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SEPARABLE TOPOLOGICAL ALGEBRAS.I

BY

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ABSTRACT. Let A be a complete topological algebra with identity and B a subalgebra of the center of A. A notion of relative topological tensor product $\hat{\otimes}_B$ for topological A modules and the resultant relative homology theory are introduced. Algebras of bidimension zero in this sense are called separable relative to B. Structure theorems are proved for such algebras under various topological assumptions on the algebra and its maximal ideal space.

0. **Introduction.** Generally speaking, our goal is to characterize algebras which are similar in structure to the algebra of endomorphisms of a vector bundle. Specifically, we study topological algebras which can be reconstructed by piecing together algebras of matrix-valued functions over the maximal ideal space of the algebra. To this end we define the notion of a topological algebra A being separable (of homological bidimension zero) with respect to a subalgebra B of its center $Z(A)$ and then attempt to prove that such an algebra must be locally, over the maximal ideal space, an algebra of $n \times n$ matrices over a commutative function algebra.

The presence of analytic phenomena for algebras related to Banach algebras will enable us in a subsequent paper to give a satisfactory conclusion to this program once we reduce to the case of (local) finite generation over the center. We carry out this reduction in §7, but are forced at several key points to impose strong topological hypotheses in order to avoid certain difficulties related to the failure of the approximation property for Banach spaces. We also prove under appropriate topological conditions that a separable algebra is the topological

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direct product of algebras A_α which are n-homogeneous in the sense that $A_\alpha/M \simeq M_n(\mathrm{C})$ for each maximal ideal $M \subset A$.

Roughly the first half of the paper consists of simply adapting much of the classical theory of separable algebras to the study of topological algebras. For the most part this amounts to replacing the algebraic relative tensor product with a completed topological relative tensor product which is introduced in §2 and then dealing with corresponding modifications of the notions of algebra, module, projecture and free module, and separability in §3 and §4. Following §5, we deal almost exclusively with algebras to which the techniques of Banach algebra are applicable. Uncovering the need for and consequences of certain topological hypotheses, we present the main results of our theory in §6 and §7.

1. The classical theory. In this introductory section we outline portions of the classical theory of separable algebras and indicate directions in which we intend to extend this theory.

The object of the classical algebraic theory is to characterize those algebras which are direct sums of full matrix algebras. The object of our theory is to characterize those topological algebras which are in some sense continuous direct sums of matrix algebras.

1.1. *Some notation.* Let A denote an algebra over the complex field. We consider the vector space $A \otimes A$ to be a bimodule over the algebra A, where the left and right actions of A on $A \otimes A$ are defined by $a(b \otimes c) = ab \otimes c$, and $(b \otimes c)a = b \otimes ca$ respectively.

The product map $a \otimes b \longrightarrow ab$ induces a linear map π: $A \otimes A \longrightarrow A$. Note that π is an A bimodule homomorphism.

There is a multiplication on $A \otimes A$ which makes it an algebra in a particularly useful way. On elementary tensors this multiplication (which we denote by $*$) is defined by $(a \otimes b) * (c \otimes d) = ac \otimes db$. Note that multiplication is reversed in the second factor. When equipped with this product, $A \otimes A$ is often denoted A^e.

1.2. *Splitting idempotents in* $M_n(\mathrm{C})$. Let $M_n(\mathrm{C})$ denote the algebra of $n \times n$ complex matrices. We choose as a basis for $M_n(\mathrm{C})$ the set of elementary matrices e_{ij}. Hence, $e_{ij}e_{jk} = e_{ik}$ and $\Sigma_{i=1}^n \, e_{ii} = 1$.

The bimodule $M_n(\mathrm{C}) \otimes M_n(\mathrm{C})$ contains elements with some surprising properties. That is, let u be any element of the form

$$u = \sum \alpha_{ij} e_{ki} \otimes e_{jk}, \tag{i}$$

where the α_{ij} are scalars with $\Sigma \, \alpha_{ii} = 1$. Then, it is not difficult to see that $\pi u = 1 \in M_n(\mathrm{C})$, and $au = ua$ for every $a \in M_n(\mathrm{C})$.

Definition. If A is an algebra, then $u \in A^e = A \otimes A$ will be called a splitting idempotent for A if $\pi u = 1$ and $au = ua$ for every $a \in A$.

A simple computation shows that the elements of the form (i) are exactly the splitting idempotents in $M_n(\mathbf{C}) \otimes M_n(\mathbf{C})$. Observe that one, and only one, of these elements has the property that it is symmetric under interchange of factors in the tensor product; this is the element $u = (1/n) \Sigma\, e_{ij} \otimes e_{ji}$. Observe also that this particular splitting idempotent is related to the normalized trace on $M_n(\mathbf{C})$ by the formula

$$\mathrm{Tr}(a) = \frac{1}{n} \sum e_{ij} a e_{ji}$$

for $a \in M_n(\mathbf{C})$.

A splitting idempotent for an algebra A is indeed an idempotent—relative to the multiplication $*$ in A^e (cf. [3, II, §1]). In case $A = M_n(\mathbf{C})$ this follows easily from equation (i).

1.3. *Separable algebras.* If A^e contains a splitting idempotent for A, then we shall say A is a separable algebra. In the terminology of homological algebra, the existence of a splitting idempotent for A means that A is a projective A bimodule (cf. [11, VII, 5.1]).

A finite direct sum of separable algebras is separable. In fact, if $A = A_1 \oplus \cdots \oplus A_n$ and $u_i \in A_i \otimes A_i$ is a splitting idempotent for A_i, then

$$u_1 \oplus \cdots \oplus u_n \in (A_1 \otimes A_1) \oplus \cdots \oplus (A_n \otimes A_n) \subset A \otimes A$$

is a splitting idempotent for A.

Since $M_n(\mathbf{C})$ is separable, any finite direct sum of complex matrix algebras is separable. The interesting thing, of course, is that the converse is also true.

A semisimple algebra A is one with the property that for each left ideal $I \subset A$ there is a complementary left ideal $J \subset A$ for which A is the vector space direct sum $I \oplus J$.

Theorem. *If A is a complex algebra, then the following statements are equivalent:*

(a) *A is separable;*

(b) *A is finite dimensional and semisimple;*

(c) *A is a finite direct sum of matrix algebras.*

We have already observed that (c) implies (a). Although the semisimplicity of A follows directly from (a) (cf. [2, IX, 7.6]) the finite dimensionality requires more sophisticated techniques (cf. [3, II, 2.1]) which we shall generalize in modified form in the ensuing sections. That (b) implies (c) follows from the Wedderburn structure theory (cf. [10, III, §3.4]) and from the fact that $\mathbf{C}$ is an algebraically closed field.

1.4. *An example, topological algebras.* Under coordinatewise multiplication C^n is an algebra. It is, of course, separable since it is a direct sum of copies of C. The splitting idempotent is unique. In fact, if $\{e_1, \cdots, e_n\}$ is a basis for C^n, $u = \Sigma\, \lambda_{ij} e_i \otimes e_j$, then the equation

$$au = \sum_{i,j} \alpha_i \lambda_{ij} e_i \otimes e_j = \sum_{i,j} \alpha_i \lambda_{ij} e_i \otimes e_j = ua$$

holds for all $a = \Sigma_i\, \alpha_i e_i$ if and only if $\lambda_{ij} = 0$ for $i \neq j$. The equation

$$\pi u = \sum_i \lambda_{ii} e_i = \sum e_{ii} = I$$

holds if and only if $\lambda_{ii} = 1$ for all i. Hence, $e_1 \otimes e_1 + \cdots + e_n \otimes e_n$ is the unique splitting idempotent for C^n.

Now consider the algebra C^∞ obtained by taking the infinite direct product of copies of C. That is, C^∞ is the algebra of all infinite sequences of complex numbers under coordinatewise multiplication. This is infinite dimensional and, hence, not separable. In fact, if one proceeds as above he will conclude that the only candidate for a splitting idempotent is

(ii) $$\mu = \sum_{i=1}^{\infty} e_i \otimes e_i$$

where e_i is the sequence which is one in the ith coordinate and zero elsewhere. However, (i) does not define an element of $C^\infty \otimes C^\infty$ since it cannot be expressed as a finite sum of elementary tensors.

Now suppose we give C^∞ a topology—the topology of coordinatewise convergence. Then there is a corresponding topology on $C^\infty \otimes C^\infty$ (the projective tensor product topology, as discussed in §2) such that the completion of $C^\infty \otimes C^\infty$ in this topology contains the element u defined by the (now convergent) series (i). This suggests that it may be worthwhile to consider a modified notion of separability for topological algebras A. In this modified version of separability we allow a splitting idempotent to be an element of the completed projective topological tensor product of A with itself, rather than insisting that it be in the algebraic tensor product.

A notion of separability for topological algebras was introduced by Taylor in [16]. He almost proves that if such an algebra is locally multiplicatively convex (l.m.c.), then it is a topological direct product of matrix algebras. We say almost because his result requires a special hypothesis concerning the approximation property for Banach spaces. Since Enflo [5] has recently shown that not every Banach space has the approximation property, this is a serious difficulty. The same difficulty will plague us here. We have been able to avoid it only by imposing very strong topological conditions on A.

1.5. *Algebras of matrix-valued functions.* Let X be a locally compact Hausdorff space and consider the algebra $C(X, M_n(\mathbf{C}))$ of all continuous $n \times n$ matrix-valued functions on X. This is a l.m.c. algebra if we give it the topology of uniform convergence on compact sets. Alternatively, one can describe this algebra as the algebra $M_n(C(X))$ of $n \times n$ matrices with entries from $C(X)$.

This algebra is not separable in the sense of (1.4). The completed projective tensor product $C(X, M_n(\mathbf{C})) \hat{\otimes} C(X, M_n(\mathbf{C}))$ can be described as a certain subspace of $C(X \times X, M_n(\mathbf{C}) \otimes M_n(\mathbf{C}))$. It is easy to see that if this space contained a splitting idempotent for $C(X, M_n(\mathbf{C}))$ it would have to be a function f on $X \times X$ with $f(x, y) = 0$ for $x \neq y$ and $f(x, x)$ a splitting idempotent for $M_n(\mathbf{C})$ for each $x \in X$. Obviously, such a function cannot be in $C(X, M_n(\mathbf{C})) \hat{\otimes} C(X, M_n(\mathbf{C}))$ unless X is discrete.

Suppose, however, we consider the space $C(X, M_n(\mathbf{C}) \hat{\otimes} M_n(\mathbf{C}))$. This is not the ordinary tensor product of our algebra $C(X, M_n(\mathbf{C}))$ with itself, but it is a "relative" tensor product.

In fact, it is $C(X, M_n(\mathbf{C})) \hat{\otimes}_{C(x)} C(X, M_n(\mathbf{C}))$. Roughly speaking, this space is obtained from $C(X, M_n(\mathbf{C})) \hat{\otimes} C(X, M_n(\mathbf{C}))$ by factoring out relations of the form $fg \otimes h \sim f \otimes gh$ for $f, h \in C(X, M_n(\mathbf{C}))$ and $g \in C(X)$. For this relative tensor product there is also a $C(X, M_n(\mathbf{C}))$ bimodule structure and a product map π: $C(X, M_n(\mathbf{C})) \hat{\otimes}_{C(x)} C(X, M_n(\mathbf{C}))$. Also, it is easy to see that any function $M \in C(X, M_n(\mathbf{C}) \otimes M_n(\mathbf{C}))$ for which $u(x)$ is a splitting idempotent for $M_n(\mathbf{C})$ will satisfy $fu = uf$ and $\pi u = 1$.

