(y)) = 0$ for $i < k$. It suffices, therefore, to show that $\tilde{H}_k(f^{-1}(y)) = 0$. We claim first that

$$H_c^k(M) \to \tilde{H}_k(f^{-1}(y))$$

is zero. For the proof, let $U$ be a neighborhood of $f^{-1}(y)$ such that

$$\alpha: H_k(U) \to H_k(M)$$

is zero, and let $V$ be a neighborhood of $f^{-1}(y)$ in $U$ such that

$$\beta: H_{k-1}(V) \to H_{k-1}(U)$$

is zero. Consider the commutative diagram below (from the universal coefficient theorem):


The notation is as in §2 of [7]. We have \( \alpha^* = 0 \) and \( \beta^# = 0 \). Since the rows of the diagram are exact, it follows that \( H^k(M) \to H^k(V) \) is zero, and hence that \( H^k(M) \to \tilde{H}^k(f^{-1}(y)) \) is zero. The claim now follows easily from the fact that \( H^k(M) \to H^k_c(M) \) is a functorial isomorphism \((k \neq 0)\), where \( \tilde{M} \) is the one-point compactification of \( M \). (See [14, pp. 331, 332].) Now consider the diagram

\[
\begin{array}{ccc}
H^k_c(M) & \to & \tilde{H}^k(f^{-1}(y)) \\
\downarrow D & & \downarrow D \\
H_{k+1}(M) & \xrightarrow{f_*} & H_{k+1}(M, M - f^{-1}(y)) \\
\downarrow f|_* & & \downarrow f_* \\
H_k(N - \{y\}) & \to & H_k(N)
\end{array}
\]

in which \( D = \text{duality isomorphism} \) and both \( f|_* \) and \( f_* \) are isomorphisms by Theorem S. The diagram commutes up to sign, and the long row is exact. It follows that \( i_* \) is an isomorphism and hence that \( \partial = 0 \). The above paragraph implies that \( j_* = 0 \), so we have

\[ 0 = H_{k+1}(M, M - f^{-1}(y)) \cong \tilde{H}^k(f^{-1}(y)). \]

Therefore, \( \tilde{H}^*(f^{-1}(y)) = 0 \).

We conclude by remarking that there exist maps \( f: S^{2k} \to S^{2k} \) which are strongly acyclic in dimensions \( \leq k - 2 \), nonalternating in dimension \( k - 1 \), but not strongly acyclic in dimension \( k - 1 \): suspend the "join" example of §6 of [7].

Remark. A technique of Soloway [13] can be used to conclude the properness of the map \( f \), by merely assuming each point-inverse of \( f \) is compact, in each of the situations (1.1), (1.2), (2.1), (2.2) and (3.2) (but definitely not in (3.1)).

Question. Suppose \( f: M^{2k+1} \to N^{2k+1} \) is a map with compact point-inverses, strongly acyclic in dimensions less than \( k \), and nonalternating in dimension \( k \). Is \( f \) proper?

4. Almost acyclic mappings. The following theorem differs from a result of Skljarenko’s [11] only in its dependence on the integer \( k \). For the statement, we take \( G \) to be a finitely generated \( R \)-module and \( A^q(f; G) \) to be the set of values \( y \) for which \( \tilde{H}^q(f^{-1}(y); G) \neq 0 \). Following Skljarenko, if \( A \) is a subset of the space
Theorem 4.1. Let \( f: X \to Y \) be a closed map between paracompact Hausdorff spaces. Suppose further that, for some integer \( k \geq 0 \), the following hold:

(i) \( H_q(X; G) \) is finitely generated for \( q < k \);
(ii) \( H_q(Y; G) \) is finitely generated for \( q < k + 1 \); and
(iii) \( \text{rd } A_q(f; G) \leq 0 \) for \( q \leq k \).

Then \( A_q(f; G) \) is finite for \( q < k \) and \( H_q(f^{-1}(y); G) \) is finitely generated for \( q \leq k \) and \( y \in Y \).