Hence, if we are willing to introduce a relative notion of tensor product and modify the definition of splitting idempotent accordingly, then we can define the concept of an algebra A being separable relative to a subalgebra B (necessarily a subalgebra of the center of A). With this relative notion of separability, $C(X, M_n(\mathbf{C}))$ is separable relative to $C(X)$.

The idea of taking tensor products with respect to a commutative ground ring rather than a field is a familiar one in algebra. The corresponding notion of separability relative to a ground ring has been studied in great detail (cf. [1], [2]). Here we shall be concerned with these notions modified so as to be appropriate to the study of topological algebras. Many of our proofs will parallel the corresponding algebraic proofs. However, the topology introduces a considerable number of special difficulties.

1.6. *Objective.* We propose, then, to study topological algebras which, relative to a ground ring that is also a topological algebra, are separable in a sense which takes account of the topology. Under appropriate topological hypotheses we will show that locally (over the maximal ideal space of the center) such an algebra must exhibit many of the special properties of the algebra of 1.5 and other algebras of the form $M_n(B)$ where B is commutative.

2. **Tensor products.** In this section we introduce and discuss a notion of relative topological tensor product which will take the role played by the relative tensor product in the classical theory of modules over an algebra.

2.1. *Projective tensor products.* Suppose X and Y are vector spaces over the complex field. Any bilinear map $\kappa: X \times Y \to Z$ to a third vector space must factor uniquely through the tensor map $\otimes: X \times Y \to X \otimes Y$ via a linear map $\rho: X \otimes Y \to Z$. In other words, κ induces a unique linear map ρ making commutative the diagram

$$\begin{tikzcd}
X \times Y \arrow[rr, "\otimes"] \arrow[dr, "\kappa"'] & & X \otimes Y \arrow[dl, "\rho"] \\
& Z &
\end{tikzcd}$$

If X and Y are l.c.s.'s, then there is exactly one locally convex topology on $X \otimes Y$ which guarantees the continuity of the tensor map and the continuity of each map induced as above by a continuous κ with range in a l.c.s. Z. Henceforth, we denote the resulting l.c.s., called the (incomplete) projective tensor product, by $X \otimes Y$.

On completing $X \otimes Y$, we obtain a separated l.c.s. $X \hat{\otimes} Y$ together with a continuous bilinear map $\hat{\otimes}: X \times Y \to X \hat{\otimes} Y$ having total range in $X \hat{\otimes} Y$ and having the property that every continuous bilinear $\kappa: X \times Y \to Z$ into a complete l.c.s. induces a unique continuous linear map $\rho: X \hat{\otimes} Y \to Z$ such that $\kappa = \rho \circ \hat{\otimes}$. Indeed, this universal property characterizes the pair $(X \hat{\otimes} Y, \hat{\otimes})$ up to isomorphism (cf. [18, Proposition 43.4]). The space $X \hat{\otimes} Y$ is called the completed projective tensor product (CPTP) of X and Y.

We set $\hat{\bigotimes}(x, y) = x \hat{\otimes} y$. The tensor product of continuous linear maps $f_1: X_1 \to Y_1$, $f_2: X_2 \to Y_2$ is the factorization $f_1 \hat{\otimes} f_2$ of the continuous bilinear map $(x_1, x_2) \to f_1(x_1) \hat{\otimes} f(x_2)$ through the tensor map.

2.2. *Algebras and modules.* Henceforth, by an algebra (complete algebra) we shall mean a separated (complete) l.c.s. A together with a continuous associative multiplication $(a, b) \to ab: A \times A \to A$ and an identity 1. A continuous map $f: A_1 \to A_2$ of algebras will be called a homomorphism if $f(1) = 1$ and $f(ab) = f(a)f(b)$.

Associated to each complete algebra A is a pair of continuous linear maps $\pi: A \hat{\otimes} A \to A$ and $\epsilon: \mathbf{C} \to A$ defined by $a \hat{\otimes} b \to ab$ and $\lambda \to \lambda \cdot 1$ respectively which render commutative the diagrams

$$\begin{tikzcd}
A \hat{\otimes} A \hat{\otimes} A \arrow[r, "\pi \hat{\otimes} 1"] \arrow[d, "1 \hat{\otimes} \pi"] & A \hat{\otimes} A \arrow[d, "\pi"] \\
A \otimes A \arrow[r, "\pi"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C} \hat{\otimes} A \arrow[d, "\epsilon \hat{\otimes} 1"] & A \arrow[l, "a \to 1 \hat{\otimes} a"'] \arrow[r, "a \to a \hat{\otimes} 1"] \arrow[d, equal] & A \hat{\otimes} C \arrow[d, "1 \hat{\otimes} \epsilon"] \\
A \hat{\otimes} A \arrow[r, "\pi"] & A & A \hat{\otimes} A \arrow[l, "\pi"']
\end{tikzcd}$$

Conversely, for any complete l.c.s. A equipped with continuous linear maps π and ϵ which render the above diagrams commutative, the map $(a, b) \to \pi(a \hat{\otimes} b)$ defines a continuous multiplication which makes A into a complete algebra with identity $\epsilon(1)$.

Definition. Let A be an algebra. By a left A module we shall mean a separated complete l.c.s. M together with a continuous bilinear map $(a, m) \to am$: $A \times M \to M$ such that $1 \cdot m = m$ and $(ab) \cdot m = a \cdot (b \cdot m)$. A left A module homomorphism $f\colon M \to M'$ from M to another left A module M' will be a continuous linear map such that $f(a \cdot m) = a \cdot f(m)$. The set of all such maps forms a linear space $\mathrm{Hom}_A (M, M')$.

Similar definitions may be given for right A modules and A bimodules.

Suppose that A and B are algebras and that B is commutative. If A is also a B bimodule such that $b \cdot a = a \cdot b$ and $(b \cdot a)(b' \cdot a') = (bb') \cdot (aa')$ for $a, a' \in A$ and $b, b' \in B$, we say that A is a B algebra. Note that a B algebra must be complete since we have insisted that modules be complete.

2.3. *Relative tensor products.* Consider a commutative algebra B and B modules M and N. Let F be the closed linear span in $M \otimes N$ of $\{bm \otimes n - n \otimes bn\colon m \in M,\ n \in N,\ b \in B\}$. The tensor product of M and N relative to B, denoted $M \hat{\otimes}_B N$, is the B module obtained by completing the factor space $(M \otimes N)/F$.

Let $\hat{\otimes}_B\colon M \times N \to M \hat{\otimes}_B N$ be the composition

$$M \times N \xrightarrow{\otimes} M \times N \xrightarrow{p} (M \times N)/F \xrightarrow{i} M \hat{\otimes}_B N$$

where p is the factor map and i the inclusion. We set $\hat{\otimes}_B (m, n) = m \hat{\otimes}_B n$.

Proposition. *For every continuous B bilinear map $\rho\colon M \times N \to L$ to a B module L, there is a unique $\hat{\rho}$ in* $\mathrm{Hom}_B (M \hat{\otimes}_B N, L)$ *such that* $\rho = \hat{\rho} \circ \hat{\otimes}_B$.

Proof. Because ρ is continuous and bilinear it induces a unique continuous linear map $\tilde{\rho}\colon M \otimes N \to L$ such that $\tilde{\rho} \cdot \otimes = \rho$. But, as ρ is B bilinear, $\tilde{\rho}$ vanishes on F and, hence induces a map $\tilde{\tilde{\rho}}\colon (M \otimes N)/F \to L$. The extension of $\tilde{\tilde{\rho}}$ to the completion of $(M \otimes N)/F$ is the required map $\hat{\rho}$.

The property expressed in this proposition determines the pair $(M \hat{\otimes}_B N, \hat{\otimes}_B)$ up to isomorphism.

Now suppose that L, M, and N are B modules. Using the associativity of the incomplete projective tensor product (cf. [14, §3]), one can prove that any continuous B trilinear map $\phi\colon L \times M \times N \to K$ into a B module K induces a B module homomorphism $\hat{\phi}_1\colon (L \hat{\otimes}_B M) \hat{\otimes}_B N \to K$ and, similarly, a B module homomorphism $\hat{\phi}_2\colon L \hat{\otimes}_B (M \hat{\otimes}_B N) \to K$. In the case where $K = L \hat{\otimes}_B (M \hat{\otimes}_B N)$ and $\phi\colon L \times M \times N \to K$ is defined by $\phi(1, m, n) = 1 \hat{\otimes}_B (m \hat{\otimes}_B n)$, the map $\hat{\phi}_1$

yields an isomorphism from $(L \hat{\otimes}_B M) \hat{\otimes}_B N$ to $L \hat{\otimes}_B (M \hat{\otimes}_B N)$. Hence, our notion of relative topological tensor product is associative.

3. Relative homology. We introduce in a series of definitions some basic ideas which enable us to discuss the essentially homological nature of the separability criteria of the next section. Throughout, we assume that B is a commutative algebra and that A is a B algebra.

3.1. *Split sequences.* A sequence of A modules and A module homomorphisms

$$S: \cdots \to E_{i-1} \xrightarrow{\alpha_{i-1}} E_i \xrightarrow{\alpha_i} E_{i+1} \to \cdots$$

is said to be a complex if $\alpha_i \circ \alpha_{i-1} = 0$ and an exact sequence if $\ker \alpha_i = \operatorname{im} \alpha_{i-1}$. If S is a complex with a contracting homotopy (a sequence $\{\beta_i \in \operatorname{Hom}(E_i, E_{i-1})\}$ such that $\alpha_{i-1} \circ \beta_i + \beta_{i+1} \circ \alpha_i =$ identity: $E_i \to E_{ii}$) then S is said to be split (cf. [2, I, §1]).

The reader may check that S is split precisely when S is exact and each E_i is an A module direct sum $\ker \alpha_i \oplus L_i$ where $\alpha_i|_{L_i}$ is a topological isomorphism from L_i to $\operatorname{Ker} \alpha_{i+1}$. Note that an exact sequence of A module homomorphisms may split as a sequence of B module homomorphisms (there is a contracting homotopy consisting of B module homomorphisms) and, yet, not split as a sequence of A module homomorphisms. Thus, we say that S is B split if it has a B module contracting homotopy. This is a condition which depends on the choice of our ground ring B. Observe that $\mathbf{C}$ is always a possible choice for a ground ring. A sequence is $\mathbf{C}$ split if it has a contracting homotopy consisting of continuous linear maps.

3.2. *Free modules.* If E is a left B module, then $A \hat{\otimes}_B E$ may be considered a left A module where

$$a_1 \hat{\otimes}_B a_2 \hat{\otimes}_B x \to a_1 a_2 \hat{\otimes}_B x: A \hat{\otimes}_B (A \hat{\otimes}_B E) \to A \hat{\otimes}_B E$$

determines the action of A on the module. A module of this form will be called a free left A module (relative to B). Similarly, for a right B module E and B bimodule F, $E \hat{\otimes}_B A$ and $A \hat{\otimes}_B F \hat{\otimes}_B A$ are called free right and free bimodules respectively. Note that this definition of free module depends on the choice of the ground ring B.

Proposition (cf. [16, §1, 1.3]). *Let E be a left B module, $A \hat{\otimes}_B E$ the corresponding free left A module. Then $\tilde{f}(a \hat{\otimes}_B x) = af(x)$ defines an isomorphism $f \to \tilde{f}$ between* $\operatorname{Hom}_B(E, F)$ *and* $\operatorname{Hom}_A(A \hat{\otimes}_B E, F)$.

Proof. For $f \in \operatorname{Hom}_B(E, F)$ we define $\tilde{f}$ by $\tilde{f}(a \hat{\otimes}_B x) = af(x)$. That this equation determines a continuous B linear map $f: A \hat{\otimes}_B E \to F$ follows from

Proposition 2.3. Obviously, $\tilde{f}$ is an A module homomorphism as well and $f \to \tilde{f}$ is a linear map from $\mathrm{Hom}_B(E, F)$ to $\mathrm{Hom}_A(A \otimes_B E, F)$. Its inverse is $g \to g_0$ where $g_0(x) = g(1 \otimes_B x)$. Hence, $f \to \tilde{f}$ is an isomorphism.

3.3. *Projective modules.* If E is a left A module, then we denote by χ the map $a \hat{\otimes}_B x \to ax: A \hat{\otimes}_B E \to E$. Note that χ is a module homomorphism with a right inverse $x \to 1 \hat{\otimes}_B x: E \to A \hat{\otimes}_B E$ which is a B module homomorphism. Observe that this implies that

$$S': 0 \to \ker \chi \to A \hat{\otimes}_B E \to E \to 0$$

is a B split exact sequence of A modules.

Definition (cf. [11, IX, §4]). Let E be an A module (left, right, or bi-). We shall say E is projective if whenever $\alpha: F \to G$ is a surjective A module homomorphism with $0 \to \ker \alpha \to F \to G \to 0$ B split, each A module homomorphism $\beta: E \to G$ lifts to an A module homomorphism $\lambda: E \to F$ with $\alpha \circ \gamma = \beta$.

Because the sequence S' above is B split, it follows that a projective left A module E is a module direct summand of the free left A module $A \hat{\otimes}_B E$. To prove the converse it suffices to show that $A \hat{\otimes}_B E$ is projective for any left B module since direct summands of projectives are obviously projective. Consider, then, a module homomorphism $\alpha: F \to G$ with a right inverse $\lambda: G \to F$ which is a B module homomorphism. Then $f \to \lambda \cdot f$ serves to lift each $f \in \mathrm{Hom}_B(E, G)$ to an element of $\mathrm{Hom}_B(E, F)$. In view of Proposition 3.2, we conclude that each element of $\mathrm{Hom}_A(A \otimes_B E, G)$ lifts to an element of $\mathrm{Hom}_A(A \otimes_B E, F)$. Analogous proofs can be given which show that projective right modules and free bimodules are precisely the module direct summands of free right modules and free bimodules respectively.

3.4. *Dimension.* Let E be a left (right, bi-) module over A. A projective resolution of length n is an exact sequence of the form $0 \to X_n \to \cdots \to X_0 \to E \to 0$ where each X_i is a projective module. If the sequence is B split, we say that the resolution is B split. The smallest integer n for which E has a B split projective resolution of length n is the homological dimension of E, h. dim E. Note that E is projective if and only if h. dim $E = 0$.

Definition (cf. [11, VI]). The left global dimension of A, l. gl. dim A, is $\sup\{\text{h. dim } E \mid E \text{ is a left } A \text{ module}\}$. The bidimension of A is the homological dimension of A as an A bimodule, that is, the length of the shortest B split projective bimodule resolution of M.

4. **Separable algebras.** It is now possible for us to extend Taylor's generalization of classically separable algebras (cf. [16, §5]) and to establish some basic structure theorems. We assume that B is a commutative algebra.

4.1. *Enveloping algebras.* If A is a B algebra the map

$$((a_1 \hat{\otimes}_B a_2), (a_3 \hat{\otimes}_B a_4)) \longrightarrow a_1a_3 \hat{\otimes}_B a_4a_2: (A \hat{\otimes}_B A) \times (A \hat{\otimes}_B A) \longrightarrow A \hat{\otimes}_B A$$

provides the B module $A \hat{\otimes}_B A$ with a jointly continuous multiplication under which it becomes a B algebra with identity $1 \hat{\otimes}_B 1$. We shall call $A \hat{\otimes}_B A$ the enveloping algebra of A (relative to B), and denote it by A^e when the base algebra B is understood The above multiplication on A will be denoted $(u, v) \longrightarrow u * v$. The embeddings $a \longrightarrow a \hat{\otimes}_B 1$ and $a \longrightarrow 1 \hat{\otimes}_B a$ induce left and right actions of A on A^e making A^e an A bimodule. On elementary tensors

$$a(a_1 \hat{\otimes}_B a_2) = (a \hat{\otimes}_B 1) * (a_1 \hat{\otimes}_B a_2) = aa_1 \hat{\otimes}_B a_2,$$

and

$$(a_1 \hat{\otimes}_B a_2)a = (1 \hat{\otimes}_B a) * (a_1 \hat{\otimes}_B a_2) = a_1 \hat{\otimes}_B a_2a.$$

Observe that each A bimodule M has the structure of a left A^e module under an action defined by $(a_1 \hat{\otimes}_B a_2, m) \longrightarrow a_1ma_2$. In particular, A itself is a left A^e module.

In order to avoid confusion, we use a special notation for the left module action of A^e on A. That is, for $\xi \in A^e$ we let $\eta(\xi) \in \mathrm{Hom}_B\,(A, A)$ be the operator which determines the action of ξ on A. Thus, if $\xi = a_1 \hat{\otimes}_B a_2$ and $a \in A$, then $\eta(\xi)(a) = a_1aa_2$. Note that $\xi \longrightarrow \eta(\xi)$ is an algebra homomorphism from A into $\mathrm{Hom}_B\,(A, A)$ and that

$$a_1\eta(\xi)(a_2) = \eta(a_1\xi)a_2, \quad \text{and} \quad [\eta(\xi)(a_2)]a_1 = \eta(\xi a_1)a_2.$$

That is, for each $a_2 \in A$, $\xi \longrightarrow \eta(\xi)(a_2)$ is an A bimodule homomorphism.

Because the multiplication $A \times A \longrightarrow A$ in A is continuous and B bilinear, it lifts to a continuous B module homomorphism π_B: $A^e \longrightarrow A$. Since

$$\begin{aligned}[\eta(a_1 \hat{\otimes}_B a_2)](\pi_B(a_3 \hat{\otimes}_B a_4)) &= [\eta(a_1 \hat{\otimes}_B a_2)](a_3a_4) = a_1a_3a_4a_2 \\ &= \pi_B(a_1a_3 \hat{\otimes}_B a_4a_2) = \pi_B((a_1 \hat{\otimes}_B a_2) * (a_3 \hat{\otimes}_B a_4)),\end{aligned}$$

π_B turns out to be a left A^e module homomorphism or, what is the same, a homomorphism of A bimodules. The maps η and π_B are related by the identity $\eta(\xi)(a) = \pi_B(\xi * (a \hat{\otimes}_B 1))$. When no confusion is likely to result we shall replace π_B with the symbol π, and the symbol $a_1 \hat{\otimes}_B a_2$ for an elementary tensor in A by $a_1 \otimes a_2$.

4.2. *Separable algebras.*

Theorem (cf. [11, VII, 5.1]). *For a B algebra A with identity* 1, *the following are equivalent*:

(1) *A is a projective A^e module;*

(2) *the multiplication map $\pi: A^e \to A$ is split as a left A module homomorphism;*

(3) *there is an element e of A^e such that $\pi e = 1$ and $ae = ea$ for all $a \in A$.*

Proof. In view of the characterization of projectives in 3.3 and the fact that $\pi_B: A \hat{\otimes}_B A \to A$ is B split by $a \to a \hat{\otimes} 1$, conditions (1) and (2) are equivalent. Assuming (2), we may choose a left inverse $\Delta \in \mathrm{Hom}_{A^e}(A, A^e)$ for π. Then $\pi(\Delta(1)) = 1$ and $a\Delta(1) = (a \otimes 1)\Delta(1) = \Delta(\eta(1 \otimes a)(1)) = \Delta(a)$. Similarly, $\Delta(1)a = \Delta(a)$. Therefore, $e = \Delta(1)$ satisfies condition (3). Conversely, given e the linear map Δ defined by $\Delta(x) = xe$ is obviously a left inverse for π in $\mathrm{Hom}_{A^e}(A, A^e)$.

A B algebra A is said to be separable relative to B if it satisfies any of the equivalent conditions of the above theorem. The algebra A is central separable or separable, for short, if it is separable relative to its center $Z(A)$. In fact, this is always the case when A is separable relative to any base algebra B. This follows from the fact that $\hat{\otimes}_{Z(A)}: A \hat{\otimes} A \to A \hat{\otimes}_{Z(A)} A$ is a jointly continuous B bilinear map inducing an algebra homomorphism $\rho: A \hat{\otimes}_B A \to A \hat{\otimes}_{Z(A)} A$ such that $\pi_{Z(A)} \circ \rho = \pi_B$. Hence, $\rho(e)$ satisfies (3) above relative to $Z(A)$ if e does relative to B.

The special element e of Theorem 4.2 we call a splitting idempotent.

4.3. *Dimension.* Note that the condition (2) of Theorem 4.2 says exactly that A is a projective A bimodule, i.e. that A has bidimension zero (cf. 3.4). Furthermore,

Proposition (cf. [2, IX, 7.6]). *If A is separable relative to B, then the left global dimension of A relative to B is zero.*

Proof. We show that each left A module M is a direct summand of $A \hat{\otimes}_B M$ and, hence, projective. We do this by producing a right inverse to the homomorphism $\pi_M: A \hat{\otimes}_B M \to M$ where π_M is induced by the map $(a, m) \to am: A \times M \to M$ defining the action of A on M. Set $\kappa(m) = (1 \hat{\otimes}_B \pi_M)(e \hat{\otimes} m)$. Then κ is a homomorphism since

$$\kappa(am) = (1 \hat{\otimes}_B \pi_M)(e \hat{\otimes} am) = (1 \hat{\otimes}_B \pi_M)(ea \hat{\otimes} m)$$

$$= (1 \hat{\otimes}_B \pi_M)(ae \otimes m) = a(1 \otimes_B \pi_M)(a \otimes m) = a\kappa(m).$$

Also, $\pi_m \circ \kappa = 1_M$ since $\pi_M \circ \kappa(m) = \pi_M(1 \hat{\otimes}_B \pi_M)(e \hat{\otimes} m) = \pi_B(e)m = 1 \cdot m = m$.

Similarly, we can show rt. gl. dim $A = 0$.

Suppose, now, that I is a closed left (right) ideal of the algebra A which is separable over B and suppose that the algebra A/I is complete and that the sequence of left (right) A modules $A^I: 0 \to I \to A \to A/I \to 0$ is B split. By the above proposition, A/I is a projective left (right) A module and, hence, the sequence A^I is also A split; that is, A is the left (right) A module direct sum of I and A/I. Using the fact that A is a left A^e module, we obtain a similar result for two sided ideals of A.