The proof is based on that of Skljarenko and uses the Leray spectral sequence of \( f \). We outline the major steps.

Lemma 4.2. Let \( \{E_{p,q}\} \) be a convergent first quadrant spectral sequence. If \( E_{2,0} = 0 \) whenever \( p > 0 \) and \( 0 < q \leq k \), then there exists an exact sequence

\[
0 \to E_{2,0} \to E_{2,0} \to H_0 \to E_{2,1,0} \to E_{2,2,0} \to \cdots \to E_{2,k+2,0},
\]

the maps \( E_{2,0} \to H_0 \to E_{2,1,0} \) being edge homomorphisms and \( E_{2,1,0} \to E_{2,2,0} \) being the map \( d_{2,1} \) of [6].

Proof. Simply apply three propositions from Chapter XV of [6]: (5.7), (5.9), and (5.9a).

Now suppose that \( f: X \to Y \) is a closed, surjective map between paracompact Hausdorff spaces and assume that \( \text{rd } A_q(f; G) \leq 0 \), \( q \leq k \). Following Skljarenko [11], we define

\[
\mathcal{G}_q = R^qf/G, \quad 1 \leq q \leq k, \quad \text{and} \quad \mathcal{G}_0 = R^0f/G/G.
\]

(Here, \( R^qf \) is the \( q \)th right derived functor of the direct image functor. See [5].) Finally, let \( \Gamma^q = \Gamma(Y, \mathcal{G}_q) \), \( 0 \leq q \leq k \), i.e., \( \Gamma^q \) is the module of sections of the sheaf \( \mathcal{G}_q \) over \( Y \).

Lemma 4.3. Under the above assumptions, there exists an exact sequence

\[
0 \to \tilde{H}_0(Y; G) \to \cdots \to \tilde{H}_0(Y; G) \to \tilde{H}_0(X; G) \to \Gamma^q \to \tilde{H}_{q+1}(Y; G) \to \cdots \to \tilde{H}_{k+2}(X; G).
\]

Proof. The proof is the same as Skljarenko's. His arguments show that \( \{E_{p,q}^{**}\} \) satisfies the hypothesis of (4.2), where \( \{E_{p,q}^{**}\} \) is the Leray spectral sequence of \( f \). This fact yields most of the required sequence, since \( E_{2,0}^{**} = \tilde{H}_0(Y; G) \), \( H_0 = \tilde{H}_0(X; G) \), and \( E_{2,1}^{**} = \tilde{H}_0(Y; R^1f/G) = \Gamma^1 \). The first few terms
are constructed in the present situation just as they are in [11].

Remark. For \( q \leq k \) the module \( \Gamma^q \) is finitely generated if and only if \( A^q(f; G) \) is finite and each \( \tilde{H}^q(f^{-1}(y); G) \) is finitely generated, \( y \in Y \). (See [11].) Hence (4.1) follows from (4.3).

The introduction of "relative dimension" is an empty generalization in the case of mappings between manifolds, as we now show that \( \text{rd} \) and \( \text{dim} \) agree on \( A^i(f; G) \). (This was suggested by D. R. McMillan, Jr., who pointed out that the same is true for \( A_i(f; G) \).)

**Theorem 4.4.** Suppose \( f: X \to Y \) is a proper map between metric spaces, with \( X \) a locally compact ANR. Then \( A^i(f; G) \) is a countable union of closed subsets of \( Y \), and hence \( \text{rd} A^i(f; G) = \text{dim} A^i(f; G) \).