4.4. **Proposition.** *If the algebra A is separable relative to B, then every A bimodule is projective relative to B. Consequently, when I is a two sided closed ideal such that $0 \to I \to A \to A/I \to 0$ is B split as a sequence of A bimodules, A is the algebra direct sum of I and A/I.*

Proof. We will prove in Proposition 4.10 that the enveloping algebra A^e is also separable relative to B. From this and Proposition 4.3 it follows that every left A^e module or, what is the same, every A bimodule is projective.

4.5. *The left ideal K.* Recall that a splitting idempotent $e \in A^e$ has the property that $ae = ea$ for all $a \in A$. We denote by K the set of all $\xi \in A^e$ such that $a\xi = \xi a$ for $a \in A$.

Proposition (cf. [1, 1.4]). *Let A be any algebra and let K be defined as above. Then*

(a) *K is the right annihilator in the algebra A^e of* $\ker(\pi\colon A^e \to A)$;

(b) *if $\eta\colon A^e \to \operatorname{Hom}_B(A, A)$ is the map of 4.1, then $\eta(\xi)a \in Z(A)$ for each $\xi \in K$, $a \in A$;*

(c) *π maps K onto an ideal of $Z(A)$ and A is separable if and only if $\pi(K) = Z(A)$.*

Proof. (a) The equation $a\xi = \xi a$ is equivalent to $(a \otimes 1 - 1 \otimes a) * \xi = 0$. Hence, K is the right annihilator of the left ideal I in A^e generated by $\{a \otimes 1 - 1 \otimes a\colon a \in A\}$. Obviously, $I \subset \ker \pi$. On the other hand, I contains every element of the form

$$a \otimes b - \pi(a \otimes b) \otimes 1 = a \otimes b - ab \otimes 1 = (a \otimes 1) * (1 \otimes b - b \otimes 1)$$

for $a, b \in A$ and, hence, every element of the form $\xi - \pi(\xi) \otimes 1$ for $\xi \in A^e$. Thus $\pi(\xi) = 0$ implies that $\xi \in I$. We conclude that $I = \ker \pi$.

(b) This follows from the computation

$$a'\eta(\xi)(a) = \eta(a'\xi)(a) = \eta(\xi a')(a) = [\eta(\xi)(a)]a' \quad \text{for } \xi \in K \text{ and } a, a' \in A.$$

(c) Since $\pi(\xi) = \eta(\xi)(1)$, π is a left A^e module homomorphism, and K is a left ideal of A^e, we conclude that $\pi(K)$ is an ideal in $Z(A)$. By definition, A is separable if and only if there is an element $e \in K$ with $\pi(e) = 1$, hence, if and only if $\pi(\kappa) = Z(A)$.

4.6. **Corollary.** *If A is separable, then $Z(A)$ is a $Z(A)$ module direct summand of A.*

Proof. If $e \in A^e$ is a splitting idempotent, then $\eta(e)$ is a $Z(A)$ module homomorphism of A into $Z(A)$. Since $\eta(e)(1) = \pi(e) = 1$, $\eta(e)$ is the identity on $Z(A)$. Thus, $\eta(e)$ is a $Z(A)$ module projection of A onto $Z(A)$.

4.7. **Corollary.** *If I is an ideal of $Z(A)$, then $IA \cap Z(A) = I$ and $\overline{IA} \cap Z(A) = \overline{I}$.*

Proof. We prove only the second equality. Let $A = A \oplus Z(A)$ be the representation provided by Corollary 4.6. Then $\overline{IA} = \overline{IA_1} \oplus IZ(A)$ and $\overline{IA_1} \cap Z(A) = 0$. Thus, $\overline{IA} \cap Z(A) = \overline{IZ(A)} = \overline{I}$.

4.8. **Corollary.** *A splitting idempotent $e \in A^e$ is, indeed, an idempotent of A^e.*

Proof. By Proposition 4.5(a), the element $e - 1 \otimes 1$ of $\ker(\pi)$ must be annihilated by multiplication on the right by any element of K. Consequently $0 = (e - 1 \otimes 1)e = e^2 - e$.

4.9. **Proposition** (cf. [1, 1.4]). *Suppose ρ is a homomorphism with dense range from the algebra A_1 separable relative to B to the complete algebra A_2. Let $\tilde{\rho}: Z(A_1) \to Z(A_2)$ be the restriction of ρ to the center of A_1. Then, A_2 is separable relative to B and $\tilde{\rho}$ also has dense range. If ρ is onto, so is $\tilde{\rho}$.*

Proof. Let e_1 be the splitting idempotent (S.I.) for A_1 and set $e_2 = (\rho \hat{\otimes}_B \rho)(e_1)$. Then, for an element a of A_1 we have

$$\rho(a)e_2 = \rho(a)[\rho \hat{\otimes}_B \rho(e_1)] = (\rho \hat{\otimes}_B \rho)(a \otimes 1) * (\rho \hat{\otimes}_B \rho)(e_1)$$

$$= (\rho \hat{\otimes}_B \rho)((a \otimes 1) * e_1) = (\rho \hat{\otimes}_B \rho)((1 \otimes a) * e_1) = \cdots = e_2\rho(a).$$

Since ρ has dense range, we conclude that $a_2e_2 = e_2a_2$ for all $a_2 \in A$. From $\rho \circ \pi = \pi \circ (\rho \hat{\otimes}_B \rho)$, we conclude that $\pi(e_2) = 1$ and, hence, that e_2 is an S.I. for A_2. Thus, A_2 is separable relative to B. If $\eta(e_i): A_i \to Z(A_i)$ $(i = 1, 2)$ are the projections of Corollary 4.6, one can show easily that $\rho \cdot \eta(e_1) = \eta(e_2)$. Thus, if $\rho(a_\alpha) \to z \in Z(A_2)$, then $\rho(\eta(e_1)(a_\alpha)) = \eta(e_2)(\rho(a_\alpha)) \to \eta(e_2)(z) = z$. This implies that $\tilde{\rho}$ has dense range. If ρ is onto, the projection of a preimage of $z \in Z(A_2)$ is also a preimage of z and, hence, $\tilde{\rho}$ is also onto.

4.10. **Proposition** (cf. [1, 1.5]). *If A is separable relative to B, then so is A^e. If $B = Z(A)$, then $b \to b \hat{\otimes}_{Z(A)} 1: Z(A) \to Z(A^e)$ is an isomorphism.*

Proof. The map $(a_1, a_2, a_3, a_4) \to a_1 \hat{\otimes}_B a_4 \hat{\otimes}_B a_2 \hat{\otimes}_B a_3$ lifts to a B module isomorphism $\rho: (A^e)^e \to (A^e)^e$. Suppose we choose a splitting idempotent e for A and set $e' = \rho(e \hat{\otimes} e)$. Then, denoting by π^e the multiplication map $(A^e)^e \to A$, we have that $\pi^e \cdot \rho = \pi \hat{\otimes}_B \pi$ and, hence, that $\pi^e(e') = \pi^e \circ \rho(e \hat{\otimes} e) = \pi e \hat{\otimes} \pi e = 1 \otimes 1$. Thus e' will be a splitting idempotent for A^e if $\xi e' = e'\xi$ for every element ξ of A. It suffices to check this when 1 is an elementary tensor. A short calculation yields the identity

$$(a_1 \otimes a_2)\rho(c_1 \otimes c_2 \otimes c_3 \otimes c_4) - \rho(c_1 \otimes c_2 \otimes c_3 \otimes c_4)(a_1 \otimes a_2)$$
$$= \rho[a_1(c_1 \otimes c_2) \otimes (c_3 \otimes c_4)a_2 - (c_1 \otimes c_2)a_1 \otimes a_2(c_3 \otimes c_4)].$$

From this and density considerations it follows that

$$(a_1 \otimes a_2)e' - e'(a_1 \otimes a_2) = \rho(a_1 e \otimes ea_2 - ea_1 \otimes a_2 e)$$
$$= \rho(a_1 e \otimes ea_2 - a_1 e \otimes ea_2) = \rho(0) = 0.$$

We let K and K^e be the right annihilators of $\ker(\pi)$ and $\ker(\pi e)$ in A^e and $(A^e)^e$ respectively. By Proposition 4.5(c) $\pi(eA^e) = Z(A)$ when A is separable relative to B. Since $eA^e \subset K$ and $\pi(ek) = \pi(k)$ for k in K, it follows that $\pi(eA^e) = Z(A)$. The analogue of this must hold for A^e which is also separable relative to B. Thus

(i) $$\pi^e[e'(A^e)^e] = Z(A^e).$$

Assume now that $B = Z(A)$. If we can demonstrate that

(ii) $$\rho^{-1}[\rho(e \mathbin{\hat{\otimes}} e)(A^e)^e] = eA^e \mathbin{\hat{\otimes}}_B eA^e,$$

then using (i) and definition of e' we obtain

$$Z(A^e) = \pi^e \circ \rho(\rho^{-1}[\rho(e \mathbin{\hat{\otimes}} e)(A^e)^e])$$
$$= \pi \mathbin{\hat{\otimes}}_{Z(A)} \pi(eA^e \otimes_{Z(A)} eA^e) \subset Z(A) \mathbin{\hat{\otimes}}_{Z(A)} Z(A).$$

Now, the restriction of π to $Z(A) \mathbin{\hat{\otimes}}_{Z(A)} Z(A)$ which is obviously contained in $Z(A^e)$ is a $Z(A)$ module inverse to the map $b \longrightarrow b \mathbin{\hat{\otimes}}_{Z(A)} 1$. Thus, it remains to prove (ii). This follows from the calculation

$$\rho^{-1}[\rho(a_1 \otimes a_2 \otimes a_3 \otimes a_4)(a_1' \otimes a_2' \otimes a_3' \otimes a_4')]$$
$$= \rho^{-1}[a_1 a_1' \otimes a_2' a_4 \otimes a_3' a_2 \otimes a_3 a_4'] = a_1 a_1' \otimes a_3' a_2 \otimes a_3 a_4' \otimes a_2' a_4$$
$$= (a_1 \otimes a_2)(a_1' \otimes a_3') \otimes (a_3 \otimes a_4)(a_4' \otimes a_2').$$

5. **Symmetry of splitting idempotents.** We call an element ξ of A symmetric if it is left fixed by the transposition map $T: A^e \longrightarrow A^e$ induced by $x \otimes_B y \longrightarrow y \otimes_B x$ on elementary tensors. Observe that although every element of the form $\Sigma\, \alpha_{ij} e_{ki} \otimes e_{jk}$ with $\Sigma\, \alpha_{ii} = 1$ qualifies as a splitting idempotent for the matrix algebra $M_n(\mathrm{C})$, only $e = (1/n)\Sigma\, e_{ij} \otimes e_{ji}$ is symmetric. We defer until §7 our partial answer to the question of whether or not a separable algebra must have a symmetric splitting idempotent. Here, we prove that if one exists it must be unique.

5.1. *Trace.* For a set $S \subset A^e$ let S^T denote $T(S)$ and, as before, let K be the right annihilator of $\delta(A) = \{a \otimes_B 1 - 1 \otimes_B a | a \in A\}$. The reader may readily verify that K^T is the left annihilator of $\delta(A)$ and that both $\pi(KK^T)$ and $\pi(K \cap K^T)$ are ideals of $Z(A)$.

Lemma. *Let A be a B algebra. For any element ξ of $K \cap K^T$ the $Z(A)$ module homomorphism $\eta(\xi)\colon A \to Z(A)$ of Proposition 4.5 satisfies $\eta(\xi)(a_1a_2) = \eta(\xi)(a_2a_1)$.*

Proof. If $\xi \in K \cap K^T$, then

$$\eta(\xi)(a_1a_2) = \pi(\xi*(a_1a_2 \otimes 1)) = \pi(\xi*(a_1 \otimes a_2)) = \pi(\xi*(a_1 \otimes 1)*(1 \otimes a_2))$$
$$= \pi(\xi*(1 \otimes a_1) * (1 \otimes a_2)) = \pi(\xi*(1 \otimes a_2a_1)) = \eta(\xi)(a_2a_1).$$

In the particular case where $\xi \in K \cap K^T$ we will denote $\eta(\xi)$ by tr_ξ and call it the trace defined by ξ.

If A is separable, then K contains a splitting idempotent e. If, in addition, e is symmetric, then $e \in K \cap K^T$ and $\mathrm{tr}_e(1) = \eta(e)(1) = \pi(e) = 1$. Hence, the projection $\mathrm{tr}_\xi = \eta(e) \in \mathrm{Hom}_{Z(A)}(A, Z(A))$ is a normalized trace whenever e is a symmetric splitting idempotent.

5.2. **Proposition.** *The restriction of π to $K \cap K^T$ is multiplicative. If $\pi(K \cap K^T)$ is not a proper ideal of $Z(A)$, then $K \cap K^T$ is a commutative subalgebra of A^e isomorphic to $Z(A)$ under π and is comprised entirely of symmetric elements of A^e.*

Proof. Since π is a bimodule homomorphism $\pi(\xi_1\xi_2) = \eta(\xi_1)\pi(\xi_2)$. If $\xi_2 \in K$, then $\pi(\xi_2) \in Z(A)$. But, the action of $\eta(\xi_1)$ on $Z(A)$ is just multiplication by $\pi(\xi_1)$. Hence π_K is multiplicative. To establish the second claim, we first verify for elements x and y of $K \cap K^T$ the equation

$$\text{(i)} \qquad \pi(y^T)x = (\pi x)y = x * y.$$

To this end, we approximate y by finite sums of elementary tensors, say $\sum_{i=1}^{n_\alpha} s_i^\alpha \otimes t_i^\alpha \to y$ where n_α is an integer for each α in a directed set $\mathfrak{A}$. Recalling that $\pi(y^T)$ is central, we obtain

$$\pi(y^T)x = \lim_\alpha \sum x * (t_i^\alpha s_i^\alpha \otimes 1) = \lim_\alpha \sum [x*(t_i^\alpha \otimes 1)] * (s_i^\alpha \otimes 1)$$
$$= \lim \sum [x * (1 \otimes t_i^\alpha)] * (s_i^\alpha \otimes 1) = \lim_\alpha \sum x * (s_i^\alpha \otimes t_i^\alpha)$$
$$= x* \lim_\alpha \sum s_i^\alpha \otimes t_i^\alpha = x * y.$$

By approximating x by finite sums of elementary tensors we can show in similar fashion that $(\pi x)y = x * y$. This proves (i), which implies that the center of $K \cap K^T$ contains every symmetric element of $K \cap K^T$.

Now, if $\pi(K \cap K^T)$ is not a proper ideal of $Z(A)$, choose $x \in K \cap K^T$ with $\pi x = 1$. Then x is a splitting idempotent. Applying (i) with $y = x$ in the calculation $x = (\pi x)x = \pi(x^T)^T x = (\pi x)x^T = x^T$, we discover that x is symmetric. Using this x in (i), we obtain

$$y = (\pi x)y = \pi(y^T)x = \pi(y^T)x^T = [\pi(y^T)x]^T = y^T.$$

Hence, each $y = (\pi y)x$ of $K \cap K^T$ is symmetric. Since $\pi y = 0$ implies $y = 0$, $\pi: K \cap K^T \to Z(A)$ is an isomorphism.

5.3. **Corollary.** *A necessary and sufficient condition that an algebra A be separable relative to B with symmetric splitting idempotent is that* $\pi(K \cap K^T) = Z(A)$. *In this case there is a unique symmetric splitting idempotent.*

Note that when A is separable relative to B, A has a symmetric splitting idempotent if and only if $\pi(K^T) = Z(A)$. For, if x and y are elements of K such that $\pi x = \pi(y^T) = 1$, $x * y$ is a splitting idempotent in $KK^T \subset K \cap K^T$.

5.4. *An example.* Consider the algebra End (E) of continuous endomorphisms of a vector bundle (E, p, X) with locally compact base X and n-dimensional fibre. Locally End (E) is an algebra of matrix-valued functions of the type considered in 1.5. If $U \subset X$ is an open set to which the restriction E_U of the vector bundle is trivial, then choosing a basis $\{e_i\}_{i=1}^n$ for the sections over U determines an isomorphism End $(E_U) \to C(U, M_n(\mathbf{C}))$. If $e_{ij} \in$ End (E_U) is defined by $e_{ij}e_k = \delta_{jk}e_i$, then by setting $e = (1/n)\sum e_{ij} \otimes_{C(U)} e_{ji}$, we construct a symmetric splitting idempotent for E_U in End $(E_U) \otimes_{C(U)}$ End (E_U). Indeed, $e = 1$, $e^2 = e = e^T$ and $se = es$ for every $s \in$ End (E_U) (since this is so for the generators e_{ij} of End (E_U) relative to its center $C(U)$).

Proposition. End (E) *is separable relative to* $C(X)$.

Proof. Let $\{U_\alpha\}_{\alpha \subset \mathfrak{A}}$ be a covering of X such that every E_{U_α} is trivial. Let e be the symmetric splitting idempotent for End (E_{U_α}). The e_α's define a global S.S.I. since the restriction of any two, say e_α and e_β, to $U_\alpha \cap U_\beta$ must yield the same S.S.I. for End $(E_{U_\alpha \cap U_\beta})$ by Corollary 5.3.

Observe that by a similar argument the algebra of smooth (analytic) endomorphisms of a smooth (analytic) vector bundle (E, p, X) is separable relative to $C^\infty(X)$ (resp. $\mathcal{O}(X)$).

5.5. **Remark.** To this point our stipulation that algebras have jointly continuous multiplication has been entirely arbitrary. Indeed, by replacing the projective with the inductive tensor product (cf. [7, I, §3, No. 1]) all our results from 2.2 through 5.3 remain valid with at most a slight modification of wording.

6. Simple separable algebras. For the algebra of matrix-valued functions $C(X, M_n(\mathbf{C}))$, evaluation at a given point of X induces a homomorphism onto $M_n(\mathbf{C})$. In this section we study factor algebras of an algebra A separable relative to B modulo a maximal ideal M and determine conditions which guarantee that A/M is a matrix algebra.

6.1. **Remarks.** Until now, the discussion has been fairly general. However, we shall soon begin imposing special topological hypotheses on our algebra A. Several circumstances make this necessary. For example, we will want factor algebras of A to be complete; this forces us to require that A be fully complete, i.e. that A be a Ptak space (cf. [14, IV, §8]). In order to use the theorem of Gelfand-Mazur (cf. [13, Proposition 2.9]) on the center of A, we shall, at certain points, need to assume A is an l.m.c. algebra—that is, an algebra whose topology is defined by a family of submultiplicative seminorms (cf. [13]).

We will call an l.c.s. X a WSD space if $X' \otimes X'$ is weak-star dense in $(X \hat{\otimes} X)'$ under the embedding $f \hat{\otimes} g \rightarrow \pi \circ (f \hat{\otimes} g)$. The condition WSD is closely related to the approximation property (AP). Taylor proved in [16] that a Banach algebra, separable relative to $\mathbf{C}$, which satisfies AP must be a finite direct product of complex matrix algebras. Replacing AP with WSD we prove a variation of this theorem which applies to certain simple non-Banach algebras. To insure the inheritance of WSD by separated quotients we are compelled in §7 to call upon nuclearity—a property which is stronger for complete l.c.s.'s than both AP and WSD (cf. [7, I. §5, No. 1]) that is preserved under nearly all l.c.s. constructions including the formation of the relative (projective) tensor product in 2.3 (cf. [18, Proposition 50.1]).

At one point, we shall need reflexivity. However, we are still left with a fairly general class, that of nuclear F-algebras, which satisfies all the topological hypotheses we impose at various stages.

6.2. *Simple algebras.* Note that each $f \in A'$ induces a map $1 \hat{\otimes} f\colon A \hat{\otimes}_{\mathbf{C}} \mathbf{C} = A$. For an algebra A and a pair $(\xi, f) \in (A \hat{\otimes} A) \times A'$, we denote by $T(\xi, f)$ the operator on A defined by $a \rightarrow (1 \hat{\otimes} f)((1 \otimes a) * \xi)$. If A is separable relative to $\mathbf{C}$ with splitting idempotent e, then the calculation

$$T(e, f)(a) = (1 \hat{\otimes} f)[(1 \otimes a) * e] = (1 \hat{\otimes} f)[(a \otimes 1) * e] = a(1 \hat{\otimes} f(e)) = aT(e, f)(1)$$

shows that $T(e, f)$ is a right multiplication operator for every continuous linear functional f on A.

If A is a Fréchet space or a complete nuclear DF space, every element of $A \hat{\otimes} A$ has an infinite sum representation $\Sigma_{i=1}^{\infty} \lambda_i a_i \otimes b_i$ for null sequences $\{a_i\}$ and $\{b_i\}$ in A and a positive sequence $\{\lambda_i\}$ in $\mathbf{C}$ with $\Sigma_{i=1}^{\infty} \lambda_i = 1$ (cf. [7, II, §3, No. 1]). Let $U = \{a \in A\colon |f(a)| < 1\}$. Since $\{b_i\}$ is a null sequence, there exists a neighborhood V of zero in A such that $b_i V \subset U$ for all i. Then since $T(\xi, f)(a) = \Sigma \lambda_i a_i f(b_i a)$, $T(\xi, f)$ maps V into the closed, convex, circled hull of the null sequence $\{a_i\}$. Since this is a compact set, $T(\xi, f)$ is a compact operator for every continuous linear functional f and every $\xi \in A^e$.

Proposition. *Let A be a simple algebra separable relative to* $\mathbf{C}$. *If A is a WSD space and either an F space or a nuclear DF space, then A is isomorphic to a full matrix algebra $M_n(\mathbf{C})$ for some integer n.*

Proof. It suffices to show that A is finite dimensional and, hence classically separable (cf. 1.3). To accomplish this we demonstrate that the ideal I of compact right multipliers is nontrivial. If otherwise, $T(e, f_2) = 0$ for some splitting idempotent e and all $f_2 \in A'$. But, then $0 = f_1[T(e, f_2(1)] = f_1 \otimes f_2(e)$ for all $f_1, f_2 \in A'$ which implies because A is a WSD space, that E cannot be separated from zero in A^e—a contradiction. Since I is nontrivial and A is simple, $I = A$. Hence, the identity operator is a compact right multiplier and A must be finite dimensional.