**Proof.** If \( \mathcal{U} \) is an open cover of \( Y \), define \( B(\mathcal{U}) \) to be the set \( \{ x \in X \mid f(x) \in U \in \mathcal{U} \} \). We claim that \( B(\mathcal{U}) \) is a closed subset of \( X \). To see this, suppose \( x \) is a limit point \( \mathcal{B}(\mathcal{U}) \), and suppose \( f(x) \in U \in \mathcal{U} \). Let \( V_1, V_2, \ldots \) be open sets in \( Y \) such that \( V_{n+1} \subseteq V_n \subseteq U \) for all \( n \) and \( f(x) = \bigcap_n V_n \). Choose points \( y_n \in V_n \cap (f(B(\mathcal{U}))) \) for each \( n \). Considering the diagram

\[
\tilde{H}^i(f^{-1}(V_n)) \to \tilde{H}^i(f^{-1}(U)) \\
\tilde{H}^i(f^{-1}(y_n)),
\]

one sees easily that \( \tilde{H}^i(f^{-1}(U)) \to \tilde{H}^i(f^{-1}(V_n)) \) is not zero for each \( n \), since \( f^{-1}(V_n) \subseteq B(\mathcal{U}) \). If we choose \( \{V_n\} \) with the additional property that image \( \tilde{H}^i(f^{-1}(V_n)) \to \tilde{H}^i(f^{-1}(V_{n+1})) \) is finitely generated for each \( n \), then it follows that the map

\[
\tilde{H}^i(f^{-1}(U)) \to \lim_n \tilde{H}^i(f^{-1}(V_n)) \cong \tilde{H}^i(f^{-1}(x))
\]

is nonzero. That \( \{V_n\} \) may be so chosen follows from an argument similar to the one for (2.1) of [9]. Therefore \( x \in B(\mathcal{U}) \).

Taking an appropriate sequence of open covers of \( Y \) shows that \( f^{-1}(A^i(f; G)) \) (and, hence, \( A^i(f; G) \)) is a countable union of closed sets.

The second part of the conclusion follows from the "Sum Theorem" for dimension.

5. Almost acyclic mappings between manifolds. In this section we let \( K \) be a field. We conjecture that \( K \) could be replaced by \( R \) in (5.1).

**Theorem 5.1.** Let \( f: M^m \to N^n \) be a map between closed, \( K \)-orientable
manifolds, $k < n$. Suppose $\dim A^q(f; K) \leq 0$ for $q < k$. Then $A^q(f; K)$ is finite whenever $q < k$ or $q \geq m - k$. Therefore, if $2k > m$, $A(f; K)$ is finite.

Proof. By (4.1), $A^q(f)$ is finite for $q < k$. (We suppress $K$ from notation in the proof.) Let

$$V = N - \bigcup_{q < k} A^q(f), \quad U = f^{-1}(V).$$

Applying Theorem 1.3 of [9], we see that $A^q(f|U) = \emptyset$ for $q < k$ and $q > m - k$ (and, hence, $A^q(f)$ is finite in these ranges). Also, Theorem 2.3 of [9] implies that $A^{m-k}(f|U)$ is a locally finite subset of $U$.

We wish to show that $A^{m-k}(f|U)$ is actually finite. Suppose $Y$ is any finite subset of $A^{m-k}(f|U)$, and let $X = f^{-1}(Y)$. By Theorem 1.1 of [9], the inclusion-induced map $H^{m-k}_c(U) \to H^{m-k}(X)$ is epic. Let $\hat{U}$ be the one-point compactification of $U$. We have the following diagram, each map being induced by inclusion.

$$
\begin{array}{ccc}
H^{m-k}_c(U) & \xrightarrow{\alpha} & H^{m-k}(\hat{U}) \\
\beta \downarrow & & \downarrow \gamma \\
H^{m-k}(X) & \xrightarrow{\beta} & H^{m-k}(U)
\end{array}
$$

The map $\alpha$ is epic, as noted above, and $\beta$ is an isomorphism (assuming $m - k \neq 0$). Therefore, $\alpha \beta$ is epic, and hence $\gamma$ is epic. Let $d$ be the dimension of $H^{m-k}(U)$. (We show in the next paragraph that $d$ is finite.) Since

$$\tilde{H}^{m-k}(X) = \bigoplus_{y \in Y} \tilde{H}^{m-k}(f^{-1}(y)),$$

the cardinality of $Y$ is no greater than $d$; therefore, $A^{m-k}(f|U)$ is finite. It follows that

$$A^{m-k}(f) \subset \left[ A^{m-k}(f|U) \cup \bigcup_{q < k} A^q(f) \right],$$

a finite set.

It remains to show that $H^{m-k}(U)$ is finitely generated. For this, it suffices to show that $H^{m-k}_c(U)$ is finitely generated. A portion of the homology sequence of $(M, U)$ looks like

$$H_{m-k+1}(M, U) \to H_{m-k}(U) \to H_{m-k}(M)$$

where

$$H_{m-k+1}(M, U) \cong \tilde{H}^{m-k+1}(f^{-1}(\bigcup_{q < k} A^q(f))) = \bigoplus_{y \in A^{m-k+1}(f)} \tilde{H}^{m-k+1}(f^{-1}(y)).$$
Since $A^{k-1}(f)$ is finite and $\tilde{H}^{k-1}(f^{-1}(y))$ is finitely generated (by (4.1)) for each $y \in N$, the result follows.

The following is an odd-dimensional analogue of the last statement in (5.1).

**Theorem 5.2.** Suppose that $f: M^{2k+1} \rightarrow N^{2k+1}$ is a map between closed, $K$-orientable manifolds. Suppose further that $\dim A^q(f; K) \leq 0$ for $q < k$ and that $\dim \chi(f^{-1}(y); K) \neq 1 \leq 0$. Then $A(f; K)$ is finite.

**Proof.** By (5.1) the sets $A^q(f)$ are finite whenever $q \neq k$. We have the inclusion

$$A^k(f) \subseteq \bigcup_{y \in f^{-1}(y)} A^q(f) \cup \bigcup_{q \neq k} A^q(f),$$

so $\dim A^k(f) \leq 0$. The result follows from (4.1).

Applying Wright's results (Theorem 1 of [17]) again we obtain what may be the ultimate generalization of his theorem, at least for maps between closed 3-manifolds.

**Corollary 5.3.** Let $f: M^3 \rightarrow N^3$ be a map between closed 3-manifolds. If there exists a zero-dimensional set $Z \subset N^3$ such that $f^{-1}(y)$ is connected and $\chi(f^{-1}(y); Z_2) \geq 1$ for each $y \in N^3 \setminus Z$, then $C_f$ is a finite set. Consequently $M^3$ is the connected sum of $N^3$ and some closed 3-manifold.

The proof of (5.3) uses Remark 1 of §2 as applied to the proof of (5.2). Note, incidentally, that there are monotone maps $b: S^3 \rightarrow -S^3$ with $\chi(b^{-1}(y)) \geq -1$ for each $y$ and $C_b$ an arc in $S^3$.

**Question.** If $f: M^3 \rightarrow N^3$ is a monotone map with $\chi(f^{-1}(y)) \geq 0$ for each $y$, must $C_f$ be finite?

6. PL mappings and the Euler characteristic. In the preceding sections local Euler characteristic assumptions were used several times, and a natural question seems to appear: If $f: X \rightarrow Y$ is a map between compact ANR's and $\chi(f^{-1}(y)) = c$ for all $y \in Y$, where $c$ is constant, what can be said about $\chi(X)$ as related to $\chi(Y)$? There is an easy answer when everything is PL, and we sketch this answer here.

**Theorem 6.1.** Suppose that $f: X \rightarrow Y$ is a PL map between (finite) polyhedra and that $Y_0$ is a subpolyhedron of $Y$. If there are integers $c, c_0$ such that $\chi(f^{-1}(y)) = c$ for $y \in Y \setminus Y_0$ and $\chi(f^{-1}(y)) = c_0$ for $y \in Y_0$, then

$$\chi(X) = c \chi(Y) + (c_0 - c) \chi(Y_0).$$

Here, $\chi(X)$ denotes the usual Euler characteristic $\chi(X; \text{rational numbers})$. 

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The proof of (6.1) requires the following calculation, in which \( \# S \) denotes the cardinality of the set \( S \) and \( \hat{\Delta} \) denotes the barycenter of \( \Delta \).