6.3. **Proposition.** *Let A be a simple algebra separable relative to* $\mathbf{C}$ *which is also a reflexive WSD space. If there is at least one nontrivial continuous submultiplicative seminorm on A, then A is isomorphic to a complex matrix algebra.*

Proof. Let us choose a nontrivial continuous submultiplicative seminorm ρ on A. Since A is simple, A is a normed algebra under the weaker topology induced by ρ. We may assume that the completion $A\rho$ of A in the ρ topology is a simple Banach algebra. If otherwise, we choose a maximal ideal M of $A\rho$. Since the image of A in $A\rho/M$ under the composition $A \to A\rho \to A\rho/M$ is dense, $M \cap A$ must be a proper ideal of A. Since A is simple $M \cap A = (0)$ and, consequently, A is a dense subalgebra of the simple algebra $A\rho/M$ which we may as well assume was A in the first place. Since $i\colon A \to A\rho$ is an injection with dense range, we conclude that both A is separable relative to $\mathbf{C}$, by Proposition 4.9, and $i^t\colon A\rho' \to A'$ has weak-* dense range. In view of the reflexivity of A, $i^t(A\rho')$ is strongly dense in A'. Thus, the composition $A\rho' \hat{\otimes} A\rho' \to A' \hat{\otimes} A' \to (A \hat{\otimes} A)'$ has weak-* dense range, since A satisfies WSD.

Let $\gamma \in (A \hat{\otimes} A)'$ separate the splitting idempotent e for A from zero. Now, some finite sum $\Sigma_{j=1}^n f_j \otimes g_j \in A\rho' \hat{\otimes} A\rho'$ must send $i \hat{\otimes} i(e)$ close enough to $\gamma(e)$ to be nonzero. But, then for some j, $f_j \hat{\otimes} g_j$ must separate $i \hat{\otimes} i(e)$ from zero. Thus, $T(i \hat{\otimes} i(e), g_j)$ is a nontrivial compact right multiplier in $A\rho$. Since $A\rho$ is simple, every element—in particular the identity—is a compact right multiplier. Therefore, $A\rho$ is finite dimensional and the desired result follows immediately.

6.4. **Proposition.** *Suppose A is a separable l.m.c. algebra with center the complexes. If A is barreled, nuclear, and fully complete, then A is simple.*

Proof. Choose a closed maximal ideal M (one exists since A is l.m.c.). The factor algebra A/M is complete (A is fully complete) and separable relative to $\mathbf{C}/M \cap \mathbf{C}) \cong \mathbf{C}$ by Proposition 4.9. Since A/M satisfies the hypotheses of Proposition 6.3, it is a matrix algebra. But the finite dimensionality of A/M forces the sequence $0 \to M \to A \to A/M \to 0$ to split relative to $\mathbf{C}$. Since A/M is a projective A bimodule by Proposition 4.4, A must be the algebra direct sum of M and A/M. This cannot happen when $Z(A) = \mathbf{C}$ unless $M = (0)$.

7. **Homogeneous algebras.** Denoting by Δ_A the set of closed maximal ideals of the separable l.m.c. algebra A, we use the bijection of an analogue of a result of Auslander and Goldman to transfer the Gelfand topology of $\Delta_{Z(A)}$ to Δ_A. We show under appropriate conditions on A and Δ_A, A is a direct product of countably many separable algebras each having the property that modulo any of its maximal ideals it is a matrix algebra of fixed dimension.

7.1. **Theorem** (cf. [1, 3.2]). *Suppose A is an l.m.c. algebra which is barreled, nuclear, and fully complete. If A is separable, then $M \to M \cap Z(A)$ is a bijective correspondence between Δ_A and $\Delta_{Z(A)}$. For every closed maximal ideal M, the factor algebra A/M is a complex matrix algebra.*

Proof. If $m \in \Delta_{Z(A)}$, then by Corollary 4.7 we have $m = \overline{mA} \cap Z(A)$. Thus, the proper ideal $\overline{mA}$ is maximal and the correspondence of the theorem surjective provided $A/(\overline{mA})$ is simple.

Observe that $A/(\overline{mA})$ is a barreled, nuclear, fully complete, l.m.c. algebra since all of these properties are retained by separated factor algebras. By Proposition 4.9, the algebra $A/(\overline{mA})$ is separable relative to its center $Z(A)/m$ which is a simple commutative l.m.c. algebra and, hence, the complexes by the theorem of Gelfand-Mazur. We conclude that $A/(\overline{mA})$ is simple since it satisfies the hypotheses of Proposition 6.4.

For $M \in \Delta_{Z(A)}$, $M \cap Z(A)$ is a proper closed ideal of $Z(A)$, for it does not contain the identity. By Proposition 4.9 the simple algebra A/M is separable (barreled, nuclear, fully complete) with center $Z = Z(A)/(M \cap Z(A))$ which must also be simple by Corollary 4.7. But Z is also an l.m.c. algebra and, hence, is $\mathbf{C}$. Thus, $M \cap Z(A) \in \Delta_{Z(A)}$ and appealing to Proposition 6.3 since A/M is a reflexive WSD space (cf. 6.1) we have that A/M is a complex matrix algebra.

An l.m.c. algebra for which all the conclusions of Theorem 7.1 are valid we shall call matrix-modular. Some of the conclusions of Theorem 7.1 remain valid under a somewhat different set of hypotheses:

7.2. **Proposition.** *Suppose A is a separable algebra which is a nuclear DF space. If A contains a dense ideal A_1 which is l.m.c. under a possibly finer topology τ, then for any closed maximal ideal M of A the factor algebra A/M is a complex matrix algebra.*

Proof. A complete DF space is fully complete (cf. [14, IV, p. 162]) and a separable algebra is complete. Thus, if M is a closed maximal ideal of A, A/M is a simple separable algebra, as before, with center $Z = Z(A)/(M \cap Z(A))$ which is simple by Corollary 4.7. Since $A_1/(M \cap A_1)$ is dense in A/M and is, in fact, an ideal, it must coincide with A/M. Furthermore, its center is a simple l.m.c. under the τ topology and, hence, is the complexes. Noting that the separated quotient of a nuclear DF space is a nuclear DF space (cf. [14, IV, p. 196]), we see that A/M is a complex matrix algebra by appealing to Proposition 6.2.

7.3. *An example.* Let G be a compact Lie group. Under convolution, the space $\mathcal{E}'(G)$ of distributions is an algebra with jointly continuous multiplication and identity δ_e, the unit point mass at the identity of G. Also, $\mathcal{E}'(G)$ is a nuclear DF space. Furthermore, Taylor (cf. [17, Proposition 7.3]) has shown that Haar measure on G can be used to construct a splitting idempotent for $\mathcal{E}'(G)$ in $\mathcal{E}'(G) \hat{\otimes} \mathcal{E}'(G)$. Hence, $\mathcal{E}'(G)$ is separable relative to $\mathbf{C}$. Note that $\mathcal{E}'(G)$ contains a dense ideal, $C^\infty(G)$, which is an F-algebra. It follows from Proposition 7.2 that modulo any maximal ideal $\mathcal{E}'(G)$ is a finite dimensional matrix algebra. In fact, because $0 \to M \to \mathcal{E}'(G) \to \mathcal{E}'(G)/M \to 0$ is $\mathbf{C}$ split for every $M \in \Delta_{\mathcal{E}'(G)}$, $\mathcal{E}'(G)$ contains the direct sum $\bigoplus_{M \in \Delta_{\mathcal{E}'(G)}} \mathcal{E}'(G)/M$. Thus, a good deal of the Peter-Weyl theory for compact Lie groups can be deduced by essentially homological methods.

7.4. *Absolute projectives.* If A is a complex algebra and M a left module over A, then, M is projective in the algebraic sense if for each surjective module homomorphism $\alpha: X \to Y$ every $\beta \in \mathrm{Hom}_A(M, Y)$ lifts to a $\gamma \in \mathrm{Hom}_A(M, X)$ with $\beta = \alpha \circ \gamma$. Note that we are not referring to topological modules here; there is no continuity requirement on homomorphisms, and no requirement that α be $\mathbf{C}$ split (such a requirement would be meaningless in the absence of topology). In order to distinguish this algebraic notion of projectivity from the one we have been using, we call such a module M absolutely projective.

A topological module over a topological algebra may or may not be absolutely projective when, ignoring topology, we consider it an algebraic module.

The algebraic analogue of the result in 3.3 is that M is absolutely projective if and only if it is a direct summand of an absolutely free module, i.e. a module of the form $A \hat{\otimes}_{\mathbf{C}} X$ (algebraic tensor product) where X is a vector space over $\mathbf{C}$. If M is finitely generated then it is absolutely projective if and only if it has the above form for some finite dimensional vector space X, i.e. if and only if it is a direct summand of $A^n = A \hat{\otimes}_{\mathbf{C}} \mathbf{C}^n$ for some n.

It seems reasonable to guess that a topological module which is finitely generated and projective must also be absolutely projective and, hence, be a direct summand of A^n; however, we have not been able to prove this.

7.5. *Homogeneity.* Let B be a commutative l.m.c. algebra with identity. Then the algebra $M_n(B)$ is matrix-modular. What is more, the matrix algebras $M_n(B)/(mM_n(B))$ are of fixed dimension as m varies throughout the maximal ideal space of B. This uniformity of dimension is shared with algebras of the form $eM_n(B)e$ for e an idempotent of $M_n(B)$. Such an algebra is called a corner over B and may be described as the algebra of endomorphisms of a finitely generated, absolutely projective B bimodule. A typical example is the algebra of endomorphisms of a vector bundle E with compact base X. The algebra End E is not only a corner over $C(X)$; it is in fact locally of the form $M_n(B)$ (i.e. if E_U is trivial, then End $(E_U) \cong M_n(C(U))$).

Definitions. (1) A topological algebra A is n-homogeneous if for every closed maximal ideal M, A/M is isomorphic to $M_n(\mathbf{C})$.

(2) A matrix-modular algebra A is said to have the local semibasis property if for any $M \in \Delta_A$ a preimage S of a vector space basis for A/M is a preimage of a basis for A/M_1 for every M_1 in a neighborhood $\mathfrak{U}(S)$ of M. The set S is called a semibasis for A over $\mathfrak{U}$ relative to the center.

Clearly, an l.m.c. algebra with the local semibasis property and a connected maximal ideal space is homogeneous. We will show that conditions on a separable l.m.c. algebra A that are only slightly stronger than those of Theorem 7.1 imply, for each continuous submultiplicative seminorm ρ on A, that $A\rho$ has the local semibasis property, that $A\rho$ is a finitely generated, absolutely projective $Z(A\rho)$ module, and that $A\rho$ is a finite direct product of homogeneous algebras. Under the additional restriction that Δ_A is locally compact it will follow (cf. 7.13) that A itself has the local semibasis property and that A is a countable direct product of homogeneous separable algebras.

Our notion of homogeneity is weaker than the standard one (cf. [8]) which entails in addition that every primitive ideal be maximal. For C^* algebras, where the two notions coincide, Fell [6] and Tomiyama and Takesaki [15] have shown that homogeneity has some strong consequences. Indeed, if A is an n-homogeneous C^* algebra, then A is a corner over its center $C(\Delta_A)$. Kaplansky in [9] conjectured that this result is valid for any semisimple algebra with identity which is homogeneous in the standard sense.