**Lemma 6.2.** If \( f: K \to \Delta^n \) is a simplicial map of the finite complex \( K \) onto an \( n \)-simplex then

\[
\sum_{i} (-1)^{i} \# \sigma^i \in K \mid f(\sigma^i) = \Delta^n \} = (-1)^n \chi(f^{-1}(\hat{\Delta}^n)).
\]

**Proof.** Let \( H^i = \{ \sigma^{n+i} \in K \mid f(\sigma^{n+i}) = \Delta^n \} \) and \( H = \bigcup_j H^j \). Now \( f^{-1}(\hat{\Delta}^n) \) has a natural triangulation as a subcomplex of a first derived subdivision of \( K \), but we want instead a cell structure determined by \( K \) as follows. Associate with each \( \sigma \in H^i \) the set \( \Gamma(\sigma) = f^{-1}(\hat{\Delta}^n) \cap \sigma \). Notice that \( \Gamma(\sigma^{n+i}) \) is a \( j \)-dimensional cell and in fact \( \Gamma = \{ \Gamma(\sigma) \mid \sigma \in H \} \) is a cell complex whose underlying space is \( f^{-1}(\hat{\Delta}^n) \). The Euler characteristic of \( f^{-1}(\hat{\Delta}^n) \) can be computed using this cell structure, and we obtain

\[
\chi(f^{-1}(\hat{\Delta}^n)) = \sum_{j} (-1)^{j} \# y^j \in \Gamma^i = \sum_{j} (-1)^{j} \# H^j = (-1)^n \sum_{j} (-1)^{n+j} \# H^j,
\]

which completes the proof.

**Proof of (6.1).** First assume that \( (K, K_0) \) and \( (L, L_0) \) are triangulations of \( (X, f^{-1}(Y_0)) \) and \( (Y, Y_0) \), respectively, such that \( f \) and \( f|f^{-1}(Y_0) \) are simplicial.

Assume as a special case that \( Y_0 = \emptyset \). Let \( \sigma^n \) be a top-dimensional simplex of \( L \), \( L_1 = L \setminus \{ \sigma^n \} \), and \( K_1 = f^{-1}(L_1) \). We have \( \chi(L) = \chi(L_1) + (-1)^n \) and, by (6.2) \( \chi(K) = \chi(K_1) + (-1)^n c \). By induction, we may assume that \( \chi(K_1) = c \chi(L_1) \), so \( \chi(K) = c \chi(L_1) + (-1)^n c = c \chi(L) \).

Now we prove the theorem assuming \( Y_0 \neq \emptyset \) by induction on the number of simplexes of \( L \setminus L_0 \). The case \( L \setminus L_0 = \emptyset \) follows from the special case above, so we proceed to the inductive step. Let \( r^n \) be a top-dimensional simplex of \( L \setminus L_0 \), \( L_1 = L \setminus r^n \), and \( K_1 = f^{-1}(L_1) \). Using (6.2) and the inductive hypothesis we have

\[
\chi(K) = c \chi(L_1) + (-1)^n c = c \chi(L_1) + (c_0 - c) \chi(L_0) + (-1)^n c
\]

\[
= c \chi(L) + (-1)^n + (c_0 - c) \chi(L_0) + (-1)^n c
\]

\[
= c \chi(L) + (c_0 - c) \chi(L_0).
\]

**Remarks.** Other similar results follow from the same kind of argument. For example, one can replace \( \chi(-; \text{rationals}) \) by \( [\chi(-; \text{rationals})]_q \) throughout, where \( [\ ]_q \) denotes equivalence class modulo \( q \) and the formula is interpreted in \( \mathbb{Z}_q \).

As another example, one can show the following: If \( f: M^n \to N^n \) is a PL map between closed, orientable PL manifolds such that \( \chi(f^{-1}(y)) = c \) for all \( y \in N^n \),
then \( \deg f = \pm c \). For a more sophisticated mod 2 version of this last statement, see [15].

Added in proof (March 10, 1974). The question at the end of \( \S 5 \) has been answered affirmatively by T. Knoblauch.

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