7.6. *A dense ideal.* Recalling the definition of relative tensor product and the proof of Proposition 4.10, we see that if a B algebra A is nuclear and has a symmetric splitting idempotent, then its enveloping algebra $A \hat{\otimes}_B A$ must also be a nuclear algebra with a symmetric splitting idempotent. But, it is not true that the enveloping algebra must be barreled or fully complete if A is. However, if A is

Fréchet, then $A \hat{\otimes} A$ and $A \hat{\otimes}_B A$ are Fréchet and, hence, barreled and fully complete (cf. [14, IV, 6.4]).

Theorem (cf. [3, II, 3.5]). *Suppose A is a nuclear l.m.c. algebra separable relative to B. Then, if $A \hat{\otimes} A$ is barreled and fully complete, $A^e K$ is dense in A^e.*

Proof. If $A^e K$ is a proper ideal, it is contained in a closed maximal ideal since A^e is l.m.c. This maximal ideal has the form IA^e for some $I \in \Delta_{Z(A^e)}$ by Theorem 7.1. Then we have that

$$A = AZ(A^e) = A(\pi K) \subset \pi(A^e K) \subset \pi\overline{(IA^e)} \subset \overline{\pi(IA^e)} = \overline{IA}.$$

This contradicts Corollary 4.7. Thus, the ideal $A^e K$ could not have been proper.

7.7. **Proposition** (cf. [1, 2.1]). *Let A satisfy the conditions of Theorem 7.6. For a continuous submultiplicative seminorm ρ let $B\rho$ be the center of the completion $A\rho$ of the normed algebra $A/\rho^{-1}(0)$. Then, the algebra $A\rho$ is a finitely generated, absolutely projective $B\rho$ bimodule.*

Proof. The right annihilator $K\rho$ of $\{x \otimes_B 1 - 1 \otimes_B x \mid x \in A\}$ in the Banach algebra $(A\rho)^e = A\rho \hat{\otimes}_B A\rho$ contains the image of $K\rho$ under the map $A^e \to (A\rho)^e$. From this and the fact that $A^e K$ is dense in A^e, it follows that the ideal $(A\rho)^e K\rho$ is dense in $(A\rho)^e$. Since there are no dense proper ideals in a Banach algebra, we conclude that $(A\rho)^e K\rho$. Representing the identity of $(A\rho)^e$ as a finite sum $\Sigma_{i=1}^n 1_i * k_i$ where $1_i \in (A\rho)^e$ and $k_i \in K\rho$, we set $a_i = \pi(1_i)$ and $f_i = \eta(k_i) \in \mathrm{Hom}_{B_\rho}(A\rho, B\rho)$ (cf. Proposition 4.5). Then, in view of the fact that $(1_i)_{|B\rho}$ is multiplication by $\pi(1_i)$ we have

$$a = \pi(1 \otimes a) = \pi[(1 \otimes 1)(1 \otimes a)] = \pi\left[\sum_1^n 1_i * k_i * (1 \otimes a)\right]$$

$$= \sum_1^n \pi[(1_i * k_i) * (1 \otimes a)] = \sum_1^n \eta(1_i * k_i)\pi(1 \otimes a)$$

$$= \sum_1^n \eta(1_i)\eta(k_i)a = \sum_{i=1}^n a_i f_i(a).$$

Thus, the map $a \to \{f_i(a)\}_{i=1}^n$ is a $B\rho$ module embedding of $A\rho$ into the free $B\rho$ module $(B\rho)^n$ with left inverse $\{b_i\}_{i=1}^n \to \Sigma_{i=1}^n b_i a_i$.

Observe for any two sided ideal I of A and any integer i between 1 and n that $f_i(I) \subset I \cap B\rho$.

7.8. **Corollary.** *Let A satisfy the conditions of Proposition 7.7. Then for every continuous submultiplicative seminorm ρ on A, $A\rho$ is matrix-modular.*

Proof. We must obtain the conclusions of Theorem 7.1 with A replaced by $A\rho$. Since the separable Banach algebra A is barreled and fully complete, it suffices for us to observe that the role played by nuclearity in the proof of Theorem 7.1 can be played equally well by the finite dimensionality of $A\rho$ over its center and that the latter property is guaranteed by Proposition 7.7.

7.9. **Proposition.** *Suppose $A = \varprojlim A\rho$ satisfies the hypotheses of Proposition 7.7. Then A has a symmetric splitting idempotent.*

Proof. If the topology of A is generated by a set of seminorms S, the topology of $A \hat{\otimes} A$ is generated by $\{p \hat{\otimes} q \mid q \in S\}$ (cf. [14, III. 6.3]). If p is submultiplicative, so is $p \hat{\otimes} p$. Since $\sup(p, q) \hat{\otimes} \sup(p, q)$ dominates $p \hat{\otimes} q$, it follows that $A \hat{\otimes} A$ is the l.m.c. algebra $\varprojlim Ap \hat{\otimes} Ap$.

Observe that the canonical linear map $\gamma\colon A \hat{\otimes} A \to A \hat{\otimes}_{Z(A)} A$ is relatively open and has dense range. Therefore, to show that $A \hat{\otimes}_{Z(A)} A = \varprojlim Ap \hat{\otimes}_{Z(Ap)} Ap$ it suffices to define, for every pair p, q, S with $p \geq q$, homomorphisms ϵ_p, ϵ_q, and ϵ_p, q which render commutative the diagram

$$\begin{tikzcd}
A \hat{\otimes} A \arrow[rr, "\gamma"] \arrow[d] & & A \hat{\otimes}_{Z(A)} A \arrow[dll, "\epsilon_p"'] \arrow[d, "\epsilon_q"] \\
Ap \hat{\otimes}_{Z(Ap)} Ap \arrow[rr, "\epsilon_{p,q}"] & & Aq \hat{\otimes}_{Z(Aq)} Aq
\end{tikzcd}$$

This easily done, for in view of Proposition 4.9

$$\ker(Ap \hat{\otimes} Ap \to Aq \hat{\otimes} Aq \to Aq \hat{\otimes}_{Z(Aq)} Aq) \supset \ker(Ap \hat{\otimes} Ap \to Ap \hat{\otimes}_{Z(Ap)} Ap) \supset \ker \gamma.$$

If each algebra $Ap \hat{\otimes}_{Z(Ap)} Ap$ contains a symmetric splitting idempotent, say ϵ_p, then so must $A \hat{\otimes}_{Z(A)} A$ since by uniqueness the bonding maps ϵ_p, q: $Ap \hat{\otimes}_{Z(Ap)} Ap \to Aq \hat{\otimes}_{Z(AQ)} Aq$ $(p \geq q)$ must send e_p to e_q. Since Ap is separable we have only to show, in view of the remark following Corollary 5.3, that $\pi(Kp)^T = Z(Ap)$. If not, $\pi(Kp)^T$ is contained in a maximal ideal m of $Z(Ap)$. The ideal map of the Banach algebra Ap must be contained in a maximal ideal M. Let ϕp denote the projection of Ap onto Ap/M which is by Corollary 7.8 a complex matrix algebra. We derive a contradiction from the commutative diagram

$$\begin{tikzcd}
(Ap)^e \arrow[r, "T"] \arrow[d, "T \circ (\phi_p \otimes \phi_p)"'] & (Ap)^e \arrow[d, "\phi_p \circ \pi"] \\
(Ap/M)^e \arrow[r, "\pi"] & Ap/M
\end{tikzcd}$$

On the one hand, $\phi \circ \pi(e^T) = 0$ for any splitting idempotent e since $\pi(e^T) \in \pi(Kp)^T \subset M$ and, on the other, the transpose of a splitting idempotent for a matrix algebra such as Ap/M cannot be in the kernel of the multiplication map since the splitting idempotent must have the form $\Sigma\, \alpha_{ij} e_{ki} \otimes e_{jk}$ with $\Sigma\, \alpha_{ii} = 1$.

7.10. Theorem. *Let A be a separable, nuclear, l.m.c. algebra. If A and $A \hat{\otimes} A$ are barreled and fully complete, then for any continuous submultiplicative seminorm ρ on A, $A\rho$ has the local semibasis property.*

Proof. Let $\{e_i\}_{i=1}^n$ be the canonical basis for $(B\rho)^n$ where, from Proposition 7.7, $a \to \Sigma_{i=1}^n f_i(a) e_i$ is a $B\rho$ module embedding γ of $A\rho$ into $(B\rho)^n$ with left inverse Δ: $\Sigma_{i=1}^n b_i e_i \to \Sigma_{i=1}^n b_i a_i$. If $x \in \Delta_{A\rho}$, then $x \cap B\rho \in \Delta_{B\rho}$. Thus, the formula

$$\widehat{\sum_{i=1}^n b_i e_i}(x) = (\hat{b}_1(x \cap B\rho), \cdots, \hat{b}_n(x \cap B\rho))$$

defines a $B\rho$ bimodule homomorphism η_x: $(B\rho)^n \to \mathbf{C}^n$ where the action of $B\rho$ on $\mathbf{C}^n$ is given by the formula

$$b_1(z_1, \cdots, z_n) b_2 = (\hat{b}_1(x \cap B\rho)\hat{b}_2(x \cap B\rho) z_1, \cdots, \hat{b}_1(x \cap B\rho)\hat{b}_2(x \cap B\rho) z_n).$$

We claim that

$$x = \ker(a \to \widehat{\gamma(a)}(x) \colon A\rho \to C^n) = \ker(a \to \hat{a}(x) \colon A\rho \to A\rho/x).$$

First we note that

$$\gamma(a)(x) = 0 \quad \text{iff } \widehat{\sum_{i=1}^n f_i(a) e_i}(x) = 0,$$

$$\text{iff } \widehat{f_i(a)}(x \cap B\rho) = 0 \in \mathbf{C}^n,\ 1 \le i \le n,$$

$$\text{iff } f_i(a) \in x \cap B\rho,\ 1 \le i \le n.$$

Thus, if $\widehat{\gamma(a)}(x) = 0$, then $a = \Sigma_{i=1}^n a_i f_i(a) \in A\rho(x \cap B\rho) \subset x$ and, conversely, if $\hat{a}(x) = 0$ (i.e. $a \in x$), then $\widehat{\gamma(a)}(x) = 0$ since $f_i(a) \in x \cap B\rho$ for $1 \le i \le n$ by the observation following Proposition 7.7.

We can now assert for each $x \in \Delta_{A\rho}$ the existence of a unique injective linear map $\phi(x)$: $A\rho/x \to \mathbf{C}^n$ making commutative the diagram

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d] & (B\rho)^n \arrow[d, "\eta_x"] \\ A\rho/x \arrow[r, "\phi(x)"] & \mathbf{C}^n \end{tikzcd}$$

From the diagram we see that

$$\begin{aligned} n &= \dim \eta_x(B\rho^n) = \dim \eta_x(\gamma A\rho) + \dim \eta_x(\ker \Delta) \\ \text{(i)} \qquad &= \dim \phi(x)(A\rho/x) + \dim \eta_x(\ker \Delta) \\ &= \dim (A\rho/x) + \dim \eta_x(\ker \Delta). \end{aligned}$$

Let us call k elements $s_1, \cdots, s_k$ of $(B\rho)^n$ linearly independent at $x_0 \in \Delta_{B\rho}$ if the elements $\hat{s}_1(x_0), \cdots, \hat{s}_k(x_0)$ form a linearly independent set in $\mathbf{C}^n$. Then for the theorem it will suffice in view of (i) to show the linear independence at a point x_0 implies linear independence at nearby points; for in that event any dim $(A\rho/x)$ elements of $A\rho$ which are linearly independent at x_0 would be a semibasis for $A\rho$ over some neighborhood of x_0. Suppose then, that $s_i = \sum g_{ij} e_j$ $(1 \le i \le k,\ 1 \le j \le n,\ g_{ij} \in B\rho)$ are linearly independent at x_0. It follows that some $k \times k$ subdeterminant of $(\hat{g}_{ij}(x_0))$ is nonzero. By continuity this same subdeterminant must not vanish in a neighborhood U of x_0. But, this is precisely the requirement that the set $\{\hat{s}_i(x)\}_{i=1}^k$ be linearly independent for $x \in U$.

If a local semibasis were a local basis, then $A\rho$ would be a locally free $B\rho$ module in the sense that $\hat{A}\rho|_U$ would have the form $\hat{B}^n_{\rho|U}$ for sufficiently small neighborhoods U of points of $\Delta_{A\rho}$. However, this may fail in an algebra where nonvanishing elements of $\hat{B}_{\rho|U}$ (such as the determinant in the preceding argument) fail to have inverses in $\hat{B}_{\rho|U}$.

7.11. **Corollary.** *For $A\rho$, $B\rho$ as in Theorem 7.10 let $k = \dim A\rho/x_0$. Then for any k elements $a_1, \cdots, a_k$ of $A\rho$ which are linearly independent at x_0 there exist a neighborhood U of x_0 in $\Delta_{A\rho}$ and an element y of $B\rho$ such that the transform $\hat{a}$ of any element a of $A\rho$ can be uniquely represented on U as $\beta_1\hat{a}_1 + \cdots + \beta_k\hat{a}_k$ where $\beta_i \in C(U)$ and $\beta_i\hat{y}|_U \in \hat{B}\rho|_U$ for all i.*

Proof. Let $a_{k+1}, \cdots, a_n$ be elements of $\ker_n \Delta$ such that $\{a_i\}_{i=1}^n$ are linearly independent at x_0 and let $a_i = \sum_{j=1}^n b_{ji} e_j$ where e_j is the standard basis of $(B\rho)^n$. Let $H = (b_{ij}) \in M_n(B\rho)$ and $y = \det(H)$. Since $\hat{y}(x_0) \neq 0$, there is a neighborhood $U \subset \Delta_{A\rho}$ of x_0 such that $\hat{y}(x) \neq 0$ for $x \in U$. Hence, $\{a_i\}_{i=1}^n$ is a semibasis for A over U and

$$\hat{a}|_U = \sum_{i=1}^{n} \beta_i \hat{a}_i|_U \sum_{j=1}^{n} \left[\sum_{i=1}^{n} \beta_i \hat{b}_{ji}|_U \right] e_j$$

for uniquely determined complex-valued functions β_i on U where $\beta_i = 0$ for $i > k$.

It remains to show that $\beta_i \in C(U)$ and $\beta_i \hat{\gamma}|_U \in \hat{\beta}\rho|_U$. By Cramer's rule the restriction of $\hat{H} = (\hat{b}_{ij})$ to U has an inverse $G = (g_{ij})$ in $GL_n(C(U))$ where $\hat{\gamma}|_U g_{ij} \in \hat{\beta}\rho|_U$ for every (i, j). Note that $\hat{\gamma}|_U G$ is an operator on $(\hat{B}\rho|_U)^n$ and that the kth component of $\hat{\gamma}|_U G\, \hat{a}|_U$ is

$$\hat{\gamma}|_U \sum_{j=1}^{n} g_{kj} \left[\sum_{i=1}^{n} \beta_i b_{ji} \right] = \hat{\gamma}|_U \left(\sum_{i=1}^{n} \beta_i \sum_{j=1}^{n} g_{kj} \hat{b}_{ji} \right) = \hat{\gamma}|_U \sum_{i=1}^{n} \beta_i \delta_{ki} = \hat{\gamma}|_U \beta_k.$$

Since $\hat{\gamma}|_U$ is invertible in $C(U) \supset \hat{B}|_U$ we have that β_k is continuous.

7.12. **Corollary.** *Let A and ρ be as in Theorem 7.10. Then, the map $x \longrightarrow \dim A\rho/x$ is locally constant on $\Delta_{A\rho}$. Furthermore, $A\rho$ is the direct sum of finitely many homogeneous Banach algebras.*

Proof. Since $\Delta_{A\rho}$ is compact the map η: $x \longrightarrow \dim (A\rho/x)^{1/2}$ which was shown to be locally constant in the proof of Theorem 7.10 must attain its maximum. For let e_m be the idempotent of $B\rho$ such that supp $\hat{e}_m = \eta^{-1}(m)$. Such an idempotent exists by the Shilov idempotent theorem. Then,

$$1 = \sum_{m \in \eta(\Delta_{A\rho})} e_m \quad \text{and} \quad A\rho = A\rho\left(\sum e_m\right) = \bigoplus_{m \in \eta(\Delta_{A\rho})} A\rho e_m.$$

The reader may check that $A\rho e_m$ is separable relative to its center $B\rho e_m$ and is $M_m(\mathrm{C})$ modulo any maximal ideal.

A topological space X is called a k-space if it carries the weak topology determined by the family of its compact subspaces. Thus, if Δ_A is a k-space and f: $\Delta_A \longrightarrow T$ is a map to the topological space T such that the restriction of f to any compact set is continuous, then f is continuous. The stipulation that Δ_A be a k-space is a significant one since Dors constructed in [4] a commutative F-algebra whose maximal ideal space is not a k-space.

7.13. **Theorem.** *Let A be a separable, nuclear, l.m.c. algebra such that A and $A \otimes A$ are both barreled and fully complete. If Δ_A is a k-space, then $x \longrightarrow \dim A/x$ is locally constant and A is the countable direct product of homogeneous separable algebras. If, furthermore, Δ_A is locally compact, then A has the local semibasis property.*

Proof. For a continuous submultiplicative seminorm ρ on A the homomorphism $\phi_\rho: A \to A\rho$ induces a continuous injection $y \to \phi_\rho^{-1}(y): \Delta_{A\rho} \to \Delta_A$ which we denote by $(\phi_\rho)^t$. From the compactness of $\Delta_{A\rho}$ it follows that $(\phi_\rho)^t$ is a homeomorphism into. Since A is a barreled l.m.c. algebra, any compact subset K of Δ_A is contained in $(\phi_\rho)^t\Delta_{A\rho}$ for some continuous submultiplicative seminorm ρ (cf. [13, 4.2]). We will show that the restriction to K of the map $x \to \dim (A/x)$ is the composition of $[(\phi_\rho)^t]^{-1}$ with the continuous map of Corollary 7.12. Since K is arbitrary and Δ_A is a k-space, it will follow that the integer-valued function $x \to \dim A/x$ is locally constant on Δ_A.

Let $y = (\phi_\rho)^t(x)$. Then the composition $A \to A\rho \to A\rho/y$ has dense range in the complex matrix algebra $A\rho/y$. From the finite dimensionality of $A\rho/y$, this map is an epimorphism. Hence, the induced map $A/x \to A/y$ is an isomorphism and $\dim (A/x) = \dim A\rho/[(\phi_\rho)^t]^{-1}(x)$ as promised.

Let $\eta(x) = \dim (A/x)^{1/2}$ for $x \in \Delta_A$. Then, for each $m \in \eta(\Delta_A)$ we may find, by the Shilov idempotent theorem, an idempotent e_m of $Z(A)$ such that $\operatorname{supp} e_m = \eta^{-1}(m)$. It follows that $e_k e_m = \delta_{km} e_m$, that $\Delta_{Ae_m} = \eta^{-1}(m)$, that Ae_m is m-homogeneous, and that $A = A(1 - e_m) \oplus Ae_m$. That the natural homomorphism is an isomorphism onto is an immediate consequence of the fact (which we now prove) that given a continuous submultiplicative seminorm ρ on A, the kernel of the canonical homomorphism $\phi_\rho: A \to A\rho = [A(1 - e_m)]\rho + [Ae_m]\rho$ contains Ae_m for all but finitely many m. In view of Corollary 7.12 we can assume without loss of generality that A is n-homogeneous. It follows that for any $x \in \Delta_{[Ae_m]\rho}$ we have

$$M_m(\mathbf{C}) \simeq Ae_m/\phi_\rho^{-1}(x) \simeq [Ae_m]\rho/x \simeq A\rho/([A(I - e_i)]\rho \oplus x) \simeq M_n(\mathbf{C}).$$

This implies that the completion of $\phi_\rho(Ae_m)$ is the trivial Banach algebra whenever $m \neq n$. Consequently, Ae_m is contained in $\ker \phi_\rho$ for all $m \neq n$.

Suppose now, that Δ_A is locally compact. For $x \in \Delta_A$ let us choose a relatively compact open neighborhood U of x in Δ_A. As before, we may choose a continuous submultiplicative seminorm ρ on A such that $(\phi_\rho)^t(\Delta\rho)$ contains the compact set U. We have shown that the diagram

$$\begin{tikzcd} A \arrow[r, "\phi_\rho"] \arrow[d] & A\rho \arrow[d] \\ A/x' \arrow[r] & A\rho/\phi_\rho(x') \end{tikzcd}$$

is commutative for every $x' \in (\phi_\rho)^t(\Delta_{A\rho})$. Thus, the image under ϕ_ρ of a preimage of a basis S for A/x is the preimage of a basis of $A\rho/\overline{\phi_\rho(x)}$ and, hence, a semibasis for $A\rho$ over some neighborhood of $\overline{\phi_\rho(x)}$ contained in $[(\phi_\rho)^t]^{-1}(U)$.

As the bottom row of (i) is an isomorphism for every x' the preimage of S is a semibasis for A over some neighborhood of x contained in U. Therefore, A has the local semibasis property.

7.14. *Epilogue.* To show that A is a matrix algebra we must produce for each $x \in \Delta_A$ a neighborhood U of x and a basis for $A_{|U}$ which is the preimage of a system of matrix units for every A/y where $y \in U$. This may not be possible in view of Corollary 7.11 which suggests that A is not locally free. Nevertheless, according to the same corollary, the "enlarged" algebra

$$\hat{A}_{|U} \otimes_{\widehat{Z(A)}_{|U}} B$$

is free where U is a small neighborhood of x and B is any of a number of extensions of $Z(A)_{|U}$ in $C(U)$. We may ask if the enlarged algebra is isomorphic to $M_n(B)$. The author has recently shown that the answer is "yes" for a canonical choice of B. Since this result is valid for any infinite dimensional separable Banach (hence, nonnuclear) algebra which is separable and finitely generated over its center, we conjecture that the seemingly natural topological hypotheses (cf. 6.1) that we have imposed are far too restrictive. Short of classical separability, are there significantly weaker conditions which together with our topological notion of separability guarantee local finite generation over the center?

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